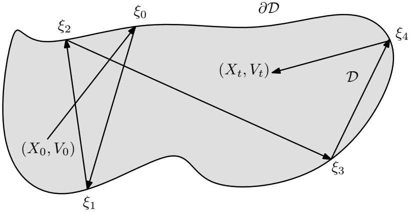
Knudsen billiards and random walks in random environment with unbounded jumps

Outline

(Comets, Popov, Schütz, Vachkovskaia, ARMA, 2009)

The model can be informally described in the following way:

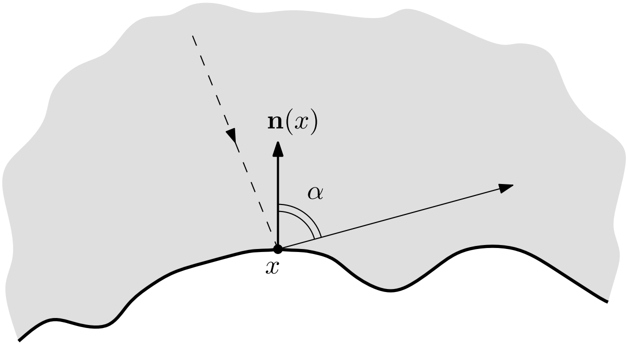
- A particle moves with constant speed inside some d-dimensional domain
- When it hits the boundary, it is reflected in some random direction, not depending on the incoming direction, and keeping the absolute value of its speed



Notations:

- $X_t \in \mathcal{D}$ is the location of the process at time t, and $V_t \in \mathbb{S}^{d-1}$ is the corresponding direction;
- $\xi_n \in \partial\mathcal{D}$, $n = 0, 1, 2, \dots$ are the points where the process hits the boundary.

Reflection: cosine reflection law:



The density of the outgoing direction is proportional to $\cos\alpha$

For the case of cosine reflection law (finite domains):

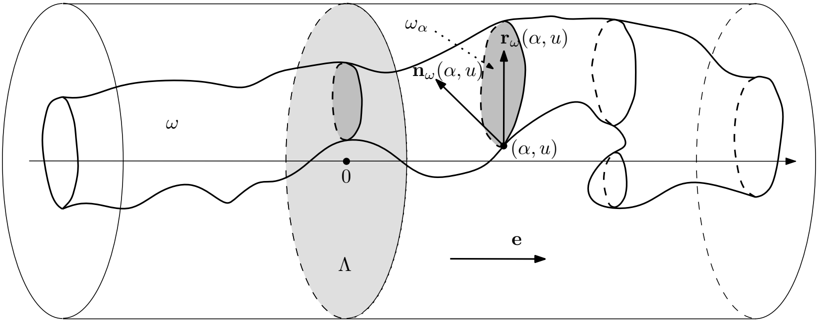
Theorem

(assuming that the boundary is Lipschitz and a.e. continuously differentiable)

- *The stationary measure of the random walk ξ_n is uniform on $\partial\mathcal{D}$.*
- *The stationary measure of the process (X_t, V_t) is the product of uniform measures on $\mathcal{D}$ and $\mathbb{S}^{d-1}$.*

Proof: follows from the reversibility (the transition density is symmetric).

(Comets, Popov, Schütz, Vachkovskaia, AP, 2010)



Stationary random tube in $\mathbb{R}^d$

Let $Z^{(m)}_{\cdot}$ be the polygonal interpolation of $n/m \mapsto m^{-1/2}\xi_n \cdot e$ (the discrete time random walk).

We denote by $\mathbb{Q}$ the stationary measure for the environment seen from the particle (there is an explicit formula for $\mathbb{Q}$).

Theorem

Assume Conditions L, P, R ("nice boundary"), and suppose that the second moment of the jump projected on the horizontal direction $\langle b \rangle_{\mathbb{Q}}$ is finite. Then, there exists a constant $\sigma > 0$ such that for $\mathbb{P}$-almost all ω, $\sigma^{-1} Z^{(m)}_{\cdot}$ converges in law, under $\mathtt{P}_{\omega}$, to Brownian motion as $m \to \infty$.

The corresponding result for the continuous time Knudsen stochastic billiard is available too. Define $\hat{Z}_t^{(s)} = s^{-1/2} X_{st} \cdot e$.

Theorem

Assume Conditions L, P, R, and suppose that $\langle b \rangle_{\mathbb{Q}} < \infty$. Denote

$$\hat{\sigma} = \frac{\sigma \Gamma(\frac{d}{2} + 1)\mathcal{Z}}{\pi^{1/2}\Gamma(\frac{d+1}{2})\langle |\omega_0| \rangle_{\mathbb{P}} d},$$

where σ is from the above Theorem and $\mathcal{Z}$ is the normalizing constant from the definition of $\mathbb{Q}$. Then, for $\mathbb{P}$-almost all ω, $\hat{\sigma}^{-1} \hat{Z}_{\cdot}^{(s)}$ converges in law to Brownian motion as $s \to \infty$.

