V.3 A TRANSFORMAÇÃO INVERSA

Para chegarmos a uma expressão para a transformação inversa, vamos explorar a relação entre as transformadas de Fourier e Laplace. Dada $f_H(t) = f(t)H(t)$, temos

$$\int_{-\infty}^{+\infty} dt\, f_H(t)\, e^{-st} e^{iKt} = \underbrace{\mathcal{L}[f_H(t)e^{iKt}](s)}_{\underbrace{\mathcal{L}[f_H(t)](s-iK)}_{\phi_K(s)}} = \underbrace{\sqrt{2\pi}\,\mathcal{F}[f_H(t)e^{-st}](K)}_{\underbrace{\sqrt{2\pi}\,\mathcal{F}[f_H(t)](K+is)}_{\sqrt{2\pi}\,F_S(K)}}$$

Temos:

$$f_H(t)e^{-st} = \mathcal{F}^{-1}[F_S(K)](t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, F_S(K)\, e^{-iKt} \overset{\substack{z=-iK+s \\ K=iz-is}}{=}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{+i\infty+s}^{-i\infty+s} \frac{dz}{(-i)} \underbrace{F_S(iz-is)}_{\mathcal{F}[f_H(t)](iz-is+is)} e^{-i(iz-is)t}$$

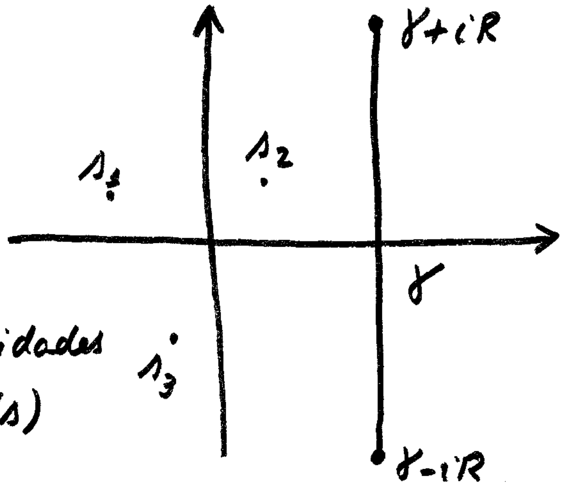
$$= \frac{1}{\sqrt{2\pi}\, i} \int_{-i\infty+s}^{+i\infty+s} dz\, \underbrace{\mathcal{F}[f_H(t)](iz)}_{\frac{1}{\sqrt{2\pi}}\mathcal{L}[f_H(t)](z)} e^{zt} e^{-st}$$

$$= e^{-st}\, \frac{1}{2\pi i} \int_{-i\infty+s}^{+i\infty+s} dz\, \mathcal{L}[f_H(t)](z)\, e^{zt}$$

o que nos sugere que, para $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, temos

$$\boxed{f_H(t) = \frac{1}{2\pi i} \int_{-i\infty+\gamma}^{+i\infty+\gamma} ds\, e^{st} F(s)}$$

para γ convenientemente escolhido. Vamos verificar que essa fórmula de fato vale, com γ escolhido de modo que todas as possíveis singularidades de $e^{st}F(s)$ fiquem à esquerda de γ, ou seja



s_i = singularidades de $e^{st}F(s)$

Vamos calcular $I_R(t)$:

$$I_R(t) = \frac{1}{2\pi i} \int_{\gamma-iR}^{\gamma+iR} ds\, e^{st} F(s) = \frac{1}{2\pi i} \int_{\gamma-iR}^{\gamma+iR} ds\, e^{st} \int_0^\infty dz\, e^{-sz} f(z) =$$

$$= \frac{1}{2\pi i} \int_0^\infty dz\, f(z) \int_{\gamma - iR}^{\gamma + iR} ds\, e^{s(t-z)} =$$

$$= \frac{1}{2\pi i} \int_0^\infty dz\, f(z) \left. \frac{e^{s(t-z)}}{t-z} \right|_{\gamma - iR}^{\gamma + iR} =$$

$$= \frac{1}{2\pi i} \int_0^\infty dz\, f(z)\, e^{\gamma(t-z)} \frac{\left(e^{iR(t-z)} - e^{-iR(t-z)}\right)}{t-z}$$

$$= \frac{1}{\pi} \int_0^\infty dz\, f(z) e^{\gamma(t-z)} \frac{\sin R(t-z)}{t-z}$$

$$= \frac{1}{\pi} \int_{-t}^\infty du\, f(t+u)\, e^{-\gamma u} \frac{\sin Ru}{u}$$

$$= \frac{1}{\pi} \int_{-t}^0 du\, f(t+u)\, e^{-\gamma u} \frac{\sin Ru}{u} + \frac{1}{\pi} \int_0^\infty du\, f(t+u) e^{-\gamma u} \frac{\sin Ru}{u}$$

