I.3 FORMA COMPLEXA

$$e^{ix} = \cos x + i \sin x \quad \rightarrow \quad \begin{cases} \cos \frac{n\pi x}{L} = \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) \\ \sin \frac{n\pi x}{L} = \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right) \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} e^{\frac{in\pi x}{L}} + \frac{a_n}{2} e^{-\frac{in\pi x}{L}} - \frac{b_n i}{2} e^{\frac{in\pi x}{L}} + \frac{b_n i}{2} e^{-\frac{in\pi x}{L}}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n - ib_n}{2}\right)}_{C_n} e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n + ib_n}{2}\right)}_{C_n^*} e^{-\frac{in\pi x}{L}}$$

$\rightarrow$ DEF: $\boxed{C_{-n} = C_n^*}$

$$\therefore \; C_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & (n = 1, 2, \ldots) \\ \frac{1}{2}(a_{-n} + ib_{-n}) & (n = -1, -2, \ldots) \\ \frac{1}{2} a_0 & (n = 0) \end{cases}$$

$$\therefore \; f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{\frac{in\pi x}{L}} + \sum_{n=-1}^{-\infty} \underbrace{\frac{1}{2}(a_{-n} + ib_{-n})}_{C_n} e^{\frac{in\pi x}{L}}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{\frac{in\pi x}{L}}}$$

OBS: $f^*(x) = \sum_{n=-\infty}^{+\infty} C_n^* e^{-\frac{in\pi x}{L}} = \sum_{n=-\infty}^{+\infty}{}' C_{-n}^* e^{\frac{in\pi x}{L}} \underset{\uparrow}{=} \sum_{n=-\infty}^{+\infty}{}' C_n e^{\frac{in\pi x}{L}} = f(x)$

$C_n^* = C_{-n}$

$\therefore f(x) \in \mathbb{R}$

$\Rightarrow \boxed{C_n = \ ?}$

$$\int_{-L}^{L} dx\, e^{-\frac{im\pi x}{L}} e^{\frac{in\pi x}{L}} = \int_{-L}^{L} dx\, e^{\frac{i(n-m)\pi x}{L}} =$$

$$= \begin{cases} \int_{-L}^{L} dx = 2L & (n=m) \\ \left. \dfrac{e^{\frac{i(n-m)\pi x}{L}}}{\frac{i(n-m)\pi}{L}} \right|_{-L}^{L} = \dfrac{L}{i(n-m)\pi}\left[e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right] = \dfrac{2L}{(n-m)\pi}\sin(n-m)\pi = 0 \end{cases}$$

$$\boxed{\int_{-L}^{L} dx\, e^{-\frac{im\pi x}{L}} e^{\frac{in\pi x}{L}} = 2L\,\delta_{mn}}$$

Com isso:

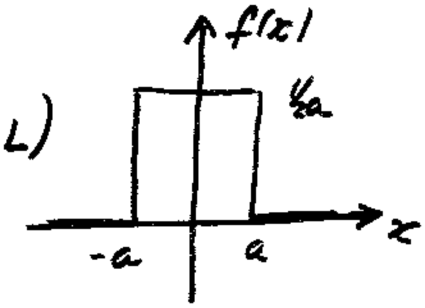
$$\int_{-L}^{L} dx\, f(x)\, e^{-\frac{im\pi x}{L}} = \sum_{n=-\infty}^{+\infty}{}' C_n \int_{-L}^{L} dx\, e^{\frac{in\pi x}{L}} e^{-\frac{im\pi x}{L}} = 2L \sum_{n=-\infty}^{+\infty} C_n \delta_{mn}$$

$$= 2L\, C_m$$

$$\therefore \quad \boxed{c_n = \frac{1}{2L}\int_{-L}^{L} dx\, f(x)\, e^{-\frac{in\pi x}{L}}}$$

