I.5.2. LEMAS AUXILIARES

• **LEMA 1**: Se F é contínua por partes em $[a,b]$, então

$$(i)\quad \lim_{K\to\infty}\int_a^b F(x)\sin Kx\,dx = 0$$

$$(ii)\quad \lim_{K\to\infty}\int_a^b F(x)\cos Kx\,dx = 0$$

DEM: Vamos dividir (a,b) em intervalos onde F é contínua. Vamos denotar um desses intervalos por $[p,q]$. Então vale (i) se

$$I = \lim_{K\to\infty}\int_p^q F(x)\sin Kx\,dx = 0$$

$$\underset{p\qquad\qquad\qquad q}{x_0\ x_1\ x_2\ \ldots\ x_{n-1}\ x_n} \qquad I = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} F(x)\sin Kx\,dx$$

$$I = \sum_{i=0}^{n-1}\left[F(x_i)\underbrace{\int_{x_i}^{x_{i+1}} dx\,\sin Kx}_{\frac{\cos Kx_i - \cos Kx_{i+1}}{K}} + \int_{x_i}^{x_{i+1}}[F(x)-F(x_i)]\sin Kx\,dx \right]$$

$$|I| \le \sum_{i=0}^{n-1}\left|F(x_i)\left(\frac{\cos Kx_i - \cos Kx_{i+1}}{K}\right)\right| + \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |F(x)-F(x_i)|\underbrace{|\sin Kx|}_{\le 1}dx$$

Seja $M = \max\limits_{x\in[p,q]} F(x)$. Então:

$$\sum_{i=0}^{n-1} |F(x_i)| \left|\frac{\cos Kx_i - \cos Kx_{i+1}}{K}\right| \le \sum_{i=0}^{n-1} M\left(\frac{\overbrace{|\cos Kx_i|}^{\le 1}+\overbrace{|\cos Kx_{i+1}|}^{\le 1}}{K}\right)$$

$$\le \sum_{i=0}^{n-1} \frac{2M}{K} = \frac{2Mn}{K}$$

Por outro lado, como F é contínua em $[p,q]$,

$|F(x) - F(x_i)| < \varepsilon'$ sempre que $|x - x_i| < \delta$

Como $x_{i+1} - x_i = \dfrac{q-p}{n}$, para $x_{i+1} - x_i < \delta$ devemos ter

$n > N$, onde $N = \left[\dfrac{q-p}{\delta}\right]$ = parte inteira de $\left(\dfrac{q-p}{\delta}\right)$

Logo, se $n > N$:

$$\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |F(x) - F(x_i)|\,dx < \varepsilon' \sum_{i=0}^{n-1} (x_{i+1} - x_i) = \varepsilon'(p-q) = \varepsilon$$

$$\therefore \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |F(x) - F(x_i)|\,dx < \varepsilon \quad \text{para } n > N$$

Voltando para I e tomando $n = N+1$:

$$|I| \leq \frac{2Mn}{K} + \varepsilon = \frac{2M(N+1)}{K} + \varepsilon$$

Para $K > K_0$, onde $K_0 = \dfrac{2M(N+1)}{\varepsilon}$, temos

$$|I| \leq \varepsilon + \varepsilon = 2\varepsilon \text{ sempre que } K > K_0$$

$$\therefore \lim_{K\to\infty} I = 0$$

$$\therefore \lim_{K\to\infty} \int_a^b F(x) \sin Kx \, dx = 0$$

Para (ii) a demonstração é análoga.