Mas do lema da pag. 28 e o da pag. 122 sabemos que

$$\lim_{K\to\infty} \int_0^b du\, F(u) \frac{\sin Ru}{u} = \lim_{K\to\infty} \int_0^\infty du\, F(u) \frac{\sin Ru}{u} = \frac{\pi}{2} F(0+)$$

Com isso, para $R \to \infty$, temos

$$\lim_{R\to\infty} \frac{1}{\pi} \int_0^\infty du\, f(t+u)\, e^{-\gamma u} \frac{\sin Ru}{u} = \frac{1}{\pi}\cdot\frac{\pi}{2} f(t+0) = \frac{1}{2} f(t^+)$$

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{-t}^0 du\, f(t+u)\, e^{-\gamma u} \frac{\sin Ru}{u} = \begin{cases} 0 & , t=0 \\ \frac{1}{2} f(t^-) & , t>0 \\ -\frac{1}{2} f(t^+) & , t<0 \end{cases}$$

de modo que

$$\lim_{R\to\infty} I_R(t) = \begin{cases} \frac{1}{2} f(0) & , t=0 \\ \frac{1}{2}[f(t^+)+f(t^-)] & , t>0 \\ 0 & , t<0 \end{cases}$$

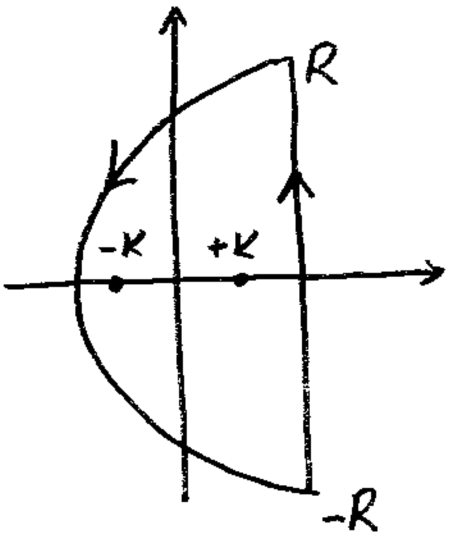
De fato:

$$\boxed{f(t)H(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s)\, ds}$$

que é chamada de <u>integral de Bromwich</u> ou <u>integral de Mellin</u> ou ainda <u>integral de Fourier-Mellin</u>.

EX

$$F(s) = \frac{K}{s^2-K^2} = \frac{K}{(s-K)(s+K)}$$



$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{K}{s^2-K^2}\,ds$$

$$= \left[\operatorname*{Res}_{s=K} + \operatorname*{Res}_{s=-K}\right]\left(\frac{e^{st}K}{s^2-K^2}\right)$$

$$= \frac{e^{Kt}K}{2K} + \frac{e^{-Kt}K}{(-2K)}$$

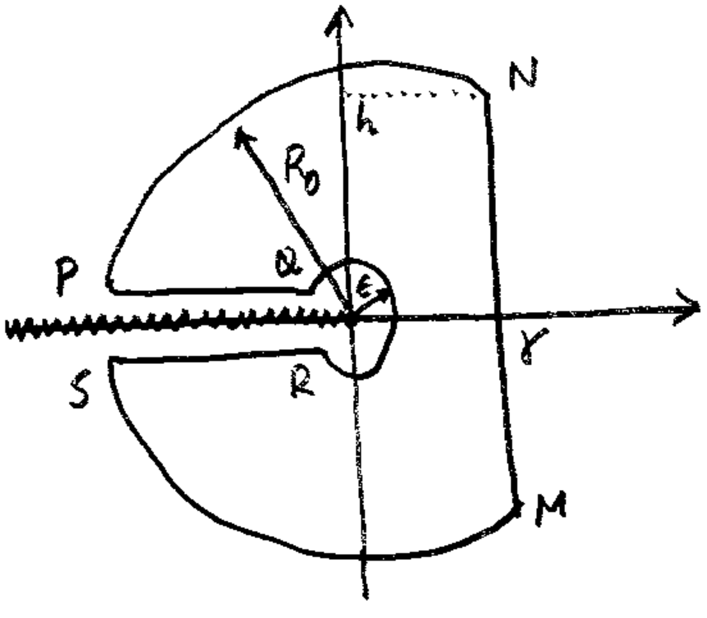
$$= \frac{e^{Kt}-e^{-Kt}}{2} = \sinh Kt$$

