Soft local times

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- we describe a method for simulating an adapted stochastic process on a general space Σ by means of a Poisson point process on $\Sigma \times \mathbb{R}_+$;
- in particular, it is useful for constructing couplings of two processes, simply by using the same realization of the Poisson point process for both of them;
- with this coupling, one can study the *range* of a stochastic process (i.e., the set $\{X_1, \dots, X_n\}$)

Consider the space of Radon point measures on $\Sigma \times \mathbb{R}_+$

$$L = \Big\{ \eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)};\, z_\lambda \in \Sigma,\, v_\lambda \in \mathbb{R}_+ \text{ and } \eta(K) < \infty \text{ for all compact } K \Big\}.$$

One can now canonically construct a Poisson point process η on the space $(L, \mathcal{D}, \mathbb{Q})$ with intensity given by $\mu \otimes dv$, where dv is the Lebesgue measure on $\mathbb{R}_+$.

Proposition

Let $g : \Sigma \to \mathbb{R}_+$ *be a measurable function with* $\int g(z)\mu(dz) = 1$*. For* $\eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)} \in L$*, we define*

$$\xi = \inf\{t \geq 0;\ \textit{there exists } \lambda \in \Lambda \textit{ such that } tg(z_\lambda) \geq v_\lambda\}.$$

Then under the law $\mathbb{Q}$ *of the Poisson point process* η*,*

- *there exists a.s. a unique* $\hat{\lambda} \in \Lambda$ *such that* $\xi g(z_{\hat{\lambda}}) = v_{\hat{\lambda}}$*,*
- $(z_{\hat{\lambda}}, \xi)$ *is distributed as* $g(z)\mu(dz) \otimes \mathrm{Exp}(1)$*,*
- $\eta' := \sum_{\lambda \neq \hat{\lambda}} \delta_{(z_\lambda, v_\lambda - \xi g(z_\lambda))}$ *has the same law as* η *and is independent of* $(\xi, \hat{\lambda})$*.*

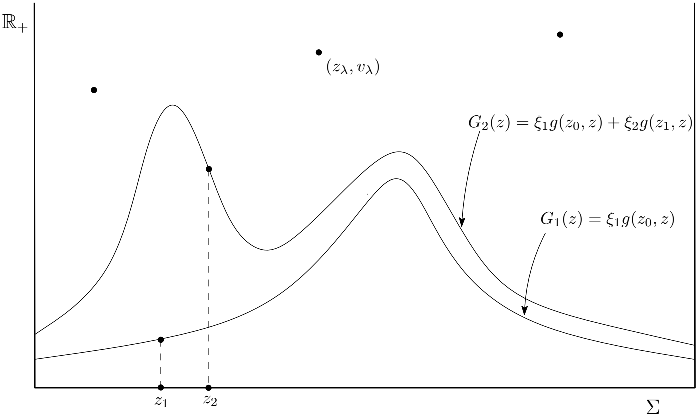

Figure: An example illustrating the definition of ξ and $\hat{\lambda}$ in the above Proposition. Observe that this construction can be iterated to obtain a realization of a stochastic process X (here, $X_1 = z_1, X_2 = z_2$). The quantity G_n is called the *soft local time* of the process X at time n.

Assume that X, Y are two stochastic processes on the same space Σ, and we want to couple them in such a way that

$$\mathbb{P}\big[\{X_1, \ldots, X_{T_1}\} \subset \{Y_1, \ldots, Y_{T_2}\}\big] \geq 1 - \varepsilon,$$

where $T_{1,2}$ are some (random) moments. The above proposition provides a way to construct such a coupling. Let $\hat{G}$ be the soft local time for the process Y, and we construct both processes using *the same* realization of the Poisson process η. Then

$$\mathbb{P}\big[\{X_1, \ldots, X_{T_1}\} \subset \{Y_1, \ldots, Y_{T_2}\}\big] \geq \mathbb{P}[G_{T_1} \leq \hat{G}_{T_2}],$$

and the last probability can be estimated in some way.

Applications:

- in [Popov-Teixeira, 2012] we used soft local times to obtain some decoupling ineuqalities for random interlacements
- in [Comets-Gallesco-Popov-Vachkovskaia, 2013], this method was used to obtain large deviation for cover times of the torus (in these two applications, the space Σ is a certain space of excursions of SRW)
- a similar method was used in an earlier paper by Tsirelson [EJP, 2006] to solve the following kind of problem: let $X_1, X_2, X_3, \ldots$ be i.i.d.r.v. $\sim U[0,1]$, and let $g > 0$ be a density on $[0,1]$. Construct (having additional randomness) a permutation σ such that $Y_1, Y_2, Y_3, \ldots$ are i.i.d.r.v. with density g, where $Y_k = X_{\sigma(k)}$. (It is an open problem to do that without additional randomness.)

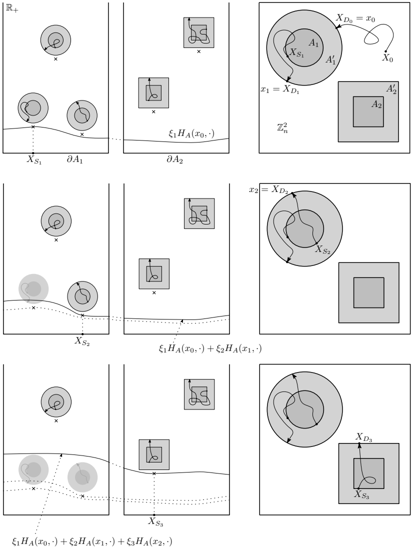

Figure: The construction of the excursions of the SRW on the torus.