II.5 SÉRIE DE FOURIER-BESSEL

A equação de Bessel de ordem ν $(\nu > 0)$ é:

$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2) y = 0}$$

Sua solução geral é da forma

$$\boxed{y = c_1 J_\nu(x) + c_2 Y_\nu(x)}$$

onde $J_\nu(x)$ é a função de Bessel de 1ª espécie de ordem ν e $Y_\nu(x)$ é a função de Bessel de 2ª espécie e ordem ν ou função de Neumann de ordem ν. Para $J_\nu(x)$ temos

$$\boxed{J_\nu(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{\Gamma(K+\nu+1)\Gamma(K+1)} \left(\frac{x}{2}\right)^{2K+\nu}}$$

onde $\Gamma(K)$ é a função gama ($\Gamma(K+1) = K!$ para $K \in \mathbb{N}$).
Vemos facilmente que

$$J_0(0) = 1 \,, \quad J_\nu(0) = 0 \ (\nu \neq 0).$$

Já para $Y_\nu(x)$ temos

$$\lim_{x \to 0^+} Y_\nu(x) = -\infty$$

devido à divergência logarítmica (termo da forma $\ln x \sum c_n x^n$).

→ **PROP**: As seguintes identidades e relações são válidas:

(i) $\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = -\frac{J_{\nu+1}(x)}{x^\nu}$ $\quad (\nu \geq 0)$

(ii) $\frac{d}{dx}\left[x^\nu J_\nu\right] = x^\nu J_{\nu-1}(x)$ $\quad (\nu \geq 1)$

(iii) $J_{\nu+1}(x) - J_{\nu-1}(x) = -2J_\nu'(x)$ $\quad (\nu \geq 1)$

(iv) $J_{\nu+1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$ $\quad (\nu \geq 1)$

(v) $\frac{J_n(x)}{x^n} = \left(-\frac{1}{x}\frac{d}{dx}\right)^n (J_0(x))$ $\quad (\nu = n \in \mathbb{N})$

DEM:

• (i)

$$\frac{d}{dx}\left[\frac{J_\nu}{x^\nu}\right] = \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k \quad (x/2)^{2k}}{\Gamma(k+\nu+1)\Gamma(k+1)}\right] = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2k\,(1/2)\,(x/2)^{2k-1}}{\Gamma(k+\nu+1)\,\underbrace{\Gamma(k+1)}_{k\Gamma(k)}}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{2k-1}}{\Gamma(k+\nu+1)\Gamma(k)} = \sum_{k=0}^{\infty} \frac{(-1)(-1)^k (x/2)^{2k+1}}{\Gamma(k+(\nu+1)+1)\,\Gamma(k+1)} =$$

$$= -\frac{1}{x^\nu}\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+(\nu+1)}}{\Gamma(k+(\nu+1)+1)\,\Gamma(k+1)} = -\frac{J_{\nu+1}}{x^\nu}$$

✓

• (ii): é análogo a (i)

• (iii) e (iv): somando e substraindo (i) e (ii):

(i) → $(x^{-\nu}J_\nu)' = -\nu x^{-\nu-1}J_\nu + x^{-\nu}J_\nu' = -x^{-\nu}J_{\nu+1}$

(ii) ⇒ $(x^\nu J_\nu)' = \nu x^{\nu-1}J_\nu + x^\nu J_\nu = x^\nu J_{\nu-1}$

• (v): segue usando (i) repetidamente quando ν é inteiro:

(i) ⇒ $\frac{J_n}{x^n} = -\frac{1}{x}\frac{d}{dx}\left(\frac{J_{n-1}}{x^{n-1}}\right)$

✓

- Vamos agora considerar o PSL associado com a eq. de Bessel. Primeiro, vamos fazer a mudança de variável

$$\boxed{x = \sqrt{\lambda}\, t}$$

Na eq. de Bessel,

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

que na forma auto-adjunta é,

$$(x y')' - \frac{\nu^2}{x} y + x y = 0$$

temos em termos da variável t,

$$\boxed{\frac{d}{dt}\left(t \frac{dY}{dt}\right) - \frac{\nu^2}{t} Y + \lambda t Y = 0}$$

onde definimos $Y(t) = y(\sqrt{\lambda}\, t)$ e usamos $\frac{d}{dt} = \sqrt{\lambda} \frac{d}{dx}$.