//

EX

$$F(s) = \frac{1}{\sqrt{s}}$$

$s=0 \Rightarrow$ ponto de ramificação



$$\underbrace{\int_M^N}_{} + \underbrace{\int_N^P}_{①} + \underbrace{\int_P^Q}_{②} + \underbrace{\int_Q^R}_{\substack{\to 0 \\ \text{para} \\ \epsilon\to 0}} + \underbrace{\int_R^S}_{②} + \underbrace{\int_S^M}_{①} = 0$$

é a integral desejada para $R_0 \to \infty$ $(h \to \infty)$

$$R_0 = \sqrt{\gamma^2+h^2}$$

(1) $\Rightarrow$ $$\int_N^P \frac{e^{st}}{\sqrt{s}}\,ds = \left.\frac{e^{st}}{t\sqrt{s}}\right|_N^P + \frac{1}{2t}\int_N^P \frac{e^{st}}{s^{3/2}}\,ds$$

$$s = x + iy = \sqrt{\gamma^2+h^2}\,e^{i\theta} = R_0 e^{i\theta}$$

$$s(N) = \gamma + ih$$
$$s(P) = R_0 e^{i\pi} = -R_0$$

$$\therefore \left.\frac{e^{st}}{t\sqrt{s}}\right|_N^P = \frac{e^{-R_0 t}}{it\sqrt{R_0}} - \frac{e^{(\gamma+ih)t}}{t\sqrt{R_0}\,e^{i\theta/2}} \xrightarrow[(t>0)]{R_0\to\infty} 0$$

$$\left|\int_N^P \frac{e^{st}}{s^{3/2}}\,ds\right| = \left|\int_N^P \frac{e^{xt}e^{iyt}R_0 e^{i\theta}\,i\,d\theta}{R_0^{3/2}\,e^{i\theta\frac{3}{2}}}\right|$$

$$\leq \int_N^P \frac{e^{xt}}{\sqrt{R_0}}\,d\theta \xrightarrow[\substack{x\to-\infty\\(t>0)}]{R_0\to\infty} 0$$

$$\therefore \lim_{R_0\to\infty}\int_N^P \frac{e^{st}}{\sqrt{s}}\,ds = 0 \qquad (t>0)$$

De forma análoga:

$$\lim_{R_0\to\infty}\int_S^M \frac{e^{st}}{\sqrt{s}}\,ds = 0 \qquad (t>0)$$

$$(2) \Rightarrow \int_P^Q \frac{e^{st}}{\sqrt{s}}\,ds \underset{\substack{s = re^{i\pi} = -r \\ \sqrt{s} = \sqrt{r}\,e^{i\pi/2} \\ = i\sqrt{r}}}{=\!=} \int_{R_0}^{0} \frac{e^{-rt}}{i\sqrt{r}}(-1)\,dr = \frac{1}{i}\int_0^{R_0} \frac{e^{-rt}}{\sqrt{r}}\,dr$$

$$\int_R^S \frac{e^{st}}{\sqrt{s}}\,ds \underset{\substack{s = re^{-i\pi} = -r \\ \sqrt{s} = \sqrt{r}\,e^{-i\pi/2} \\ = -i\sqrt{r}}}{=\!=} \int_0^{R_0} \frac{e^{-rt}}{(-i)\sqrt{r}}(-1)\,dr = \frac{1}{i}\int_0^{R_0} \frac{e^{-rt}}{\sqrt{r}}\,dr$$

$$R_0 \to \infty \Rightarrow \int_0^{\infty} \frac{e^{-rt}}{\sqrt{r}}\,dr = \int_0^{\infty} \frac{e^{-y^2 t}\,2y\,dy}{y} = 2\int_0^{\infty} e^{-ty^2}\,dy = 2\cdot\frac{1}{2}\sqrt{\frac{\pi}{t}}$$

