IV.4 APLICAÇÕES DA TRANSFORMADA DE FOURIER NA SOLUÇÃO DE EQUAÇÕES DIFERENCIAIS

EX

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$

← oscilador harmônico amortecido

$\alpha > 0$, $\omega_0^2 = K/m$

$f(t) = F(t)/m$

$$\mathcal{F}[f(t)] = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, f(t)\, e^{i\omega t}$$

$$\mathcal{F}[x(t)] = X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, x(t)\, e^{i\omega t}$$

$$\mathcal{F}[x'(t)] = -i\omega X(\omega)$$

$$\mathcal{F}[x''(t)] = (-i\omega)^2 X(\omega) = -\omega^2 X(\omega)$$

← hipótese implícita $\left(\lim_{t\to\pm\infty} x(t) = \lim_{t\to\pm\infty} x'(t) = 0\right)$

$\therefore$ ED $\Rightarrow$ $$-\omega^2 X - 2\alpha i\omega X + \omega_0^2 X = F$$

$$\therefore\ X(\omega) = \frac{F(\omega)}{(\omega_0^2 - \omega^2) - 2\alpha\omega i}$$

$$\therefore\ x(t) = \mathcal{F}^{-1}[X(\omega)] = \mathcal{F}^{-1}\left[F(\omega)\cdot\frac{1}{(\omega_0^2-\omega^2) - 2\alpha\omega i}\right]$$

Escrevendo

$$\frac{1}{(\omega_0^2-\omega^2)-2\alpha\omega i} = \mathcal{F}[K(t)]$$

vemos que

$$x(t) = \mathcal{F}^{-1}\Big[\mathcal{F}[f(t)]\cdot\mathcal{F}[K(t)]\Big]$$

e pelo teorema da convolução

$$x(t) = (f * K)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\xi\, f(\xi)\,K(t-\xi)$$

Nosso problema agora consiste em encontrar

$$K(t) = \mathcal{F}^{-1}\left[\frac{1}{(\omega_0^2-\omega^2)-2\alpha\omega i}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\, \frac{e^{-i\omega t}}{\underbrace{(\omega_0^2-\omega^2)-2\alpha\omega i}_{Z(\omega)}}$$

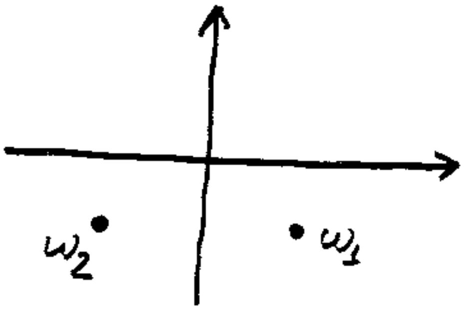
$$Z(\omega)=0 \Rightarrow \omega^2+2\alpha i\omega-\omega_0^2=0$$

$$\omega = \frac{-2\alpha i \pm\sqrt{-4\alpha^2+4\omega_0^2}}{2} = -\alpha i \pm\sqrt{\omega_0^2-\alpha^2}$$

$\boxed{(1)\ \omega_0 > \alpha}$

$$\omega_1 = \sqrt{\omega_0^2 - \alpha^2} - \alpha i$$

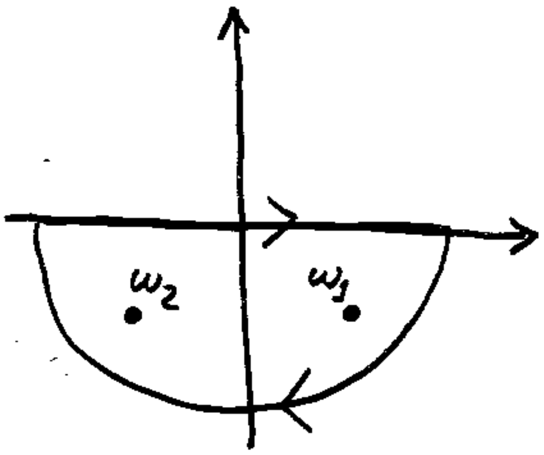
$$\omega_2 = -\sqrt{\omega_0^2 - \alpha^2} - \alpha i$$