A equação acima está na forma do PSL com

$$p(t) = t \,, \quad q(t) = \frac{\nu^2}{t} \,, \quad \rho(t) = t$$

ou seja, trata-se de um PSL <u>SINGULAR</u> para $t \to 0$.

Nesse caso, as condições apropriadas são:

- Y, Y' limitadas para $t \to 0$
- $c_1 Y(a) + c_2 Y'(a) = 0$

<u>OBS</u>: Se Y e Y' devem ser limitadas para $t \to 0$, devemos ter y e y' limitadas para $x \to 0$, o que implica que devemos descartar a função de Bessel de 2ª espécie pois esta <u>NÃO</u> é limitada.

$$\begin{cases} \dfrac{d}{dt}\left(t\dfrac{dY}{dt}\right) - \dfrac{\nu^2}{t}Y + \lambda t Y = 0 & , 0<t<a \\ Y, Y' \text{ limitadas para } t \to 0 \quad (i) & \\ Y(a) = 0 \quad (ii) & \end{cases}$$

$(i) \Rightarrow Y(t) = y(\sqrt{\lambda}\,t) = J_\nu(\sqrt{\lambda}\,t)$

$(ii) \Rightarrow Y(a) = 0 \Rightarrow J_\nu(\sqrt{\lambda}\,a) = 0$

$$\boxed{\alpha_{\nu m} = m\text{-ésimo zero de } J_\nu \quad (m = 1, 2, \ldots)}$$

$\therefore \boxed{\sqrt{\lambda}\,a = \alpha_{\nu m}} \Rightarrow$ auto-valores $\boxed{\lambda_m = \dfrac{\alpha_{\nu m}^2}{a^2}}$ $(m = 1, 2, \ldots)$

$\therefore Y_m(t) = J_\nu(\sqrt{\lambda}\,t) = J_\nu\left(\dfrac{\alpha_{\nu m}t}{a}\right) \Rightarrow$ auto-funções $\boxed{Y_m(t) = J_\nu\left(\dfrac{\alpha_{\nu m}t}{a}\right)}$ $(m = 1, 2, \ldots)$

➡ relação de ortogonalidade: $\rho(t) = t$, $I = [0, a]$

$$\boxed{\int_0^a dt\, t\, J_\nu\left(\frac{\alpha_{\nu m}t}{a}\right) J_\nu\left(\frac{\alpha_{\nu n}t}{a}\right) = 0 \quad , m \neq n}$$

➡ normalização:

$$(\Delta)\quad \frac{d}{dt}\left(t\frac{d}{dt}J_\nu(\sqrt{\lambda}\,t)\right) - \frac{\nu^2}{t}J_\nu(\sqrt{\lambda}\,t) + \lambda t J_\nu(\sqrt{\lambda}\,t) = 0$$

$$(\Delta\Delta)\quad \frac{d}{dt}\left(t\frac{d}{dt}J_\nu(\sqrt{\mu}\,t)\right) - \frac{\nu^2}{t}J_\nu(\sqrt{\mu}\,t) + \mu t J_\nu(\sqrt{\mu}\,t) = 0$$

Fazendo $(\Delta)\,J_\nu(\sqrt{\mu}\,t) - (\Delta\Delta)\,J_\nu(\sqrt{\lambda}\,t) \Rightarrow$

$$\frac{d}{dt}\left(t\frac{d}{dt}J_\nu(\sqrt{\lambda}\,t)\right)J_\nu(\sqrt{\mu}\,t) - \frac{d}{dt}\left(t\frac{d}{dt}J_\nu(\sqrt{\mu}\,t)\right)J_\nu(\sqrt{\lambda}\,t) +$$

$$+ \lambda t J_\nu(\sqrt{\lambda}\,t)J_\nu(\sqrt{\mu}\,t) - \mu t J_\nu(\sqrt{\mu}\,t)J_\nu(\sqrt{\lambda}\,t) = 0$$

e daí:

$$(\mu-\lambda)\int_0^a dt\, t\, J_\nu(\sqrt{\mu}\,t)J_\nu(\sqrt{\lambda}\,t) = \int_0^a dt\, \frac{d}{dt}\left(t\frac{d}{dt}J_\nu(\sqrt{\lambda}\,t)\right)J_\nu(\sqrt{\mu}\,t)$$