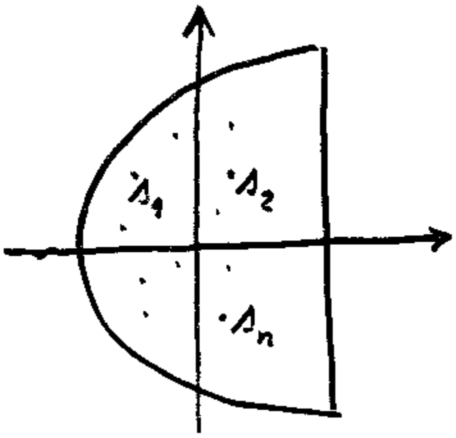
$$\therefore \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{\sqrt{s}}\,ds + \frac{1}{2\pi i}\left[\frac{1}{i}\sqrt{\frac{\pi}{t}} + \frac{1}{i}\sqrt{\frac{\pi}{t}}\right] = 0$$

$$\therefore f(t) = -\frac{1}{2\pi i}\,\frac{2}{i}\sqrt{\frac{\pi}{t}} = \frac{1}{\sqrt{\pi t}}$$

$$\therefore f(t) = \frac{1}{\sqrt{\pi t}}$$

//

EX $F(s) = \dfrac{A(s)}{B(s)}$, $B(s) = (s-s_1)\dots(s-s_n)$, $s_1 \neq \dots \neq s_n$



$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A(s)\,e^{st}\,ds}{(s-s_1)\dots(s-s_n)}$$

$$= \sum_{i=1}^{n} \operatorname*{Res}_{s=s_i}\left[\frac{A(s)\,e^{st}}{(s-s_1)\dots(s-s_n)}\right]$$

$$= \sum_{i=1}^{n} \lim_{s\to s_i}\left[\frac{(s-s_i)\,A(s)\,e^{st}}{(s-s_1)\dots(s-s_n)}\right]$$

$$= \sum_{i=1}^{n}\left[\frac{A(s_i)\,e^{s_i t}}{(s_i-s_1)\dots(s_i-s_{i-1})(s_i-s_{i+1})\dots(s_i-s_n)}\right]$$

MAS:

$$B'(s) = (s-s_2)(s-s_3)\dots(s-s_n) + (s-s_1)(s-s_3)\dots(s-s_n) + \cdots$$
$$\cdots + (s-s_1)(s-s_2)\dots(s-s_{n-1})$$

$\therefore$

$$B'(s_i) = (s_i-s_1)(s_i-s_2)\dots(s_i-s_{i-1})(s_i-s_{i+1})\dots(s_i-s_n)$$

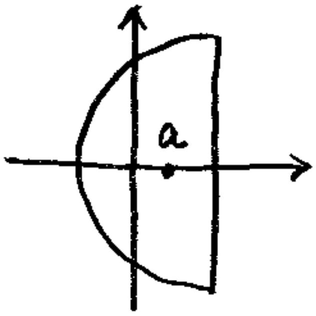
Logo:

$$f(t) = \sum_{i=1}^{n} \frac{A(s_i)\,e^{s_i t}}{B'(s_i)}$$

//

EX

$$F(s) = \frac{A(s)}{(s-a)^n}$$



$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A(s)e^{st}}{(s-a)^n}\,ds$$

$$= \operatorname*{Res}_{s=a}\left[\frac{A(s)e^{st}}{(s-a)^n}\right]$$

$$= \lim_{s\to a} \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left[(s-a)^n\frac{A(s)e^{st}}{(s-a)^n}\right]$$

$$= \lim_{s\to a} \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}\left[A(s)e^{st}\right]$$

$$= \lim_{s\to a} \frac{1}{(n-1)!}\sum_{K=0}^{n-1}\binom{n-1}{K} A^{(K)}(s)\, t^{n-1-K} e^{st}$$

$$= \sum_{K=0}^{n-1} \frac{1}{(n-1-K)!\,K!} A^{(K)}(a)\, t^{n-1-K} e^{at}$$

Para $F(s) = \dfrac{A(s)}{(s-a_1)^{n_1}\dots(s-a_i)^{n_i}}$ usamos a decomposição em frações parciais e aí a expressão acima para cada termo.