• <u>LEMA 2</u>: $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

<u>DEM</u>: $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = F(0)$, onde $\displaystyle F(t) = \int_0^\infty dx\, e^{-tx} \frac{\sin x}{x} \quad (t \geq 0)$

Seja $S(x) = \dfrac{\sin x}{x}$. Vamos primeiro mostrar que

$|S(x)| \leq 1$ para $x \geq 0$.

$$S(x) = \frac{x - x^3/3! + x^5/5! \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots$$

$$= 1 - \underbrace{\left(\frac{x^2}{3!} - \frac{x^4}{5!}\right)}_{\substack{\geq 0 \\ \text{para } 0 \leq x \leq 1}} - \underbrace{\left(\frac{x^6}{7!} - \frac{x^8}{9!}\right)}_{\substack{\geq 0 \\ \text{para } 0 \leq x \leq 1}} - \underbrace{(\cdots)}_{\substack{\geq 0 \\ \text{para } 0 \leq x \leq 1}} \cdots$$

$$\therefore \; S(x) \leq 1 \quad \text{para} \quad 0 \leq x \leq 1$$

Mas $S(0) = 1$ e o primeiro zero de $S(x)$ é $x_0 = \pi > 1$. Como $S(x)$ é contínua (definida pela série acima)

$$0 < S(x) \leq 1 \quad \text{para} \quad 0 \leq x \leq 1$$

Por outro lado, para $|x| > 1$:

$$|S(x)| = \left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{|x|} < \frac{1}{|x|} < 1 \qquad (|x| > 1)$$

$$\therefore \quad |S(x)| \leq 1 \quad \text{para} \quad x \geq 0$$

Voltando para $F(t)$, temos:

$$|F(t)| \leq \int_0^\infty dx \, |e^{-tx}| \underbrace{\left|\frac{\sin x}{x}\right|}_{\leq 1} \leq \int_0^\infty dx \, e^{-tx} = \left.\frac{e^{-tx}}{-t}\right|_0^\infty = \frac{1}{t}$$

$$\therefore |F(t)| \leq \frac{1}{t} \quad \Rightarrow \lim_{t \to \infty} F(t) = 0$$

Por outro lado:

$$F'(t) = \int_0^\infty dx\,(-x)\,e^{-tx}\frac{\sin x}{x} = -\int_0^\infty dx\, e^{-tx}\sin x$$

$$\int dx\, e^{-tx}\sin x = -e^{-tx}\cos x - t\int dx\, e^{-tx}\cos x = -e^{-tx}\cos x - t e^{-tx}\sin x - t^2\int dx\, e^{-tx}\sin x$$

$$\therefore (1+t^2)\int_0^\infty dx\, e^{-tx}\sin x = -e^{-tx}\cos x\Big|_0^\infty - t e^{-tx}\sin x\Big|_0^\infty = e^0\cos 0 = 1$$

$$\therefore \int_0^\infty dx\, e^{-tx}\sin x = \frac{1}{1+t^2}$$

$$\therefore F'(t) = -\frac{1}{1+t^2} \Rightarrow F(t) = -\tan^{-1} t + C$$

$$0 = \lim_{t\to\infty} F(t) = -\lim_{t\to\infty}\tan^{-1} t + C = -\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2}$$

$$\therefore \int_0^\infty dx\,\frac{\sin x}{x} = F(0) = -\underbrace{\tan^{-1} 0}_{0} + \frac{\pi}{2} = \frac{\pi}{2}$$

<u>OBS</u>: Se escolhessemos $\lim_{t\to\infty}\tan^{-1} t = \frac{3\pi}{2}$, por exemplo, então $C = \frac{3\pi}{2}$ e como $\tan^{-1} 0 = \pi$ nesse caso teríamos $\frac{3\pi}{2} - \pi = \frac{\pi}{2}$

• <u>LEMA 3</u>: Se F é contínua por partes em $[0,b]$ e tem derivada à direita $F'_+(0)$, então

$$\lim_{K\to\infty}\int_0^b F(x)\frac{\sin Kx}{x}\,dx = \frac{\pi}{2}F(0+)$$

<u>DEM</u>:

$$I(K)=\int_0^b F(x)\frac{\sin Kx}{x}\,dx = F(0+)\int_0^b \frac{\sin Kx}{x}\,dx + \int_0^b [F(x)-F(0+)]\frac{\sin Kx}{x}\,dx$$