(Comets, Popov, Schütz, Vachkovskaia, JSP, 2010)

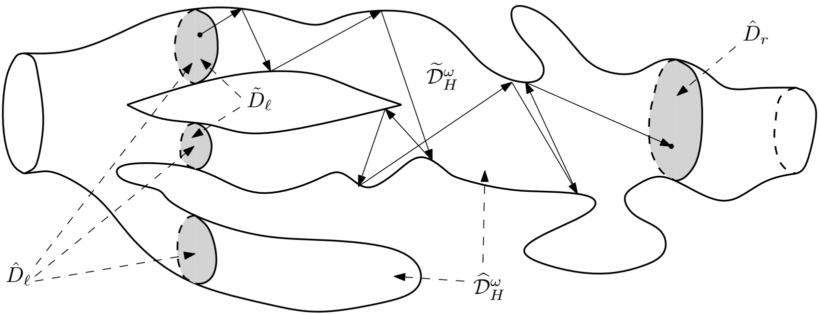
Let $\widehat{\mathcal{D}}_H^\omega$ be the part of the random tube ω which lies between 0 and H:

$$\widehat{\mathcal{D}}_H^\omega = \{z \in \omega : z \cdot \mathbf{e} \in [0, H]\}.$$

$$\hat{D}_\ell = \{0\} \times \omega_0,$$
$$\hat{D}_r = \{H\} \times \omega_H,$$

$\tilde{\omega}_0$ is the set of points of ω_0, from where the particle can reach $\hat{D}_r$ by a path which stays within $\widehat{\mathcal{D}}_H^\omega$ and $\tilde{D}_\ell := \{0\} \times \tilde{\omega}_0$

$\mathfrak{C}_H$: the event that the particle crosses the tube without going back to $\tilde{D}_\ell$



On the definition of $\tilde{D}_\ell, \widetilde{\mathcal{D}}_H^\omega$, and the event $\mathfrak{C}_H$ (a trajectory crossing the tube is shown)

Results ($\mathcal{T}_H$ is the total lifetime of the particle):

- $\mathrm{E}_\omega \mathcal{T}_H \sim \dfrac{\gamma_d |\mathbb{S}^{d-1}| \left\langle |\omega_0| \right\rangle_{\mathbb{P}}}{2|\tilde{\omega}_0|} \cdot H$
- $\mathrm{P}_\omega[\mathfrak{C}_H] \sim \dfrac{\gamma_d |\mathbb{S}^{d-1}| \hat{\sigma}^2 \left\langle |\omega_0| \right\rangle_{\mathbb{P}}}{2|\tilde{\omega}_0|} \cdot \dfrac{1}{H}$
- $\mathrm{E}_\omega(\mathcal{T}_H \mid \mathfrak{C}_H) \sim \frac{1}{3\hat{\sigma}^2} \cdot H^2$

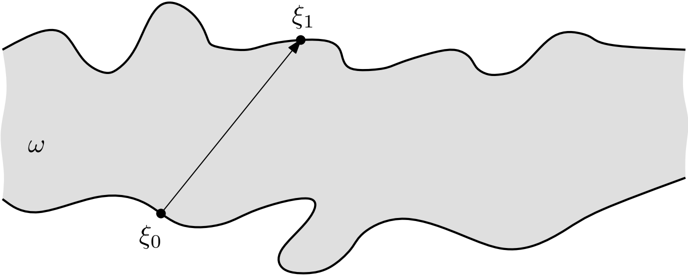
As a consequence,

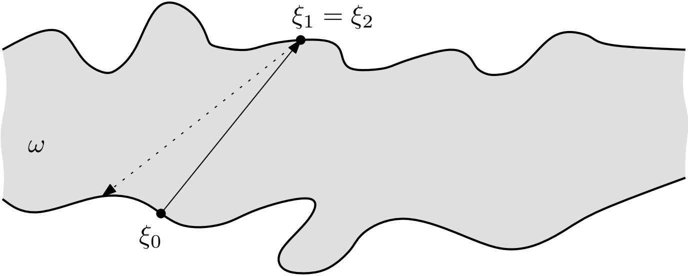
$$\mathrm{E}_\omega(\mathcal{T}_H \mathbb{I}\{\mathfrak{C}_H^c\}) \sim \frac{H}{3} \gamma_d |\mathbb{S}^{d-1}| \left\langle |\omega_0| \right\rangle_{\mathbb{P}} \left\langle |\tilde{\omega}_0|^{-1} \right\rangle_{\mathbb{P}} \sim 2\mathrm{E}_\omega(\mathcal{T}_H \mathbb{I}\{\mathfrak{C}_H\})$$