$$- \int_0^a dt\, \frac{d}{dt}\left(t\frac{d}{dt}J_\nu(\sqrt{\mu}\,t)\right)J_\nu(\sqrt{\lambda}\,t)$$

$$= t\frac{d}{dt}\left(J_\nu(\sqrt{\lambda}\,t)\right)J_\nu(\sqrt{\mu}\,t)\Big|_0^a - \cancel{\int_0^a dt\, t\,\frac{d}{dt}\left(J_\nu(\sqrt{\lambda}\,t)\right)\frac{d}{dt}\left(J_\nu(\sqrt{\mu}\,t)\right)}$$

$$- t\frac{d}{dt}\left(J_\nu(\sqrt{\mu}\,t)\right)J_\nu(\sqrt{\lambda}\,t)\Big|_0^a + \cancel{\int_0^a dt\, t\,\frac{d}{dt}\left(J_\nu(\sqrt{\mu}\,t)\right)\frac{d}{dt}\left(J_\nu(\sqrt{\lambda}\,t)\right)}$$

$$(\mu-\lambda)\int_0^a dt\, t\, J_\nu(\sqrt{\mu}\, t) J_\nu(\sqrt{\lambda}\, t) = a\sqrt{\lambda}\, J_\nu'(\sqrt{\lambda}\, a) J_\nu(\sqrt{\mu}\, a) - a\sqrt{\mu}\, J_\nu'(\sqrt{\mu}\, a) J_\nu(\sqrt{\lambda}\, a)$$

Tomando, por exemplo, $\sqrt{\lambda} = \frac{\alpha_{\nu m}}{a}$, temos $J_\nu(\sqrt{\lambda}\, a) = J_\nu\left(\frac{\alpha_{\nu m}}{a} a\right) = 0$.
Logo:

$$\left(\mu - \frac{\alpha_{\nu m}^2}{a^2}\right)\int_0^a dt\, t\, J_\nu(\sqrt{\mu}\, t) J_\nu\left(\frac{\alpha_{\nu m} t}{a}\right) = \alpha_{\nu m} J_\nu'(\alpha_{\nu m}) J_\nu(\sqrt{\mu}\, a)$$

Note que tomando $\sqrt{\mu} = \frac{\alpha_{\nu n}}{a}$, temos a relação de ortogonalidade para $n \neq m$. Derivando a expressão acima com relação a $\sqrt{\mu}$:

$$2\sqrt{\mu}\int_0^a dt\, t\, J_\nu(\sqrt{\mu}\, t) J_\nu\left(\frac{\alpha_{\nu m} t}{a}\right) + \left(\mu - \frac{\alpha_{\nu m}^2}{a^2}\right)\int_0^a dt\, t^2 J_\nu'(\sqrt{\mu}\, t) J_\nu\left(\frac{\alpha_{\nu m} t}{a}\right)$$

$$= \alpha_{\nu m}\, a\, J_\nu'(\alpha_{\nu m}) J_\nu'(\sqrt{\mu}\, a)$$

Tomando agora $\sqrt{\mu} = \frac{\alpha_{\nu m}}{a}$, temos:

$$2\frac{\alpha_{\nu m}}{a}\int_0^a dt\, t\, J_\nu^2\left(\frac{\alpha_{\nu m}}{a} t\right) = \alpha_{\nu m}\, a \left[J_\nu'(\alpha_{\nu m})\right]^2$$

$$\boxed{\int_0^a dt\, t\, J_\nu^2\left(\frac{\alpha_{\nu m}}{a} t\right) = \frac{a^2}{2}\left[J_\nu'(\alpha_{\nu m})\right]^2}$$

Mas temos as relações:

$$J_{\nu+1} - J_{\nu-1} = -2\,J_\nu' \quad , \quad J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu$$

$$\therefore \quad J_{\nu+1} - \left(\frac{2\nu J_\nu}{x} - J_{\nu+1}\right) = -2\,J_\nu'$$

$$2\,J_{\nu+1} - \frac{2\nu}{x} J_\nu = -2 J_\nu'$$

$$\therefore \quad 2\,J_{\nu+1}(\alpha_{\nu m}) - \frac{2\nu}{\alpha_{\nu m}} \underbrace{J_\nu(\alpha_{\nu m})}_{=0} = -2\,J_\nu'(\alpha_{\nu m})$$

$$\therefore \quad \boxed{J_\nu'(\alpha_{\nu m}) = -J_{\nu+1}(\alpha_{\nu m})}$$

e daí:

$$\boxed{\int_0^a dt\; t\, J_\nu^2\!\left(\frac{\alpha_{\nu m}}{a} t\right) = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu m})\right]^2}$$