MAS:

$$\lim_{K\to\infty}\int_0^b \frac{\sin Kx}{x}\,dx = \lim_{K\to\infty}\int_0^{Kb}\frac{\sin t}{t}\,dt = \int_0^\infty \frac{\sin t}{t}\,dt \overset{\text{LEMA 2}}{=} \frac{\pi}{2}$$

$$\lim_{K\to\infty}\int_0^b \underbrace{\left[\frac{F(x)-F(0+)}{x}\right]}\sin Kx\,dx \overset{\text{LEMA 1}}{=} 0$$

p/ $x\to 0$ é igual a $F'_+(0)$, que supomos que existe (logo essa quantidade é contínua por partes)

$$\therefore \lim_{K\to\infty} I(K) = F(0+)\frac{\pi}{2} + 0$$

✓

• LEMA 4: Seja F uma função contínua por partes em (a,b) e que tem derivadas laterais à esquerda e à direita em um ponto x_0 tal que $a<x_0<b$. Então

$$\lim_{K\to\infty}\int_a^b F(x)\,\frac{\sin K(x-x_0)}{x-x_0}\,dx = \pi\;\frac{F(x_0+0)+F(x_0-0)}{2}$$

DEM:

$$I(K)=\int_a^b F(x)\,\frac{\sin K(x-x_0)}{x-x_0}\,dx$$

$$=\int_a^{x_0} F(s)\,\frac{\sin K(s-x_0)}{s-x_0}\,ds+\int_{x_0}^{b} F(t)\,\frac{\sin K(t-x_0)}{t-x_0}\,dt$$

$$\downarrow x_0-s=x \qquad\qquad \downarrow t-x_0=x$$

$$=-\int_{x_0-a}^{0}\underbrace{F(x_0-x)}_{G_1(x)}\,\frac{\sin Kx}{x}\,dx+\int_0^{b-x_0}\underbrace{F(x_0+x)}_{G_2(x)}\,\frac{\sin Kx}{x}\,dx$$

$$=\int_0^{x_0-a} G_1(x)\,\frac{\sin Kx}{x}\,dx+\int_0^{b-x_0} G_2(x)\,\frac{\sin Kx}{x}\,dx$$

$$\therefore \lim_{K\to\infty} I(K)=\underbrace{\lim_{K\to\infty}\int_0^{x_0-a} G_1(x)\,\frac{\sin Kx}{x}\,dx}_{\Downarrow\ \text{LEMA 3}}+\underbrace{\lim_{K\to\infty}\int_0^{b-x_0} G_2(x)\,\frac{\sin Kx}{x}\,dx}_{\Downarrow\ \text{LEMA 3}}$$

$$=\frac{\pi}{2}\,G_1(0+)\qquad+\qquad\frac{\pi}{2}\,G_2(0+)$$

Mas: $G_1(x) = F(x_0 - x) \Rightarrow G_1(0+) = F(x_0 - 0)$

$G_2(x) = F(x_0 + x) \Rightarrow G_2(0+) = F(x_0 + 0)$

$$\therefore \lim_{K \to \infty} I(K) = \frac{\pi}{2}\left[F(x_0 - 0) + F(x_0 + 0)\right]$$

I.5.3. DEMONSTRAÇÃO DO TEOREMA DE FOURIER

De ⊛ na pág. 19:

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} d\xi\, f(\xi)\left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\xi - x)\right]$$

de modo que a N-ésima soma parcial é:

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} d\xi\, f(\xi)\left[\frac{1}{2} + \sum_{n=1}^{N} \cos n(\xi - x)\right]$$

Por outro lado:

$$\sum_{n=1}^{N} \cos nu = \sum_{n=1}^{N}\left(\frac{e^{inu} + e^{-inu}}{2}\right)$$

$$\therefore \quad 2\sum_{n=1}^{N} \cos nu = \sum_{n=1}^{N} (e^{iu})^n + \sum_{n=1}^{N} (e^{-iu})^n$$

mas: $a^{N+1}-1=(a-1)(1+a+a^2+\dots+a^N) \Rightarrow \sum_{n=0}^{N} a^n = \dfrac{a^{N+1}-1}{a-1} \quad (a\neq 1)$

$$\therefore\ 2\sum_{n=1}^{N} \cos nu = \sum_{n=0}^{N} (e^{iu})^n - 1 + \sum_{n=0}^{N} (e^{-iu})^n - 1$$

