$\boxed{\text{EX}}$ II

$$\mathcal{M}\left[\frac{1}{1+x^a}\right] = \int_0^\infty dx\,\frac{x^{p-1}}{1+x^a} \underset{\substack{\uparrow \\ y=x^a \\ (a>0)}}{=} \frac{1}{a}\int_0^\infty dy\,\frac{y^{\frac{p}{a}-1}}{1+y} = \frac{1}{a}\,\frac{\pi}{\sin\frac{p}{a}\pi}$$

Então:

$$\mathcal{M}[f(xe^{i\theta})] = \int_0^\infty dx\,\frac{x^{p-1}}{1+(xe^{i\theta})^a} = \int_0^\infty dx\,\frac{x^{p-1}}{1+x^a e^{i\theta a}} =$$

$$= \int_0^\infty dx\,\frac{x^{p-1}}{1+x^a\cos\theta a + i x^a \sin\theta a} = \int_0^\infty dx\,\frac{x^{p-1}(1+x^a\cos\theta a - i x^a \sin\theta a)}{(1+x^a\cos\theta a)^2 + (x^a\sin\theta a)^2} =$$

$$= \int_0^\infty dx\,\frac{x^{p-1}(1+x^a\cos\theta a - i x^a\sin\theta a)dx}{1+2x^a\cos\theta a + x^{2a}}$$

$$= e^{-ip\theta}\int_0^\infty dx\,\frac{x^{p-1}}{1+x^a} = e^{-ip\theta}\,\frac{\pi}{a\sin\frac{p}{a}\pi}$$

$\therefore$

$$\cos\theta p\,\frac{\pi}{a\sin\frac{p}{a}\pi} = \mathcal{M}\left[\frac{1+x^a\cos\theta a}{1+2x^a\cos\theta a + x^{2a}}\right]$$

$$\sin\theta p\,\frac{\pi}{a\sin\frac{p}{a}\pi} = \mathcal{M}\left[\frac{x^a\sin\theta a}{1+2x^a\cos\theta a + x^{2a}}\right]$$

//

EX

$$\begin{cases} \nabla^2 u = 0 \ , \quad 0 \le r < \infty \ , \quad 0 \le \theta \le \alpha \\ u(r,0) = f(r) \ , \ u(r,\alpha) = g(r) \\ \lim\limits_{r\to\infty} |u(r,\alpha)| < +\infty \end{cases}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\therefore \quad r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \dots (*)$$

$$\mathcal{U}(p,\theta) = \int_0^\infty dr \, r^{p-1} u(r,\theta) = \mathcal{M}_r[u(r,\theta)]$$

$$\mathcal{M}\left[r\frac{\partial u}{\partial r}\right] = -p\,\mathcal{M}[u](p) = -p\,\mathcal{U}(p,\theta)$$

$$\mathcal{M}\left[r^2\frac{\partial^2 u}{\partial r^2}\right] = p(p+1)\,\mathcal{M}[u](p) = p(p+1)\,\mathcal{U}(p,\theta)$$

$$(*) \Rightarrow \quad p(p+1)\mathcal{U} - p\,\mathcal{U} + \mathcal{U}_{\theta\theta} = 0$$

$$\therefore \quad \mathcal{U}_{\theta\theta} + p^2\,\mathcal{U} = 0$$

$$\therefore \ \mathcal{U}(p,\theta) = A(p)\cos p\theta + B(p)\sin p\theta$$

Se: $F(p) = \mathcal{M}[f(r)] = \mathcal{M}[u(r,0)]$

$G(p) = \mathcal{M}[g(r)] = \mathcal{M}[u(r,\alpha)]$

temos

$$F(p) = A(p)$$

$$G(p) = A(p)\cos p\alpha + B(p)\sin p\alpha \Rightarrow B(p) = \frac{G(p) - F(p)\cos p\alpha}{\sin p\alpha}$$

$$\therefore \quad U(p,\theta) = F(p)\cos p\theta + \left[\frac{G(p) - F(p)\cos p\alpha}{\sin p\alpha}\right]\sin p\theta$$

$$U(p,\theta) = F(p)\frac{\sin p(\alpha-\theta)}{\sin p\alpha} + G(p)\frac{\sin p\theta}{\sin p\alpha}$$

mas do exemplo anterior:

$$\frac{\sin p\theta}{\sin p\alpha} = M\Bigg[\underbrace{\frac{\pi}{\alpha}\,\frac{r^{\pi/\alpha}\sin\theta\pi/\alpha}{1 + 2r^{\pi/\alpha}\cos\theta\pi/\alpha + r^{2\pi/\alpha}}}_{h(r,\theta)}\Bigg]$$

e do teorema da convolução:

$$u(r,\theta) = \int_0^\infty d\rho\,\rho^{-1}\left[f(\rho)\,h\!\left(\frac{r}{\rho},\alpha-\theta\right) + g(\rho)\,h\!\left(\frac{r}{\rho},\theta\right)\right]$$