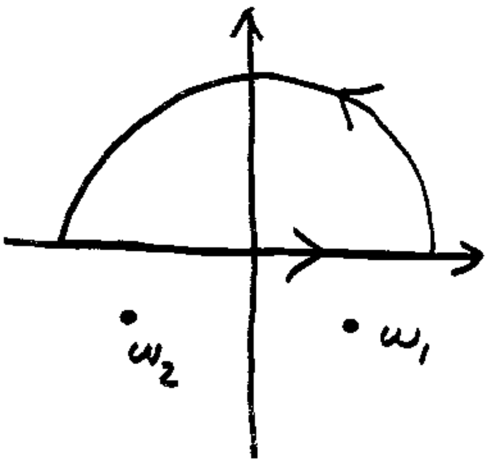
$$\lim_{y \to \pm\infty} e^{-i(iy)t} = \lim_{y \to \pm\infty} e^{yt} = \begin{cases} \infty & \begin{cases} y>0,\ t>0 \\ y<0,\ t<0 \end{cases} \\ 0 & \begin{cases} y>0,\ t<0 & (*) \\ y<0,\ t>0 & (**) \end{cases} \end{cases}$$

(*) → fechar caminho por cima p/ $t<0$

(**) → fechar caminho por baixo p/ $t>0$



$t>0$



$t<0$

$\boxed{t>0}$

$$\int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t}}{(\omega_0^2-\omega^2)-2\alpha\omega i} \overset{\text{sentido horário}}{=} -2\pi i \left[\operatorname*{Res}_{\omega=\omega_1} + \operatorname*{Res}_{\omega=\omega_2}\right] \frac{e^{-i\omega t}}{\underbrace{Z(\omega)}_{-(\omega-\omega_1)(\omega-\omega_2)}}$$

$$= -2\pi i \left[\frac{e^{-i\omega_1 t}}{\omega_2-\omega_1} + \frac{e^{-i\omega_2 t}}{\omega_1-\omega_2}\right] = 2\pi i \left[\frac{e^{-i\omega_1 t}}{\omega_1-\omega_2} + \frac{e^{-i\omega_2 t}}{\omega_2-\omega_1}\right]$$

$$= \frac{2\pi i}{\omega_1 - \omega_2}\left[e^{-i(\sqrt{\omega_0^2-\alpha^2} - \alpha i)t} - e^{-i(-\sqrt{\omega_0^2-\alpha^2} - \alpha i)t} \right]$$

$$= \frac{2\pi i}{2\sqrt{\omega_0^2-\alpha^2}} e^{-\alpha t}\left[e^{-i\sqrt{\omega_0^2-\alpha^2}\,t} - e^{+i\sqrt{\omega_0^2-\alpha^2}\,t} \right]$$

$$= \frac{2\pi e^{-\alpha t}}{\sqrt{\omega_0^2-\alpha^2}}\left[-\frac{e^{i\sqrt{\omega_0^2-\alpha^2}\,t} - e^{-i\sqrt{\omega_0^2-\alpha^2}\,t}}{2i} \right]$$

$$= \frac{2\pi\, e^{-\alpha t} \sin\sqrt{\omega_0^2-\alpha^2}\,t}{\sqrt{\omega_0^2-\alpha^2}}$$

$\boxed{t<0}$ $$\oint \frac{e^{-i\omega t}}{Z(\omega)} = 0 \Rightarrow \int_{-\infty}^{+\infty} (\cdots) = 0 \quad \therefore K(t) = 0 \ , \ t<0$$

$$\therefore K(t) = \begin{cases} \dfrac{\sqrt{2\pi}\, e^{-\alpha t} \sin\sqrt{\omega_0^2-\alpha^2}\,t}{\sqrt{\omega_0^2-\alpha^2}} & , \ t>0 \\ 0 & , \ t<0 \end{cases}$$

$$\therefore x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi\, K(\xi) f(t-\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} d\xi\, K(\xi) f(t-\xi)$$

$$x(t) = \int_0^{\infty} d\xi\, \frac{e^{-\alpha\xi} \sin\left(\sqrt{\omega_0^2-\alpha^2}\,\xi\right)}{\sqrt{\omega_0^2-\alpha^2}} f(t-\xi)$$

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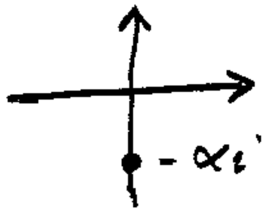
$\boxed{(2)\ \omega_0 = \alpha}$ $\quad \omega_1 = \omega_2 = -\alpha i$