Resumindo:

$$\boxed{\int_0^a dt\; t\, J_\nu\!\left(\frac{\alpha_{\nu m}}{a} t\right) J_\nu\!\left(\frac{\alpha_{\nu n}}{a} t\right) = \delta_{mn}\, \frac{a^2}{2} J_{\nu+1}^2(\alpha_{\nu n})}$$

$\boxed{EX}$ Expansão de $f(x) = x^\nu$ em série de Fourier-Bessel de ordem ν

$$f(x) = \sum_{n=1}^{\infty} c_n J_\nu\left(\frac{\alpha_{\nu n}}{a} x\right) \Rightarrow \int_0^a dx\, x f(x) J_\nu\left(\frac{\alpha_{\nu m} x}{a}\right) = \sum_{n=1}^{\infty} c_n \underbrace{\int_0^a dx\, x\, J_\nu\left(\frac{\alpha_{\nu n} x}{a}\right) J_\nu\left(\frac{\alpha_{\nu m} x}{a}\right)}_{\frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu n})]^2 \delta_{nm}}$$

$$\therefore\ c_n = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu n})]^2} \int_0^a dx\, x\, f(x) J_\nu\left(\frac{\alpha_{\nu n} x}{a}\right)$$

$$\therefore f(x) = x^\nu \Rightarrow \quad c_n = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu n})]^2} \underbrace{\int_0^a dx\, x^{\nu+1} J_\nu\left(\frac{\alpha_{\nu n} x}{a}\right)}_{I}$$

<u>MAS</u>: $\frac{d}{dx}[x^\nu J_\nu] = x^\nu J_{\nu-1}$

$$\therefore\ I = \left(\frac{a}{\alpha_{\nu n}}\right)^{\nu+2} \int_0^{\alpha_{\nu n}} d\left(\frac{\alpha_{\nu n}}{a} x\right) \left(\frac{\alpha_{\nu n} x}{a}\right)^{\nu+1} J_\nu\left(\frac{\alpha_{\nu n}}{a} x\right)$$

$$= \left(\frac{a}{\alpha_{\nu n}}\right)^{\nu+2} \int_0^{\alpha_{\nu n}} dy\, y^{\nu+1} J_\nu(y) = \left(\frac{a}{\alpha_{\nu n}}\right)^{\nu+2} \int_0^{\alpha_{\nu n}} dy \frac{d}{dy}[y^{\nu+1} J_{\nu+1}]$$

$$= \left(\frac{a}{\alpha_{\nu n}}\right)^{\nu+2} y^{\nu+1} J_{\nu+1}(y)\Big|_0^{\alpha_{\nu n}} = \left(\frac{a}{\alpha_{\nu n}}\right)^{\nu+2} (\alpha_{\nu n})^{\nu+1} J_{\nu+1}(\alpha_{\nu n})$$

$$= \frac{a^{\nu+2}}{\alpha_{\nu n}} J_{\nu+1}(\alpha_{\nu n})$$

$$\therefore\ c_n = \frac{2}{a^2 J_{\nu+1}^2(\alpha_{\nu n})} \cdot \frac{a^{\nu+2}}{\alpha_{\nu n}} J_{\nu+1}(\alpha_{\nu n}) = \frac{2a^\nu}{\alpha_{\nu n} J_{\nu+1}(\alpha_{\nu n})}$$

$$\therefore\ x^\nu = \sum_{n=1}^{\infty} \frac{2a^\nu}{\alpha_{\nu n} J_{\nu+1}(\alpha_{\nu n})} J_\nu\left(\frac{\alpha_{\nu n}}{a} x\right), \quad 0 \le x \le a$$

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II.6 WAVELETS ("ONDALETAS")

Vamos considerar

$$f_n(x) = \chi_{[n,n+1)}(x) = \begin{cases} 1 & , n \le x < n+1 \\ 0 & , x < n,\ x \ge n+1 \end{cases}$$