VI.2 TRANSFORMADA DE HANKEL

Vamos considerar uma função de duas variáveis $f(x,y)$ e a sua transformada de Fourier $F(K,L)$, ou seja,

$$F(K,L) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\; f(x,y)\,e^{i(Kx+Ly)} \,,$$

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}dK\int_{-\infty}^{+\infty}dL\; F(K,L)\,e^{-i(Kx+Ly)} \,.$$

Agora vamos passar para coordenadas polares. Por abuso de notação, vamos denotar $f(r,\theta) = f(x(r,\theta), y(r,\theta))$ e $F(\rho,\varphi) = F(K(\rho,\varphi), L(\rho,\varphi))$. Uma vez que essas funções tem periodicidade angular, ou seja, temos $f(r,\theta+2\pi) = f(r,\theta)$ e $F(\rho,\varphi+2\pi) = F(\rho,\varphi)$, podemos nessa variável tomar uma série de Fourier, de modo que

$$f(r,\theta) = \sum_{n=-\infty}^{+\infty} f_n(r)\,e^{in\theta} \quad , \quad f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,e^{-in\theta}\,d\theta \,.$$

$$F(\rho,\varphi) = \sum_{n=-\infty}^{+\infty} F_n(\rho)\,e^{in\varphi} \quad , \quad F_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} F(\rho,\varphi)\,e^{-in\varphi}\,d\varphi \,.$$

Em coordenadas polares temos

$$F(\rho,\varphi) = \frac{1}{2\pi}\int_0^\infty dr\, r \int_0^{2\pi} d\theta\, f(r,\theta)\, e^{i[(\rho\cos\varphi)(r\cos\theta)+(\rho\sin\varphi)(r\sin\theta)]}$$

$$= \frac{1}{2\pi}\int_0^\infty dr\, r \int_0^{2\pi} d\theta\, f(r,\theta)\, e^{i\rho r\cos(\varphi-\theta)}$$

e portanto:

$$F_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{-in\varphi}\, \frac{1}{2\pi}\int_0^\infty dr\, r\int_0^{2\pi} d\theta\, f(r,\theta)\, e^{i\rho r\cos(\varphi-\theta)}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dr\, r\int_0^{2\pi} d\varphi\int_0^{2\pi} d\theta\, e^{-in\varphi}\, e^{i\rho r\cos(\varphi-\theta)}\sum_{m=-\infty}^{+\infty} f_m(r)\, e^{im\theta}$$

$$\overset{\substack{\varphi=\alpha+\theta\\ \downarrow}}{=} \frac{1}{(2\pi)^2}\int_0^\infty dr\, r\sum_{m=-\infty}^{+\infty} f_m(r)\int_0^{2\pi} d\alpha\int_0^{2\pi} d\theta\, e^{-in(\alpha+\theta)}\, e^{i\rho r\cos\alpha}\, e^{im\theta}$$

onde usamos, na integração em α, o fato da periodicidade 2π, o que implica que $\int_{0+\theta}^{2\pi+\theta} d\alpha\,(\ldots) = \int_0^{2\pi} d\alpha\,(\ldots)$.

Continuando:

$$F_n(\rho) = \frac{1}{2\pi}\int_0^\infty dr\, r \sum_{m=-\infty}^{+\infty}{}' f_m(r) \int_0^{2\pi} d\alpha\, e^{-in\alpha} e^{i\rho r\cos\alpha} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-in\theta} e^{im\theta}}_{\delta_{mn}}$$

$$= \frac{1}{2\pi}\int_0^\infty dr\, r\, f_n(r) \int_0^{2\pi} d\alpha\, e^{-in\alpha} e^{i\rho r\cos\alpha}$$

Mas a função geratriz para as funções de Bessel é:

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{+\infty}{}' t^n J_n(z)$$

ou, tomando $t = i e^{i\varphi}$

$$e^{iz\cos\varphi} = \sum_{n=-\infty}^{+\infty}{}' i^n e^{in\varphi} J_n(z)$$

de modo que

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{-in\varphi} e^{iz\cos\varphi} = i^n J_n(z)$$

e portanto:

$$F_n(\rho) = \int_0^\infty dr\, r\, f_n(r)\, i^n J_n(\rho r)$$

De modo análogo:

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-in\theta}\, \frac{1}{2\pi}\int_0^{\infty} d\rho\,\rho \int_0^{2\pi} d\varphi\, F(\rho,\varphi)\, e^{-i\rho r\cos(\theta-\varphi)}$$

$$= \frac{1}{2\pi}\int_0^{\infty} d\rho\,\rho \int_0^{2\pi} d\alpha\, e^{-in\alpha} e^{-i\rho r\cos\alpha} \sum_{m=-\infty}^{+\infty} F_m(\rho)\, \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{-in\varphi} e^{im\varphi}}_{\delta_{mn}}$$

$$= \int_0^{\infty} d\rho\,\rho\, F_n(\rho)\, \underbrace{\frac{1}{2\pi}\int_0^{2\pi} d\alpha\, e^{-in\alpha} e^{-i\rho r\cos\alpha}}_{(-i)^n J_n(\rho r)}$$

$$= \int_0^{\infty} d\rho\,\rho\, F_n(\rho)\,(-i)^n J_n(\rho r)$$

Resumindo, temos:

$$F_n(\rho) = i^n \int_0^{\infty} dr\, r\, J_n(\rho r)\, f_n(r)$$

$$f_n(r) = (-i)^n \int_0^{\infty} d\rho\, \rho\, J_n(\rho r)\, F_n(\rho)$$

o que nos sugere definir a <u>transformada de Hankel de ordem ν</u> $\mathcal{H}_\nu$ e sua <u>inversa</u> $\mathcal{H}_\nu^{-1}$ como

$$\boxed{\begin{aligned} \mathcal{H}_\nu[f(x)] &= \int_0^\infty dx\, x\, J_\nu(kx) f(x) = F_\nu(k) \\ \mathcal{H}_\nu^{-1}[F_\nu(k)] &= \int_0^\infty dk\, k\, J_\nu(kx) F_\nu(k) = f(x) \end{aligned}}$$

EX II $\mathcal{H}_0[e^{-ax}] = ?$

$$\mathcal{H}_0[e^{-ax}] = \int_0^\infty dx\, x\, J_0(kx)\, e^{-ax} = \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{2^{2n}(n!)^2} \int_0^\infty dx\, x^{2n+1} e^{-ax} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{2^{2n}(n!)^2} \; \frac{\Gamma(2n+2)}{a^{2n+2}}$$

mas da fórmula de duplicação de Legendre:

$$\sqrt{\pi}\, \Gamma(2z+1) = 2^{2z}\, \Gamma\left(z+\tfrac{1}{2}\right) \Gamma(z+1)$$

temos

$$\Gamma(2n+2) = \Gamma\left[2\left(n+\tfrac{1}{2}\right)+1\right] = \frac{2^{2(n+\frac{1}{2})}}{\sqrt{\pi}}\, \Gamma\left(n+\tfrac{1}{2}+\tfrac{1}{2}\right) \Gamma\left(n+\tfrac{1}{2}+1\right)$$

$$\therefore \mathcal{H}_0[e^{-ax}] = \sum_{n=0}^{\infty} \frac{(-1)^n k^{2n}}{2^{2n}[\Gamma(n+1)]^{\cancel{2}}} \; \frac{2^{2n}\cdot 2}{\sqrt{\pi}} \; \frac{\cancel{\Gamma(n+1)}\, \Gamma(n+3/2)}{a^{2n+2}} =$$

$$= \frac{1}{a^2} \sum_{n=0}^{\infty} \frac{(-1)^n (k/a)^{2n}}{\Gamma(n+1)} \; \frac{\Gamma(n+3/2)}{\underbrace{\Gamma(3/2)}_{\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}}} = \frac{1}{a^2} \sum_{n=0}^{\infty} (-1)^n (3/2)_n \frac{\left[(k/a)^2\right]^n}{n!}$$

$$= \frac{1}{a^2} \frac{1}{\left[1+(k/a)^2\right]^{3/2}} = \frac{a}{(a^2+k^2)^{3/2}} \;//$$

Uma das propriedades que mais nos interessa no caso de uma transformada integral envolve a derivada de uma função. Nesse caso temos

$$\mathcal{H}_\nu[f'(x)] = \int_0^\infty dx\, x\, J_\nu(kx) f'(x) =$$

$$= \underbrace{x J_\nu(kx) f(x)\Big|_0^\infty}_{\substack{=0 \\ \text{se } \lim_{x\to\infty} f(x) = 0}} - \int_0^\infty dx\, [x J_\nu(kx)]' f(x)$$

Sabemos, porém, que

$$2 J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$$

$$\nu J_\nu(x) = \frac{x}{2} J_{\nu+1}(x) + \frac{x}{2} J_{\nu-1}(x)$$

Logo:

$$(x J_\nu(x))' = J_\nu(x) + x J_\nu'(x) =$$

$$= \frac{x}{2\nu}\left(J_{\nu+1}(x) + J_{\nu-1}(x)\right) + \frac{x}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)$$

$$= \frac{x}{2} J_{\nu+1}(x)\left(\frac{1-\nu}{\nu}\right) + \frac{x}{2} J_{\nu-1}(x)\left(\frac{1+\nu}{\nu}\right)$$