EX

$$f(x) = \begin{cases} \frac{1}{2a}, & |x| \leq a \\ 0, & |x| > a \end{cases} \qquad (a < L)$$



$$c_n = \frac{1}{2L}\int_{-L}^{L} dx\, f(x)\, e^{-in\pi x/L} = \frac{1}{2L}\int_{-a}^{a} dx\, \frac{1}{2a}\, e^{-i\frac{n\pi x}{L}}$$

$$= \frac{1}{2L}\cdot\frac{1}{2a}\,\frac{e^{-in\pi x/L}}{(-in\pi/L)}\Bigg|_{-a}^{a} = \frac{1}{2\cdot 2a(-in\pi)}\left(e^{-in\pi a/L} - e^{in\pi a/L}\right)$$

$$= \frac{1}{2an\pi}\left(\frac{e^{in\pi a/L} - e^{-in\pi a/L}}{2i}\right) = \frac{\sin(n\pi a/L)}{2n\pi a}$$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{+\infty}{}' \frac{\sin(n\pi a/L)}{2n\pi a}\, e^{in\pi a/L} \quad //$$

ou ainda:

$$f(x) = \frac{1}{2L} + \sum_{n=1}^{\infty}{}' \frac{\sin(n\pi a/L)}{2n\pi a}\, e^{in\pi a/L} + \sum_{n=1}^{\infty}{}' \frac{\sin((-n)\pi a/L)}{2(-n)\pi a}\, e^{i(-n)\pi a/L}$$

$$= \frac{1}{2L} + \sum_{n=1}^{\infty} \frac{\sin(n\pi a/L)}{2n\pi a}\left(e^{in\pi a/L} + e^{-in\pi a/L}\right)$$

$$= \frac{1}{2L} + \sum_{n=1}^{\infty} \frac{\sin(n\pi a/L)}{n\pi a}\cos(n\pi a/L) \quad //$$

I.4 PROPRIEDADES DE PARIDADE: SÉRIES EM SENO E CO-SENO

$$\boxed{f(-x) = \begin{cases} f(x) \Rightarrow f \text{ par} \\ -f(x) \Rightarrow f \text{ ímpar} \end{cases}}$$

Denotando uma função par por $f_+(x)$ e ímpar por $f_-(x)$, temos:

$$\boxed{f_\pm(-x) = \pm f_\pm(x)}$$

➡ PROP:
$$\int_{-L}^{L} f(x)dx = \begin{cases} 2\int_0^L f(x)dx \,, & f \text{ par} \\ 0 \,, & f \text{ ímpar} \end{cases}$$

DEM:
$$\int_{-L}^{L} f_\pm(x)dx = \int_{-L}^{0} f_\pm(x)dx + \int_0^L f_\pm(x)dx =$$
$$= -\int_{L}^{0} f_\pm(-x)\,dx + \int_0^L f_\pm(x)dx =$$
$$= \int_0^L f_\pm(-x)dx + \int_0^L f_\pm(x)dx =$$
$$= \pm\int_0^L f_\pm(x)\,dx + \int_0^L f_\pm(x)\,dx = \begin{cases} 2\int_0^L f_+(x)dx & p/\, f_+ \\ 0 & p/\, f_- \end{cases}$$ ✔

Por outro lado, é fácil verificar que

$f(x)$	$g(x)$	$f(x)g(x)$
par	par	par
ímpar	par	ímpar
ímpar	ímpar	par

Logo, da prop:

$$\underbrace{a_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x) \underbrace{\cos n\pi x/L}_{\text{par}}}_{=0 \text{ se } f(x) \text{ é } \underline{\text{ímpar}}} \quad ; \quad \underbrace{b_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x) \underbrace{\sin n\pi x/L}_{\text{ímpar}}}_{=0 \text{ se } f(x) \text{ é } \underline{\text{par}}}$$

Logo:

(i) Se $f(x) = f_+(x)$ é par:

$$\boxed{f_+(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad , \quad a_n = \frac{2}{L}\int_0^L dx\, f(x)\cos\frac{n\pi x}{L}}$$

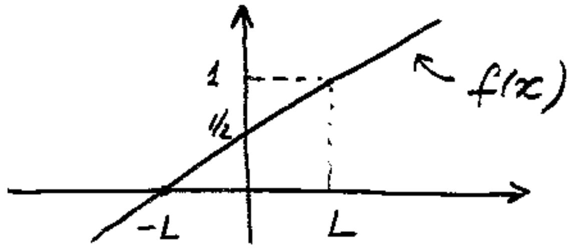
série de Fourier em co-senos

(ii) Se $f(x) = f_-(x)$ é ímpar:

$$\boxed{f_-(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad , \quad b_n = \frac{2}{L}\int_0^L dx\, f(x)\sin\frac{n\pi x}{L}}$$

série de Fourier em senos

EX $\quad f(x) = \dfrac{x}{2L} + \dfrac{1}{2}$



(a) série de Fourier:

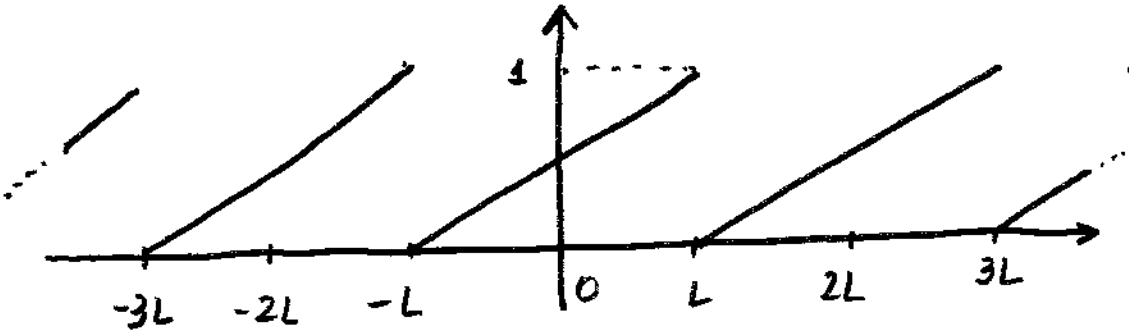
$$a_0 = \frac{1}{L}\int_{-L}^{L} dx \left(\frac{x}{2L} + \frac{1}{2}\right) = \frac{1}{L}\left(\frac{x^2}{4L} + \frac{x}{2}\right)\Bigg|_{-L}^{L} = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} dx \cos\frac{n\pi x}{L}\left(\frac{x}{2L} + \frac{1}{2}\right) = \underbrace{\frac{1}{2L^2}\int_{-L}^{L} dx\, x \cos\frac{n\pi x}{L}}_{=0\ (x \text{ é ímpar})} + \underbrace{\frac{1}{2L}\int_{-L}^{L} dx \cos\frac{n\pi x}{L}}_{=0} = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} dx \sin\frac{n\pi x}{L}\left(\frac{x}{2L} + \frac{1}{2}\right) = \frac{1}{2L^2}\int_{-L}^{L} dx\, x \sin\frac{n\pi x}{L} + \underbrace{\frac{1}{2L}\int_{-L}^{L} dx \sin\frac{n\pi x}{L}}_{=0}$$

$$= \frac{1}{2L^2}\left[\frac{-x\cos(n\pi x/L)}{(n\pi/L)}\Bigg|_{-L}^{L} + \underbrace{\frac{1}{(n\pi/L)}\int_{-L}^{L} dx \cos\frac{n\pi x}{L}}_{=0}\right] = \frac{1}{2L^2}\,\frac{(-2L^2)(-1)^n}{n\pi} = \frac{(-1)^{n+1}}{n\pi}$$

$$\therefore \quad F(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{L}$$



$F(x) =$ extensão periódica de $f(x)$, $-L \le x \le L$

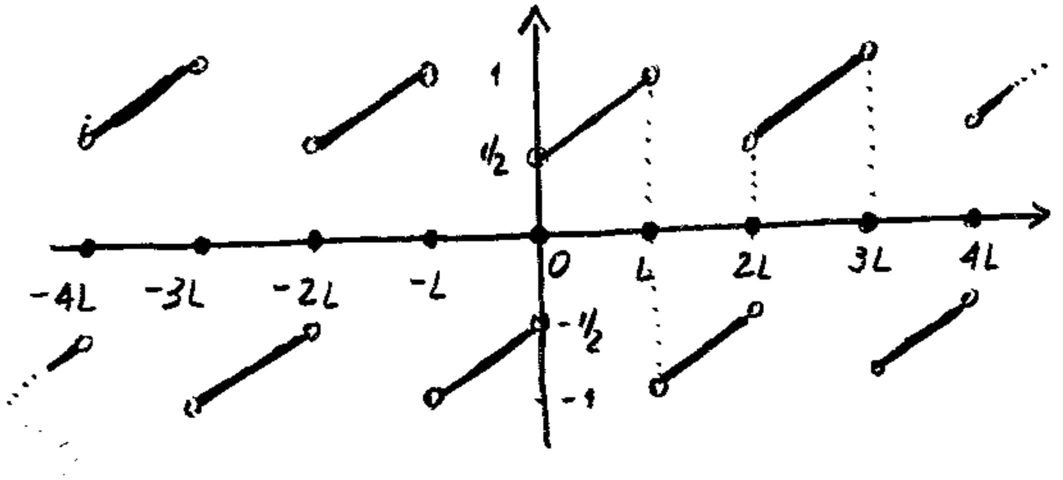
(b) série de Fourier em senos:

$$b_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\left(\frac{x}{2L}+\frac{1}{2}\right) = \frac{1}{L^2}\int_0^L dx\, x \sin\frac{n\pi x}{L} + \frac{1}{L}\int_0^L dx \sin\frac{n\pi x}{L}$$

$$= \frac{1}{L^2}\left[-\frac{x\cos(n\pi x/L)}{(n\pi/L)}\Big|_0^L + \frac{L}{n\pi}\int_0^L dx \cos\frac{n\pi x}{L}\right] + \frac{1}{L}\,\frac{(-\cos n\pi x/L)}{(n\pi/L)}\Big|_0^L$$

$$= -\frac{1}{n\pi}\cos n\pi - \frac{1}{n\pi}\cos n\pi + \frac{1}{n\pi} = \frac{1-2(-1)^n}{n\pi}$$

$$F_-(x) = \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n\pi}\right)\sin\frac{n\pi x}{L}$$



$F_-(x) =$
= extensão periódica ímpar de $f(x)$, $-L \le x \le L$

$$F_-(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, -L, L \\ -f(-x), & -L < x < 0 \end{cases}$$

$$F_-(x+2L) = F_-(x)$$

(c) série de Fourier em co-senos:

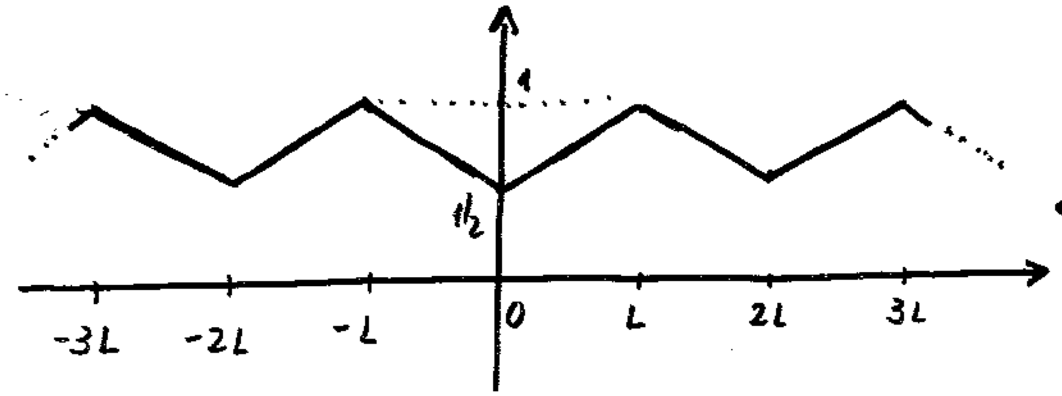
$$a_0 = \frac{2}{L}\int_0^L dx\left(\frac{x}{2L}+\frac{1}{2}\right) = \frac{2}{L}\left(\frac{x^2}{4L}+\frac{x}{2}\right)\Big|_0^L = \frac{3}{2}$$

$$a_n = \frac{2}{L}\int_0^L dx\left(\frac{x}{2L}+\frac{1}{2}\right)\cos\frac{n\pi x}{L} = \frac{1}{L^2}\int_0^L dx\, x\cos\frac{n\pi x}{L} + \frac{1}{L}\int_0^L dx\cos\frac{n\pi x}{L}$$

$$= \frac{1}{L^2}\left[\cancelto{0}{\frac{x\sin(n\pi x/L)}{(n\pi/L)}\Big|_0^L} - \frac{L}{n\pi}\int_0^L dx\sin\frac{n\pi x}{L}\right] + \frac{1}{L}\frac{1}{(n\pi/L)}\cancelto{0}{\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L}$$

$$= \frac{1}{Ln\pi}\,\frac{\cos(n\pi x/L)}{(n\pi/L)}\Big|_0^L = \frac{1}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$F_+(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{\left[(-1)^n-1\right]}{n^2\pi^2}\cos\frac{n\pi x}{L}$$



$F_+(x) =$
$=$ extensão periódica par de $f(x)$, $-L \le x \le L$

$$F_+(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0 \end{cases}$$

$$F_+(x+2L) = F_+(x)$$

I.5 TEOREMA DE FOURIER

Para facilitar a notação, vamos tomar $L = \pi$. Então

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi\right] + \sum_{n=1}^{\infty}\left[\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\cos n\xi\,d\xi\right)\cos nx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\sin n\xi\,d\xi\right)\sin nx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(\xi)\left[\cos n\xi \cos nx + \sin n\xi \sin nx\right]d\xi$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\xi\, f(\xi) + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} d\xi\, f(\xi)\cos n(\xi - x)} \quad (*)$$

TEOREMA (FOURIER): Seja $f(x)$ uma função contínua por partes e com derivadas laterais no intervalo $(-\pi, \pi)$ e periódica com período 2π. Então sua série de Fourier $(*)$ converge para o valor

$$\frac{1}{2}\left[f(x+0) + f(x-0)\right]$$

em $-\infty < x < +\infty$.