Evidentemente

$$\int_{-\infty}^{+\infty} dx\, f_n(x) f_m(x) = 0 \quad , \; m \ne n$$

$$\int_{-\infty}^{+\infty} dx\, f_n^2(x) = \int_n^{n+1} dx = 1$$

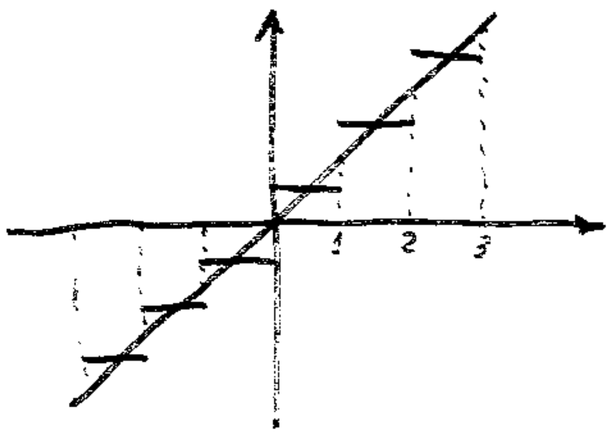
$\{f_n(x)\}$ é ortogonal mas não é completo

EX II

$$f(x) = x = \sum_n c_n f_n \quad \therefore \quad c_n = \int_{-\infty}^{+\infty} dx\, x f_n = \int_n^{n+1} dx\, x$$

$$= \frac{x^2}{2}\Big|_n^{n+1} = \frac{(n+1)^2 - n^2}{2} = \frac{2n+1}{2}$$

$$\therefore \quad x = \sum_{n=-\infty}^{+\infty} \left(\frac{2n+1}{2}\right) f_n$$



← não é completo pois o desvio total quadrático não tende a zero quando $N \to \infty$.

• outro conjunto:

$$\phi(x) = \chi_{[0,1)}(x)$$

$$\phi_n(x) = \phi(x-n) = \chi_{[0,1)}(x-n) = \chi_{[n,n+1)}(x)$$

Dilatação $\Rightarrow$ $\phi(x) \longmapsto \phi(2^m x)$ (dilatação binária (2^m))

$$\int_{-\infty}^{+\infty} dx\, \phi(2^m x)\phi(2^m x) = 2^{-m} \underbrace{\int_{-\infty}^{+\infty} d(2^m x)\, \phi(2^m x)\phi(2^m x)}_{1} = 2^{-m}$$

$\therefore$ normalização $\Rightarrow$ $2^{m/2}\phi(2^m x)$

Translação $\Rightarrow$ $\phi(2^m x) \longmapsto \phi(2^m x - n) = \phi(2^m(x - n2^{-m}))$

(translação diádica $n2^{-m}$)

$$\boxed{\phi_{mn}(x) = 2^{m/2}\phi(2^m x - n)}$$

OBS: $\{\phi_{mn}\}$ deve ser completo por causa da dilatação.

MAS não é ortogonal...

De fato, $\phi(x)$ e $\phi(2x)$ não são ortogonais:

$$\phi(x)=\begin{cases}0\ , & x<0\ ,\ x\geq 1\\ 1\ , & 0\leq x<1\end{cases}\quad ,\quad \phi(2x)=\begin{cases}0\ , & 2x<0\ ,\ 2x\geq 1\\ 1\ , & 0\leq 2x<1\end{cases}$$

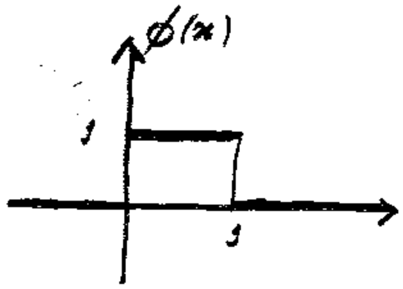
$$\therefore \int_{-\infty}^{+\infty} dx\, \phi(x)\,\phi(2x) = \int_{0}^{1/2} dx.\, 1.1 = \frac{1}{2} \neq 0$$

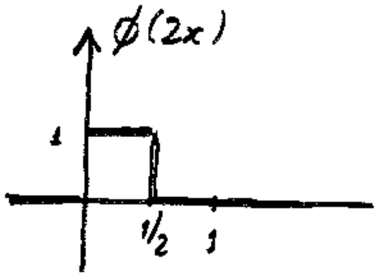
$\therefore$ $\phi(x)$ e $\phi(2x)$ não são ortogonais

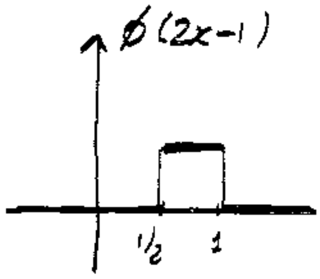
<u>MAS</u> $\psi(x) = \phi(2x) - \phi(2x-1)$ é !