//

V.4 ALGUMAS APLICAÇÕES

EX EDO de 2ª ordem

$$\begin{cases} x'' + 2\lambda x' + \omega_0^2 x = f(t) & (\omega_0 > \lambda) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\mathcal{L}[x'] = s\mathcal{L}[x] - x(0) = sX(s) - x_0$$

$$\mathcal{L}[x''] = s^2\mathcal{L}[x] - sx(0) - x'(0) = s^2X(s) - sx_0 - v_0$$

$$\therefore \; s^2X(s) - sx_0 - v_0 + 2\lambda sX(s) - 2\lambda x_0 + \omega_0^2X(s) = F(s)$$

$$(s^2 + 2\lambda s + \omega_0^2)X(s) - (2\lambda + s)x_0 - v_0 = F(s)$$

$$X(s) = \frac{(2\lambda + s)x_0 + v_0}{s^2 + 2\lambda s + \omega_0^2} + \frac{F(s)}{s^2 + 2\lambda s + \omega_0^2}$$

mas: $s^2 + 2\lambda s + \omega_0^2 = 0 \Rightarrow s = \dfrac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega_0^2}}{2}$

$$= -\lambda \pm i\underbrace{\sqrt{\omega_0^2 - \lambda^2}}_{\omega} \qquad (\omega_0 > \lambda)$$

$$\therefore \; s^2 + 2\lambda s + \omega_0^2 = \left(s + \lambda + i\sqrt{\omega_0^2 - \lambda^2}\right)\left(s + \lambda - i\sqrt{\omega_0^2 - \lambda^2}\right)$$

$$= (s - \sigma_+)(s - \sigma_-) \quad ; \quad \sigma_\pm = -\lambda \pm i\omega$$

$$\therefore X = X_1 + X_2 \quad , \quad X_1 = \frac{(2\lambda+s)x_0+v_0}{(s-\sigma_+)(s-\sigma_-)} \quad , \quad X_2 = \frac{F(s)}{(s-\sigma_+)(s-\sigma_-)}$$

$$x_1(t) = \mathcal{L}^{-1}[X_1] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{[(2\lambda+s)x_0+v_0]}{(s-\sigma_+)(s-\sigma_-)}e^{st}ds$$

$$= \left[\operatorname*{Res}_{s=\sigma_+} + \operatorname*{Res}_{s=\sigma_-}\right]\left(\frac{[(2\lambda+s)x_0+v_0]e^{st}}{(s-\sigma_+)(s-\sigma_-)}\right)$$

$$= \frac{[(2\lambda+\sigma_+)x_0+v_0]e^{\sigma_+ t}}{\sigma_+-\sigma_-} + \frac{[(2\lambda+\sigma_-)x_0+v_0]e^{\sigma_- t}}{\sigma_--\sigma_+}$$

$$= \frac{[(\lambda+i\omega)x_0+v_0]e^{-\lambda t}e^{i\omega t}}{2i\omega} + \frac{[(\lambda-i\omega)x_0+v_0]e^{-\lambda t}e^{-i\omega t}}{-2i\omega}$$

$$= \frac{e^{-\lambda t}}{\omega}\left[(\lambda x_0+v_0)\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right) + x_0\omega\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)\right]$$

$$= e^{-\lambda t}\left[x_0\cos\omega t + \left(\frac{\lambda x_0+v_0}{\omega}\right)\sin\omega t\right]$$

$$x_2(t) = \mathcal{L}^{-1}\left[F(s)\cdot\frac{1}{(s-\sigma_+)(s-\sigma_-)}\right] = (f*\varphi)(t)$$

onde $\mathcal{L}[\varphi(t)] = \dfrac{1}{(s-\sigma_+)(s-\sigma_-)}$

$$\therefore \varphi(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{(s-\sigma_+)(s-\sigma_-)}\,ds =$$

$$= \frac{e^{\sigma_+ t}}{\sigma_+-\sigma_-} + \frac{e^{\sigma_- t}}{\sigma_--\sigma_+} = \frac{e^{-\lambda t}e^{i\omega t}}{2i\omega} + \frac{e^{-\lambda t}e^{-i\omega t}}{-2i\omega}$$