$$= \frac{(e^{iu})^{N+1}-1}{e^{iu}-1} - 1 + \frac{(e^{-iu})^{N+1}-1}{e^{-iu}-1} - 1$$

$$= \frac{e^{iu(N+1)} \cancel{-1} - e^{+iu} \cancel{+1}}{e^{iu}-1} + \frac{e^{-iu(N+1)} \cancel{-1} - e^{-iu} \cancel{+1}}{e^{-iu}-1}$$

$$= \frac{e^{iu(N+1)} - e^{iu}}{e^{iu/2}(e^{iu/2}-e^{-iu/2})} + \frac{e^{-iu(N+1)} - e^{-iu}}{e^{-iu/2}(e^{-iu/2}-e^{iu/2})}$$

$$= \frac{e^{iu(N+1/2)} - e^{+iu(1-1/2)} - e^{-iu(N+1/2)} + e^{-iu(1-1/2)}}{e^{iu/2}-e^{-iu/2}}$$

$$= \frac{e^{iu(N+1/2)} - e^{-iu(N+1/2)}}{e^{iu/2}-e^{-iu/2}} - \frac{e^{iu/2}-e^{-iu/2}}{e^{iu/2}-e^{-iu/2}}$$

$$= \frac{\sin (N+1/2)u}{\sin u/2} - 1$$

$$\therefore\ 1+2\sum_{n=1}^{N}{}'\cos nu = \frac{\sin(N+1/2)u}{\sin u/2}$$

$$\therefore\quad \boxed{S_N(x) = \int_{-\pi}^{\pi} d\xi\, f(\xi)\, \frac{\sin(N+1/2)(\xi-x)}{2\pi \sin(\xi-x)/2}}$$

$$\boxed{S_N(x) = \int_{-\pi}^{\pi} d\xi\, f(\xi)\, D_N(\xi-x)}$$

2 → DEF: O núcleo de Dirichlet $D_N(u)$ é definido como

$$\boxed{D_N(u) = \frac{\sin(N+\frac{1}{2})u}{2\pi \sin\frac{u}{2}}}$$

continuando...

Como $f(\xi)$ tem período 2π, assim como $\sin(N+\frac{1}{2})u$ e $\sin\frac{u}{2}$, então $S_N(x)$ pode também ser escrito como

$$S_N(x) = \frac{1}{\pi}\int_{a}^{a+2\pi} d\xi\, f(\xi)\, \frac{\sin(N+\frac{1}{2})(\xi-x)}{2\sin\frac{(\xi-x)}{2}}$$

onde escolhemos a de modo que $a \le x \le a+2\pi$.

Escrevendo:

$$S_N(x) = \frac{1}{\pi}\int_a^{a+2\pi} d\xi\, F(\xi)\, \frac{\sin (N+1/2)(\xi-x)}{\xi-x}$$

onde:

$$F(\xi) = f(\xi)\, \frac{\frac{1}{2}(\xi-x)}{\sin\frac{1}{2}(\xi-x)}$$

e usando o lema 4:

$$\lim_{N\to\infty} S_N(x) = \frac{1}{\pi}\left[\pi \quad \frac{F(x+0)+F(x-0)}{2}\right]$$

$$F(x+0) = \lim_{\xi\to x^+} F(\xi) = \lim_{\xi\to x^+} f(\xi) \underbrace{\lim_{\xi\to x^+} \frac{\frac{1}{2}(\xi-x)}{\sin\frac{1}{2}(\xi-x)}}_{=1} = f(x+0)$$

$$F(x-0) = \lim_{\xi\to x^-} F(\xi) = \lim_{\xi\to x^-} f(\xi) \underbrace{\lim_{\xi\to x^-} \frac{\frac{1}{2}(\xi-x)}{\sin\frac{1}{2}(\xi-x)}}_{=1} = f(x-0)$$

$$\therefore \lim_{N\to\infty} S_N(x) = \frac{f(x+0)+f(x-0)}{2}$$

✓