$\therefore Z(\omega) = -(\omega + \alpha i)^2$

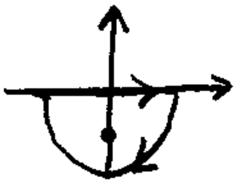
$$\oint \frac{e^{-i\omega t}}{-(\omega+\alpha i)^2} d\omega = -2\pi i \operatorname*{Res}_{\omega = -\alpha i} \left(\frac{e^{-i\omega t}}{-(\omega+\alpha i)^2} \right) =$$

$$= 2\pi i \lim_{\omega \to -\alpha i} \frac{d}{d\omega} \left((\omega+\alpha i)^2 \frac{e^{-i\omega t}}{(\omega+\alpha i)^2} \right)$$

$$= 2\pi i \lim_{\omega \to -\alpha i} \left(-it e^{-i\omega t} \right) = 2\pi t \, e^{-i(-\alpha i)t}$$

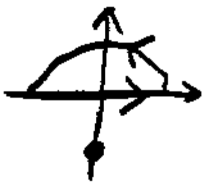
$$= 2\pi t \, e^{-\alpha t}$$

$\boxed{t>0}$



$\therefore K(t) = \frac{1}{\sqrt{2\pi}} \cdot 2\pi t \, e^{-\alpha t} \quad , \quad t>0$

$\boxed{t<0}$



$\oint = 0 \quad \therefore \quad K(t) = 0$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi \, K(\xi) f(t-\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} d\xi \, K(\xi) f(t-\xi)$$

$$x(t) = \int_0^{\infty} d\xi \, e^{-\alpha\xi} \xi f(t-\xi)$$

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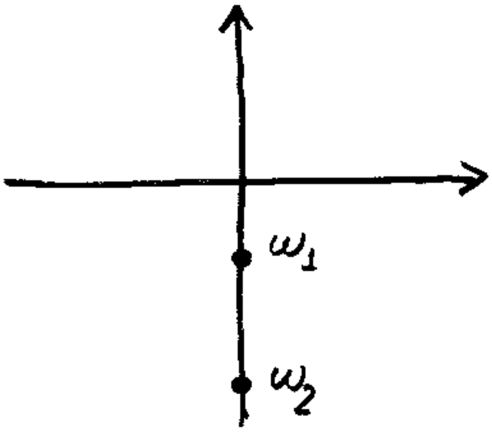
$\boxed{(3)\ \omega_0 < \alpha}$

$$\omega_1 = i\sqrt{\alpha^2-\omega_0^2} - \alpha i = -i\left(\alpha - \sqrt{\alpha^2-\omega_0^2}\right)$$

$$\omega_2 = -i\sqrt{\alpha^2-\omega_0^2} - \alpha i = -i\left(\alpha + \sqrt{\alpha^2+\omega_0^2}\right)$$



Os cálculos são análogos ao caso $\omega_0 > \alpha$ exceto que tomamos $\sqrt{\omega_0^2-\alpha^2} = \sqrt{(-1)(\alpha^2-\omega_0^2)} = i\sqrt{\alpha^2-\omega_0^2}$ e usando $\sin(iA) = i\sinh A$, segue

$$x(t) = \int_0^\infty d\xi\, e^{-\alpha\xi}\, \frac{\sinh\left(\sqrt{\alpha^2-\omega_0^2}\,\xi\right)}{\sqrt{\alpha^2-\omega_0^2}}\, f(t-\xi) \quad //$$

$\boxed{EX}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$ ← eq. de onda $(-\infty < x < \infty)$

$$\mathcal{U}(K,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, u(x,t)\, e^{iKx} = \mathcal{F}[u(x,t)]$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, \mathcal{U}(K,t)\, e^{-iKx} = \mathcal{F}^{-1}[\mathcal{U}(K,t)]$$

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (-iK)^2\mathcal{F}[u] \qquad \left(\lim_{x\to\mp\infty} u(x,t) = 0\right)$$

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2}{\partial t^2}\mathcal{F}[u]$$

tomando $\mathcal{F} \Rightarrow$ $$(-iK)^2 \mathcal{U} = \frac{1}{v^2}\frac{\partial^2 \mathcal{U}}{\partial t^2}$$

$$\therefore \frac{\partial^2 \mathcal{U}}{\partial t^2} + v^2 K^2 \mathcal{U} = 0$$

$$\mathcal{U} = A(K)\, e^{ivKt} + B(K)\, e^{-ivKt}$$

$$\therefore u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, A(K)\, e^{ivKt} e^{-iKx} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, B(K)\, e^{-ivKt} e^{-iKx}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, A(K)\, e^{-iK(x-vt)} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, B(K)\, e^{-iK(x+vt)}$$