$$= \frac{e^{-\lambda t}\sin\omega t}{\omega} \qquad (t>0)$$

$$\therefore x_2(t) = \int_{-\infty}^{+\infty} d\tau\, f_H(\tau)\,\varphi_H(t-\tau) = \int_0^t d\tau\, f(\tau)\,\frac{e^{-\lambda(t-\tau)}\sin\omega(t-\tau)}{\omega}$$

$$\therefore x(t) = x_0 e^{-\lambda t}\cos\omega t + \frac{(v_0+\lambda x_0)}{\omega}e^{-\lambda t}\sin\omega t +$$

$$+ \frac{1}{\omega}\int_0^t e^{-\lambda(t-\tau)}\sin\omega(t-\tau)\, f(\tau)\,d\tau$$

//

EX II **EDP: Eq. de Difusão**

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{K^2}\dfrac{\partial^2 u}{\partial x^2} \qquad , \; t\geq 0,\; x\geq 0 \\ u(x,0) = T_0 \\ u(0,t) = T_1 \\ \lim\limits_{x\to\infty}|u(x,t)| < +\infty \end{cases}$$

$$\mathcal{U}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt$$

$$\mathcal{L}[u_t] = s\,\mathcal{L}[u] - u(x,0) = s\,\mathcal{U} - T_0$$

$$\therefore \quad s\,\mathcal{U} - T_0 = \frac{1}{K^2}\frac{d^2\mathcal{U}}{dx^2} \qquad (*)$$

$$\mathcal{U}(0,s) = \int_0^\infty e^{-st} u(0,t)\,dt = T_1 \int_0^\infty e^{-st}\,dt = T_1 \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{T_1}{s}$$

$$\lim_{x\to\infty} \mathcal{U}(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st} u(x,t)\,dt = \text{finito}$$

$$(*) \Rightarrow \quad \frac{d^2\mathcal{U}}{dx^2} - sK^2\,\mathcal{U} = -K^2 T_0$$

eq. homogênea $\Rightarrow \mathcal{U} = c_1 e^{-K\sqrt{s}\,x} + c_2 e^{K\sqrt{s}\,x} \qquad (s>0)$

solução particular $\Rightarrow \mathcal{U}_p = \dfrac{T_0}{s}$

$$\therefore \quad \mathcal{U} = \frac{T_0}{s} + c_1 e^{-K\sqrt{s}\,x} + c_2 e^{K\sqrt{s}\,x}$$

solução finita para $x \to \infty \Rightarrow c_2 = 0$

$$\mathcal{U} = \frac{T_0}{s} + c_1 e^{-K\sqrt{s}\,x}$$

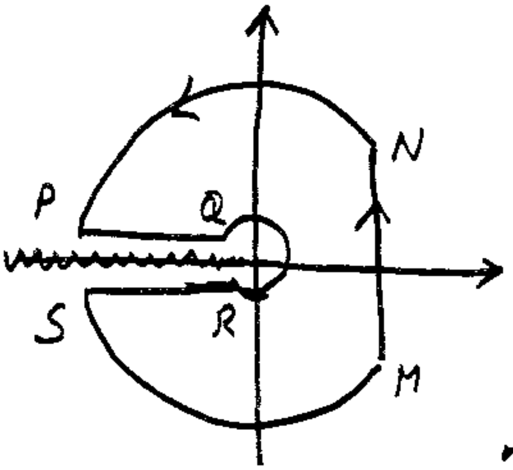
$$x=0 \Rightarrow \frac{T_1}{s} = \frac{T_0}{s} + c_1 \quad \therefore \quad c_1 = \frac{T_1 - T_0}{s}$$

$$\therefore \mathcal{U}(x,s) = \frac{T_0}{s} + \frac{(T_1 - T_0)}{s} e^{-K\sqrt{s}\,x}$$

Como já sabemos que $\mathcal{L}[1] = 1/s$, só falta calcular

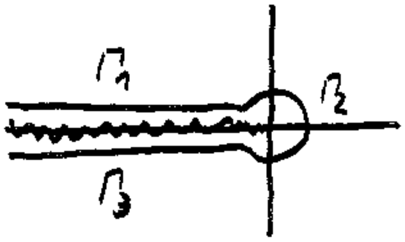
$$\varphi(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{-K\sqrt{s}\,x}\, e^{st}}{s}\, ds$$