Antes de provarmos o teorema de Fourier precisamos explicitar o que entendemos por derivadas laterais e provar alguns lemas auxiliares.

I.5.1. DERIVADAS LATERAIS

notação:

$$f(x_0+0) = \lim_{x \to x_0^+} f(x) \qquad \text{(limite à direita)}$$

$$f(x_0-0) = \lim_{x \to x_0^-} f(x) \qquad \text{(limite à esquerda)}$$

DEF: Derivada à direita $f'_+(x_0)$:

$$f'_+(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0+0)}{h} \qquad (h>0)$$

Derivada à esquerda $f'_-(x_0)$:

$$f'_-(x_0) = \lim_{h \to 0} \frac{f(x_0-0) - f(x_0-h)}{h} \qquad (h>0)$$

• OBS: ATENÇÃO: NÃO CONFUNDIR AS NOTAÇÕES! f'_+ (f'_-) NÃO é a derivada de uma função par (ímpar)!

• OBS: Note o uso de $f(x_0+0)$ e $f(x_0-0)$ e não $f(x_0)$.

➡ Sejam:

f, f' = contínuas por partes

$[a,b]$ = intervalo onde f e f' são contínuas e tem limites laterais

∴ em $[a,b]$ vale o teorema do valor médio:

$\exists\, \theta$ $(0<\theta<1)$ tal que para $0<\lambda<b-a$

$$\frac{f(a+\lambda)-f(a+0)}{\lambda} = f'(a+\theta\lambda)$$

Então:

$$\lim_{\lambda\to 0} \frac{f(a+\lambda)-f(a+0)}{\lambda} = f'_+(a) = \lim_{\lambda\to 0} f'(a+\theta\lambda) = f'(a+0)$$

Analogamente: $f'_-(b) = f'(b-0)$

∴ em cada ponto de um intervalo fechado no qual f e f' são contínuas por partes, as derivadas laterais de f (do interior do intervalo) existem e são as mesmas que os correspondentes limites laterais de f'.

EX

$$f(x) = \begin{cases} \sin x & , x \geq 0 \\ x^2 & , x < 0 \end{cases}$$

$$f'_+(0) = \lim_{h \to 0} \frac{\sin h - 0}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0} \frac{0 - (-h)^2}{h} = 0$$

$\therefore$ $f'(x)$ é contínua por partes e $f'_+(0) = f'(0+0)$ e $f'_-(0) = f'(0-0)$.

EX

$$f(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$$\left.\begin{array}{l} \lim\limits_{\Delta x \to 0^+} \dfrac{1 - 1}{\Delta x} = 0 \\ \lim\limits_{\Delta x \to 0^-} \dfrac{0 - 1}{\Delta x^-} = +\infty \end{array}\right\} \nexists\, f'(0)$$

MAS: $f'_+(0) = \lim\limits_{h \to 0} \dfrac{1 - 1}{h} = 0$

$f'_-(0) = \lim\limits_{h \to 0} \dfrac{0 - 0}{h} = 0$

$\therefore$ $f'_+(0) = f'_-(0) = 0$ mas $\nexists\, f'(0)$.

EX

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , (x \neq 0) \\ 0 & , (x = 0) \end{cases}$$

$$\lim_{x \to 0} x^2 \sin 1/x = \lim_{x \to 0} x \frac{\sin 1/x}{1/x} = 0$$

$$f'_+(0) = \lim_{h \to 0} \frac{h^2 \sin 1/h - 0}{h} = 0$$

$$f'_-(0) = \lim_{h \to 0} \frac{0 - (-h)^2 \sin (1/-h)}{h} = 0$$

$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \Rightarrow \nexists\, f'(0+)$ e $\nexists\, f'(0-)$ $\quad \therefore \nexists\, f'(0)$

$$\therefore f'_+(0) \neq f'(0+) \text{ e } f'_-(0) \neq f'(0-)$$