Se: $$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, A(K)\, e^{-iKx}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dK\, B(K)\, e^{-iKx}$$

então:

$$\boxed{u(x,t) = \psi(x-vt) + \phi(x+vt)}$$

← "solução de d'Alembert"

onda viajando para direita com velocidade v

onda viajando para esquerda com velocidade v

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IV.5 TRANSFORMADAS EM SENO E CO-SENO DE FOURIER

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, f(x)\, e^{ikx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dx\, f(x)\, e^{ikx} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\, f(x)\, e^{ikx}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\, f(-x)\, e^{-ikx} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\, f(x)\, e^{ikx}$$

$\boxed{f(x) \text{ par}}$

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\, f(x)\, e^{-ikx} + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} dx\, f(x)\, e^{ikx}$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} dx\, f(x)\left[\frac{e^{ikx}+e^{-ikx}}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} dx\, f(x)\cos kx = \mathcal{F}_c[f(x)] = F_c(k)$$

$$\boxed{F_c(k) = \mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} dx\, f(x)\cos kx}$$

← TRANSFORMADA EM CO-SENO DE FOURIER

Como $\cos kx$ é par, é imediato que $F_c(k)$ é par. Logo, repetindo o raciocínio acima para $\mathcal{F}^{-1}[F_c(k)]$ vamos encontrar que $\mathcal{F}^{-1}[F_c(k)] = \mathcal{F}_c^{-1}[F_c(k)]$, onde

$$\boxed{f(x) = \mathcal{F}_c^{-1}[F_c(k)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} dk\, F_c(k)\cos kx}$$

← TRANSFORMADA INVERSA EM CO-SENO DE FOURIER

$\boxed{f(x) \text{ ímpar}}$

$$\mathcal{F}[f(x)] = -\frac{1}{\sqrt{2\pi}}\int_0^\infty dx\, f(x)\, e^{-ikx} + \frac{1}{\sqrt{2\pi}}\int_0^\infty dx\, f(x)\, e^{ikx}$$

$$= \frac{2i}{\sqrt{2\pi}}\int_0^\infty dx\, f(x)\left[\frac{e^{ikx}-e^{-ikx}}{2i}\right]$$

$$= i\sqrt{\frac{2}{\pi}}\int_0^\infty dx\, f(x)\sin kx = i\,\mathcal{F}_s[f(x)] = i\,F_s(k)$$

$$\boxed{F_s(k) = \mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty dx\, f(x)\sin kx}$$

← TRANSFORMADA EM SENO DE FOURIER

Como $\sin kx$ é ímpar, segue que $F_s(k)$ é ímpar.

$$f(x) = \mathcal{F}^{-1}[F(k)] = i\,\mathcal{F}^{-1}[F_s(k)] = i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, F_s(k)\, e^{-ikx}$$

$$= i\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} dk\, F_s(k)\, e^{-ikx} + \int_0^\infty dk\, F_s(k)\, e^{-ikx}\right]$$

$$= i\frac{1}{\sqrt{2\pi}}\left[\int_0^\infty dk\, F_s(-k)\, e^{ikx} + \int_0^\infty dk\, F_s(k)\, e^{-ikx}\right]$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty dk\left[\frac{e^{ikx}-e^{-ikx}}{2i}\right]F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_s(k)\sin kx$$

$$\boxed{f(x) = \mathcal{F}_s^{-1}[F_s(k)] = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_s(k)\sin kx}$$

← TRANSFORMADA INVERSA EM SENO DE FOURIER

• OBS: Um dos aspectos negativos das transformadas em seno ou em co-seno é a perda da dualidade existente na transformada de Fourier. Por exemplo, já não vale a propriedade que derivar em um espaço equivale a multiplicação pela variável do espaço recíproco. De fato, supondo $\lim_{x\to\infty} f(x) = 0$, podemos ver facilmente através de integração por partes que:

$$\mathcal{F}_C[f'(x)] = -\sqrt{\frac{2}{\pi}}\, f(0) + k\,\mathcal{F}_S[f(x)]$$

$$\mathcal{F}_S[f'(x)] = -k\,\mathcal{F}_C[f(x)]$$

Além disso, não é imediato como podemos relacionar as transformadas em seno e co-seno das funções $f(x)$ e $f(x-a)$ pois no primeiro caso só consideramos os valores de $f(s)$ para $s>0$ enquanto no segundo devemos considerar os valores de $f(s)$ para $s>-a$.