Para isso vamos considerar a integral



$$\frac{1}{2\pi i} \oint \frac{e^{-K\sqrt{z}\,x}\, e^{zt}}{z}\, dz = 0$$

Repetindo os argumentos já utilizados no cálculo de uma integral ao longo do mesmo caminho, concluímos que a integral que nos interessa pode ser escrita em termos da integral no caminho abaixo:



$$-\varphi(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{-K\sqrt{z}\,x}\, e^{zt}}{z}\, dz$$

$$= \frac{1}{2\pi i}\left[\int_{\infty}^{0} \frac{e^{-iK\sqrt{r}\,x}\, e^{-rt}}{r}\, dr + \int_{0}^{\infty} \frac{e^{iK\sqrt{r}\,x}\, e^{-rt}}{r}\, dr \right]$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_2} \frac{e^{-iK\sqrt{z}\,x}\, e^{-zt}}{z}\, dz$$

Aqui, entretanto, ao contrário do outro exemplo, a integral ao longo de Γ_2 não se anula para $\epsilon \to 0$. De fato, tomando $z = \epsilon e^{i\theta}$ temos $dz = z\, i\, d\theta$ e como $e^A = 1 + A + \frac{A^2}{2!} + \dots$ temos

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{e^{-iK\sqrt{z}\,x}\, e^{-zt}}{z}\, dz = \frac{1}{2\pi i}\int_{+\pi-\theta(\epsilon)}^{-\pi-\theta(\epsilon)} \frac{(1+\theta(\epsilon))}{\epsilon e^{i\theta}}\, \epsilon e^{i\theta}\, i\, d\theta =$$

$$= -1 + \theta(\epsilon) \xrightarrow{\epsilon\to 0} -1$$

Sendo assim.

$$-\varphi(t) = -1 + \frac{1}{2\pi i}\left[\int_0^\infty \frac{e^{-rt}}{r}\left(e^{ik\sqrt{r}\,x} - e^{-ik\sqrt{r}\,x}\right) dr\right]$$

$$= -1 + \frac{1}{\pi}\int_0^\infty dr\, \frac{e^{-rt} \sin K\sqrt{r}\,x}{r}$$

$$= -1 + \frac{2}{\pi}\int_0^\infty dy\, \frac{e^{-y^2} \sin \frac{Kx}{\sqrt{t}}\, y}{y}$$

Agora precisamos calcular

$$I(\alpha) = \int_0^\infty dy\, \frac{e^{-y^2} \sin \alpha y}{y}, \quad I(0) = 0$$

$$I'(\alpha) = \int_0^{\infty} dy\, e^{-y^2} \cos\alpha y = \frac{1}{2} \int_{-\infty}^{+\infty} dy\, e^{-y^2} \cos\alpha y =$$

$$= \frac{1}{4}\left[\int_{-\infty}^{+\infty} dy\, e^{-y^2} e^{i\alpha y} + \int_{-\infty}^{+\infty} dy\, e^{-y^2} e^{-i\alpha y} \right]$$

$$= \frac{1}{4}\left[e^{-\alpha^2/4} \underbrace{\int_{-\infty}^{+\infty} dy\, e^{-(y-i\alpha/2)^2}}_{\sqrt{\pi}} + e^{-\alpha^2/4} \underbrace{\int_{-\infty}^{+\infty} dy\, e^{-(y+i\alpha/2)^2}}_{\sqrt{\pi}} \right]$$

$$= \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

$$\therefore I(\alpha) = \frac{\sqrt{\pi}}{2} \int_{\alpha_0}^{\alpha} e^{-K^2/4} dK \quad \Rightarrow I(0) = 0 \Leftrightarrow \alpha_0 = 0$$

$$\therefore I(\alpha) = \frac{\sqrt{\pi}}{2} \int_0^{\alpha} e^{-K^2/4} dK = \sqrt{\pi} \int_0^{\alpha/2} e^{-K^2} dK = \frac{\pi}{2} \operatorname{erf}\left(\frac{\alpha}{2}\right)$$

$$\therefore -\varphi(t) = -1 + \frac{2}{\pi}\frac{\pi}{2} \operatorname{erf}\left(\frac{Kx}{2\sqrt{t}}\right) = -\operatorname{erfc}\left(\frac{Kx}{2\sqrt{t}}\right)$$

$$\therefore u(x,t) = T_0 + (T_1 - T_0) \operatorname{erfc}\left(\frac{Kx}{2\sqrt{t}}\right)$$