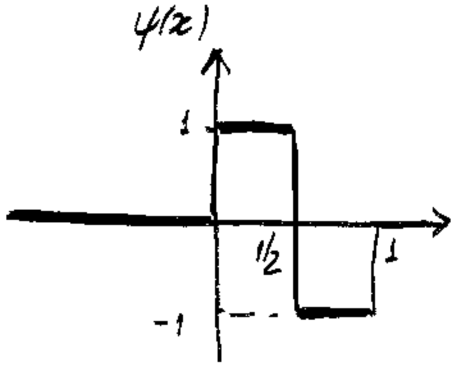
$$\phi(2x)=\begin{cases}0\ , & x<0\ ,\ x\geq 1/2\\ 1\ , & 0\leq x<\frac{1}{2}\end{cases}\quad ,\quad \underbrace{\phi(2x-1)}_{\phi(2(x-1/2))}=\begin{cases}0\ , & x<1/2\ ,\ x\geq 1\\ 1\ , & \frac{1}{2}\leq x<1\end{cases}$$

translação por 1/2



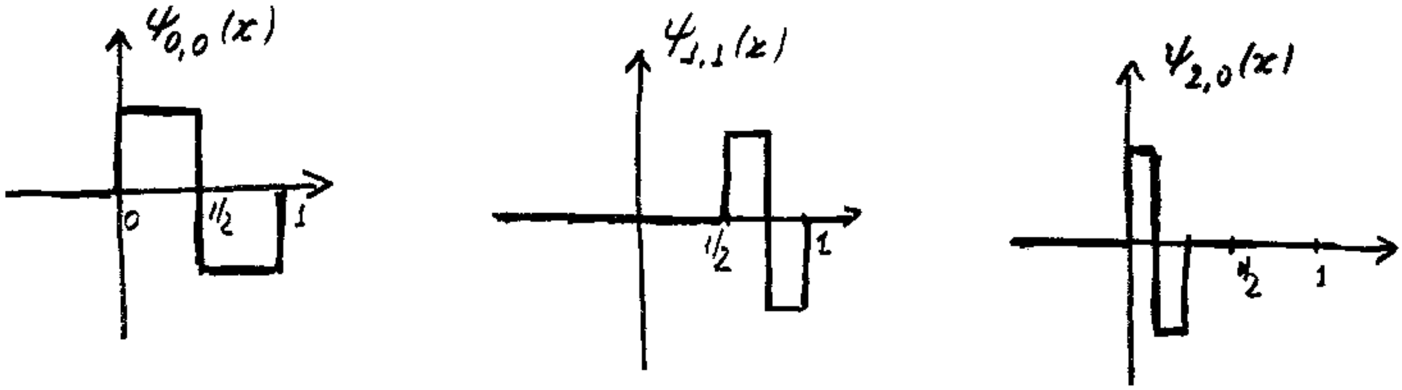






$$\boxed{\psi_{mn}(x) = 2^{m/2}\,\psi(2^m x - n)}$$ ⟵ WAVELETS DE HAAR

(é ortogonal e completo)



Assim: $$f(x) = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_{mn}\,\psi_{mn}(x)$$

$$\langle \psi_{mn}, \psi_{m'n'} \rangle = \int_{-\infty}^{+\infty} dx\, \psi_{mn}(x)\,\psi_{m'n'}(x) = \delta_{mm'}\,\delta_{nn'}$$

$$\therefore \quad c_{mn} = \langle f, \psi_{mn} \rangle = \int_{-\infty}^{+\infty} dx\, f(x)\,\psi_{mn}(x)$$