Já quanto à convolução, podemos notar que a inversão do produto $F_C(k)\,G_C(k)$, onde $F_C(k) = \mathcal{F}_C[f(x)]$ e $G_C(k) = \mathcal{F}_C[g(x)]$, se faz através de $\mathcal{F}_C^{-1}$ pois o produto de duas funções pares é uma função par. Por outro lado, o produto de duas funções ímpares é uma função par. Logo, a inversão de $F_S(k)\,G_S(k)$ deve ser feita usando $\mathcal{F}_c^{-1}$. Com isso:

$$\mathcal{F}_c^{-1}\left[F_c(k)G_c(k)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_c(k)G_c(k)\cos kx =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_c(k)\sqrt{\frac{2}{\pi}}\int_0^\infty d\xi\, g(\xi)\cos k\xi \cos kx$$

$$= \frac{1}{\pi}\int_0^\infty d\xi\, g(\xi)\int_0^\infty dk\, F_c(k)\cos k(x-\xi) + \frac{1}{\pi}\int_0^\infty d\xi\, g(\xi)\int_0^\infty dk\, F_c(k)\cos k(x+\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi\, g(\xi)\, f(x-\xi) + \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi\, g(\xi)\, f(x+\xi)$$

$$\mathcal{F}_c^{-1}\left[F_s(k)G_s(k)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_s(k)G_s(k)\cos kx =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, F_s(k)\sqrt{\frac{2}{\pi}}\int_0^\infty d\xi\, g(\xi)\sin k\xi \cos kx$$

$$= \frac{1}{\pi}\int_0^\infty d\xi\, g(\xi)\int_0^\infty dk\, F_s(k)\sin k(\xi-x) + \frac{1}{\pi}\int_0^\infty d\xi\, g(\xi)\int_0^\infty dk\, F_s(k)\sin k(\xi+x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi\, g(\xi)\, f(\xi-x) + \frac{1}{\sqrt{2\pi}}\int_0^\infty d\xi\, g(\xi)\, f(\xi+x)$$

Isso tudo mostra que as transformadas em seno e co-seno não são tão "boas" quanto a transformada de Fourier. Veremos, entretanto, que quando consideramos apenas o intervalo $0 \le x < \infty$, existe uma melhor opção que as transformadas em seno e co-seno que é a transformada de Laplace.

EX $\quad \dfrac{d^2x}{dt^2} - \alpha^2 x = 0 \quad , \ 0 \le t < \infty \ , \ \dfrac{dx}{dt}(0) = b , \ \lim_{t\to\infty} x(t) = 0$

Vamos usar a transformada em co-seno para resolver esse problema.

$$\mathcal{F}_C\left[\frac{d^2x}{dt^2}\right] = -\sqrt{\frac{2}{\pi}}\,\frac{dx}{dt}(0) + K\,\mathcal{F}_S\left[\frac{dx}{dt}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,b + K\left(-K\,\mathcal{F}_C[x]\right)$$

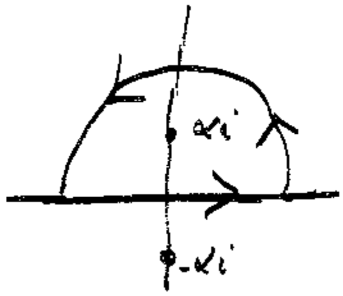
$$= -b\sqrt{\frac{2}{\pi}} - K^2\mathcal{F}_C[x]$$

$$\therefore \quad -b\sqrt{\frac{2}{\pi}} - K^2\mathcal{F}_C[x] - \alpha^2\mathcal{F}_C[x] = 0$$

$$\therefore \mathcal{F}_C[x] = \frac{-b\sqrt{2/\pi}}{K^2+\alpha^2}$$

$$\therefore x(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty dK\left(\frac{-b\sqrt{2/\pi}}{K^2+\alpha^2}\right)\cos Kt = -b\,\frac{2}{\pi}\int_0^\infty dK\,\frac{\cos Kt}{K^2+\alpha^2}$$

$$= -\frac{b}{\pi}\int_{-\infty}^{+\infty} dK\,\frac{\cos Kt}{K^2+\alpha^2} = -\frac{b}{\pi}\,\mathrm{Re}\left[\int_{-\infty}^{+\infty} dK\,\frac{e^{iKt}}{K^2+\alpha^2}\right]$$



$$= -\frac{b}{\pi}\,\mathrm{Re}\left[2\pi i\,\mathrm{Res}_{K=\alpha i}\left(\frac{e^{iKt}}{K^2+\alpha^2}\right)\right] =$$

$$= -\frac{b}{\pi}\cdot 2\pi i\,\frac{e^{i(\alpha i)t}}{2\alpha i} = -\frac{b}{\alpha}\,e^{-\alpha t}$$

$$\therefore x(t) = -\frac{b}{\alpha}\,e^{-\alpha t}$$

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