EX

A transformada de Laplace é particularmente útil para estudarmos expansões assintóticas. Dizemos que $\sum_{k=0}^{\infty}{}' \frac{a_k}{z^k}$ é uma expansão assintótica de $f(z)$ se

$$\lim_{|z|\to\infty} z^n\left[f(z) - \sum_{k=0}^{n}{}' \frac{a_k}{z^k}\right] = 0 \qquad (n = 0, 1, 2, \ldots)$$

e escrevemos

$$f(z) \approx \sum_{k=0}^{\infty}{}' \frac{a_k}{z^k}$$

Vamos supor que $f(t)$ tem a seguinte expansão em série de Taylor:

$$f(t) = \sum_{n=0}^{\infty} a_n\, t^{\lambda_n - 1}$$

onde $0 < \lambda_0 < \lambda_1 < \ldots$, e que seja de ordem exponencial para $t > a$,

$$|f(t)| < c\,e^{bt}$$

Seja agora

$$f_n(t) = f(t) - \sum_{k=0}^{n}{}' a_k\, t^{\lambda_k - 1}$$

Em função desses comportamentos para $t \to 0$ e $t > a$, existem constantes tais que

$$|f_n(t)| \leq C_n \, e^{bt} \, t^{\lambda_{n+1}-1}$$

Por outro lado,

$$\mathcal{L}[f_n(t)] = \mathcal{L}[f(t)] - \sum_{k=0}^{n}{}' a_k \mathcal{L}[t^{\lambda_k - 1}]$$

$$= \mathcal{L}[f(t)] - \sum_{k=0}^{n}{}' a_k \frac{\Gamma(\lambda_k)}{s^{\lambda_k}}$$

de modo que, se

$$\lim_{|s|\to\infty} |s|^{\lambda_n} |\mathcal{L}[f_n(t)]| = 0$$

então temos

$$F(s) = \mathcal{L}[f(t)] \approx \sum_{k=0}^{\infty} a_k \frac{\Gamma(\lambda_k)}{s^{\lambda_k}}$$

De fato:

$$|\mathcal{L}[f_n(t)]| \leq \int_0^\infty |e^{-st}||f_n(t)|dt \leq \int_0^\infty e^{-Re(s)t} C_n e^{bt} t^{\lambda_{n+1}-1} dt$$

$$\leq C_n \int_0^\infty e^{-|s|\cos\phi \, t} e^{bt} t^{\lambda_{n+1}-1} dt = C_n \frac{\Gamma(\lambda_{n+1})}{(|s|\cos\phi - b)^{\lambda_{n+1}}}$$

onde devemos ter $|s|\cos\phi - b > 0$, o que necessariamente implica que $\cos\phi > 0$, ou seja, $\arg(s) < \frac{\pi}{2}$.

Com isso

$$\lim_{|s|\to\infty} |s|^{\lambda_n} |\mathcal{L}[f_n(t)]| \leq \lim_{|s|\to\infty} \frac{|s|^{\lambda_n} \; c_n \Gamma(\lambda_{n+1})}{(|s|\cos\phi - b)^{\lambda_{n+1}}} = 0$$

$\lambda_{n+1} > \lambda_n$

Portanto, provamos o chamado <u>LEMA DE WATSON</u>, ou seja, se $f(t) = \sum_{n=0}^{\infty} a_n t^{\lambda_n - 1}$ e é de ordem exponencial, então a sua transformada de Laplace $F(s)$ tem, para $\arg(s) < \pi/2$, a seguinte expansão assintótica:

$$F(s) = \mathcal{L}[f(t)] \approx \sum_{k=0}^{\infty} a_k \frac{\Gamma(\lambda_k)}{s^{\lambda_k}}$$

Por exemplo, para $f(t) = \ln(1+\sqrt{t})$, temos

$$\ln(1+\sqrt{t}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\sqrt{t})^n}{n}$$

e daí:

$$\mathcal{L}[\ln(1+\sqrt{t})] \approx \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\Gamma\left(\frac{n}{2}+1\right)}{s^{\frac{n}{2}+1}}$$

//