EX Expressar $f(x) = \sin 2\pi x$ em série de Fourier-Haar

$$c_{mn} = 2^{m/2}\int_{-\infty}^{+\infty} dx\, f(x)\,\psi(2^m x - n) = 2^{m/2}\int_{2^{-m}n}^{2^{-m}(n+1)} dx\, f(x)\,\psi(2^m x - n)$$

$$= 2^{m/2}\left[\int_{2^{-m}n}^{2^{-m}(n+1/2)} dx\, f(x) - \int_{2^{-m}(n+1/2)}^{2^{-m}(n+1)} dx\, f(x)\right]$$

$$\therefore C_{mn} = 2^{m/2}\left[\int_{2^{-m}n}^{2^{-m}(n+1/2)} dx \sin 2\pi x - \int_{2^{-m}(n+1/2)}^{2^{-m}(n+1)} dx \sin 2\pi x\right]$$

$$= \frac{2^{m/2}}{2\pi}\left[\cos 2\pi\, 2^{-m} n - \cos 2\pi\, 2^{-m}(n+1/2) + \cos 2\pi\, 2^{-m}(n+1) - \cos 2\pi\, 2^{-m}(n+1/2)\right]$$

$\therefore$

$$C_{mn} = \frac{2^{m/2}}{2\pi}\left[\cos 2^{-m+1} n\pi - 2\cos 2^{-m+1}(n+1/2)\pi + \cos 2^{-m+1}(n+1)\pi\right]$$

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• <u>OBS</u>: "NOMENCLATURA"

$\phi(x)$ = função de escala (scaling function, father wavelet)

$\psi(x)$ = wavelet-mãe (mother wavelet)

$\{\psi_{mn}(x)\}$ = base de wavelets

• <u>OBS</u>: outras wavelets

(i) $\psi(x) = (1-x^2)\, e^{-x^2/2}$ (wavelet "chapéu mexicano")

(ii) $\psi(x) = e^{ik_0 x}\, e^{-x^2/2}$ (k_0 fixo) (wavelet de Morlet)

(iii) $\psi(x) = \dfrac{\sin(\pi x/2)}{(\pi x/2)} \cos(3\pi x/2)$ (wavelet de Shannon)

II.7 SISTEMAS ORTOGONAIS DE FUNÇÕES DE VÁRIAS VARIÁVEIS

produto escalar: $\langle f, g \rangle = \iint_R dx\,dy\, \rho(x,y)\, f(x,y)\, g(x,y)$

EX Fourier dupla

$$f(x,y) = \frac{a_0}{4} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \big(a_{mn} \cos mx \cos ny + b_{mn} \cos mx \sin ny + c_{mn} \sin mx \cos ny + d_{mn} \sin mx \sin ny \big)$$

onde:

$$a_0 = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx\,dy\, f(x,y), \quad a_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx\,dy\, f(x,y) \cos mx \cos ny$$

etc...

ou na forma complexa:

$$f(x,y) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{mn}\, e^{imx} e^{iny}$$

onde:

$$c_{mn} = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx\,dy\, f(x,y)\, e^{-imx} e^{-iny}$$

EX III Fourier em várias variáveis

$$\vec{x} = (x_1, \dots, x_N) \quad , \quad \vec{x}\cdot\vec{y} = \sum_{i=1}^{N} x_i y_i$$

$$f(\vec{x}) = f(x_1, \dots, x_N) \quad , \quad \vec{m} = (m_1, \dots, m_N)$$

$$f(\vec{x}) = \underbrace{\sum_{m_1=-\infty}^{+\infty} \cdots \sum_{m_N=-\infty}^{+\infty}}_{\sum_{\vec{m}\in \mathbb{Z}_N}} \underbrace{C_{m_1 \dots m_N}}_{C_{\vec{m}}} \underbrace{e^{i m_1 x_1} \cdots e^{i m_N x_N}}_{e^{i\vec{m}\cdot\vec{x}}}$$

$$\therefore \boxed{f(\vec{x}) = \sum_{\vec{m}\in \mathbb{Z}_N} C_{\vec{m}}\, e^{i\vec{m}\cdot\vec{x}}}$$

onde :

$$\therefore C_{m_1 \dots m_N} = \frac{1}{(2\pi)^N} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dx_1 \dots dx_N \, f(x_1, \dots, x_N) e^{-i m_1 x_1} \dots e^{-i m_N x_N}$$

$$\therefore \boxed{C_{\vec{m}} = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} d\vec{x}\, f(\vec{x})\, e^{-i\vec{m}\cdot\vec{x}}}$$

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