Sendo assim

$$\mathcal{H}_\nu[f'(x)] = -k \int_0^\infty dx \left[\frac{x}{2} J_{\nu+1}(kx)\left(\frac{1-\nu}{\nu}\right) + \frac{x}{2} J_{\nu-1}(kx)\left(\frac{1+\nu}{\nu}\right)\right] f(x)$$

$$\boxed{\mathcal{H}_\nu[f'(x)] = -k\left(\frac{1-\nu}{2\nu}\right)\mathcal{H}_{\nu+1}[f(x)] - k\left(\frac{1+\nu}{2\nu}\right)\mathcal{H}_{\nu-1}[f(x)]} \quad (\nu \geq 1)$$

Não é, porém, nesse caso que a transformada de Hankel assume seu papel mais simples. De fato, temos

$$\mathcal{H}_\nu\left[f''(x) + \frac{1}{x}f'(x) - \frac{\nu^2}{x^2}f(x)\right] =$$

$$= \int_0^\infty \left[f''(x) + \frac{1}{x}f'(x) - \frac{\nu^2}{x^2}f(x)\right] J_\nu(kx)\,x\,dx =$$

$$= \underbrace{f'(x)J_\nu(kx)x\Big|_0^\infty}_{=0 \text{ (hipótese)}} + \underbrace{f(x)J_\nu(kx)\Big|_0^\infty}_{=0 \text{ (hipótese)}} +$$

$$+ \int_0^\infty \left[-f'(x)\left(J_\nu(kx)x\right)' - f(x)J_\nu'(kx) - \frac{\nu^2}{x}f(x)J_\nu(kx)\right]dx =$$

$$= \underbrace{-f(x)\left(J_\nu(kx)x\right)'\Big|_0^\infty}_{=0 \text{ (hipótese)}} + \int_0^\infty \left[f(x)\underbrace{\underbrace{\left(J_\nu(kx)x\right)''}_{\left(J_\nu'(kx)x + J_\nu(kx)\right)'}}_{J_\nu''(kx)x + 2J_\nu'(kx)} - f(x)J_\nu'(kx) - \frac{\nu^2}{x}f(x)J_\nu(kx)\right]dx =$$

$$= \int_0^\infty \left[f(x)J_\nu''(kx)x + 2f(x)J_\nu'(kx) - f(x)J_\nu'(kx) - \frac{\nu^2}{x}f(x)J_\nu(kx)\right]dx$$

$$= \int_0^\infty f(x)\left[J_\nu''(kx) + \frac{1}{x}J_\nu'(kx) - \frac{\nu^2}{x^2}J_\nu(kx)\right]x\,dx \qquad (*)$$

mas, da equação de Bessel para $z = Kx$, segue

$$J_\nu''(Kx) + J_\nu'(Kx) + \left(K^2 - \frac{\nu^2}{x^2}\right) J_\nu(Kx) = 0$$

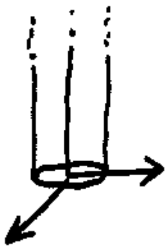
e portanto:

$$(*) = \int_0^\infty f(x)\left[-K^2 J_\nu(Kx)\right] x\,dx = -K^2 \mathcal{H}_\nu[f(x)]$$

$$\therefore \boxed{\mathcal{H}_\nu\left[f''(x) + \frac{1}{x} f'(x) - \frac{\nu^2}{x^2} f(x)\right] = -K^2 \mathcal{H}_\nu[f(x)]}$$

EX

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0$$



$0 \le r \le a$
$z \ge 0$
$0 \le \theta < 2\pi$

simetria axial: $u = u(r,z) \therefore u_{\theta\theta} = 0$

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0$$

Denotando: $\mathcal{U}(K,z) = \mathcal{H}_0[u(r,z)] = \int_0^\infty dr\, r\, u(r,z) J_0(Kr)$, temos

$$\mathcal{H}_0\left[u_{rr} + \frac{1}{r} u_r\right] + \mathcal{H}_0[u_{zz}] = -K^2 \mathcal{U} + \mathcal{U}_{zz} = 0$$

$$\therefore \mathcal{U}(K,z) = A(K) e^{-Kz} + B(K) e^{Kz}$$

$$\lim_{z\to\infty} \mathcal{U}(K,z) = \mathcal{H}_0\left[\lim_{z\to\infty} u(r,z)\right] < +\infty \quad (\underline{\text{hipótese}}) \Rightarrow B(K) = 0$$

$$\mathcal{U}(K,0) = A(K) = \mathcal{H}_0[u(r,0)]$$

$$\therefore u(r,z) = \mathcal{H}_0^{-1}[\mathcal{U}(K,z)] = \int_0^\infty dK\, K\, A(K) e^{-Kz} J_0(Kr)$$

//