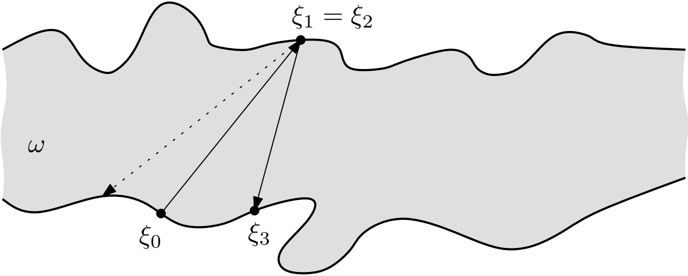
(Comets, Popov, `arXiv:1009.0048`; to appear in AIHP-PS)

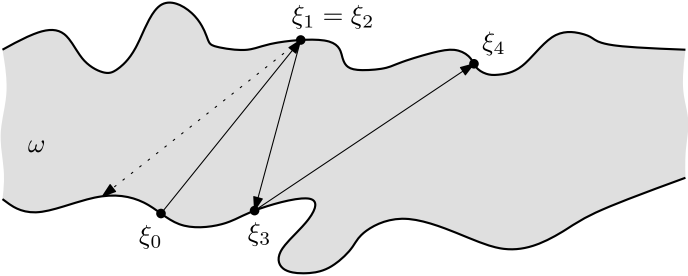
Informal definition:

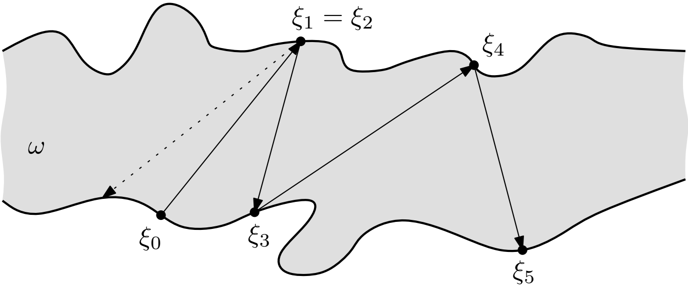
- the process lives in the infinite random tube
- the jumps in the positive direction are always accepted
- the jumps in the negative direction are accepted with probability $e^{-\lambda u}$, where u is the horizontal size of the attempted jump.

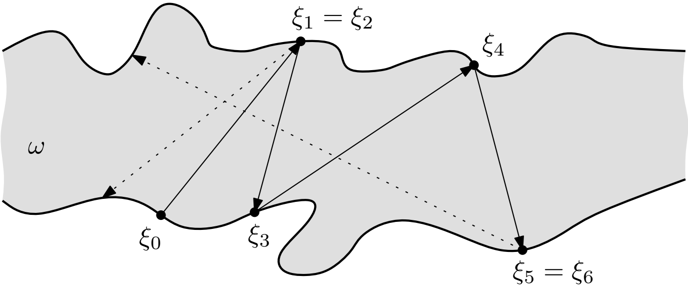

ξ_1
ω
ξ_0

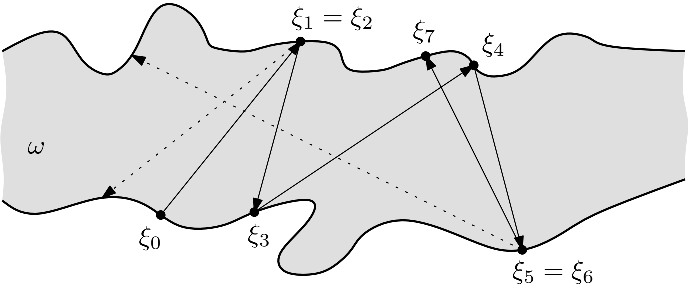

$\xi_1 = \xi_2$
ω
ξ_0


$\xi_1 = \xi_2$
ω
ξ_0
ξ_3


$\xi_1 = \xi_2$
ξ_4
ω
ξ_0
ξ_3


$\xi_1 = \xi_2$
ξ_4
ω
ξ_0
ξ_3
ξ_5


$\xi_1 = \xi_2$
ξ_4
ω
ξ_0
ξ_3
$\xi_5 = \xi_6$


$\xi_1 = \xi_2$
ξ_7
ξ_4
ω
ξ_0
ξ_3
$\xi_5 = \xi_6$

Main result: LLN

Theorem

Assume that $d \geq 3$. There exists a positive deterministic $\hat{v}$ such that for $\mathbb{P}$-almost every ω

$$\frac{\xi_n \cdot \mathbf{e}}{n} \to \hat{v} \qquad \text{as } n \to \infty,\ \mathrm{P}_\omega\text{-a.s.}$$

Main difficulty: although the random walk is still reversible, it is unclear how to obtain an explicit form for the invariant measure for environment seen from the particle.

So, we first consider an analogous process in discrete space.

Consider a Markov chain $(S_n, n = 0, 1, 2, \dots)$ in $\mathbb{Z}$ with transition probabilities

$$\mathtt{P}_\omega^{x_0}[S_{n+1} = x + y \mid S_n = x] = \omega_{xy} \text{ for all } n \geq 0,\ \mathtt{P}_\omega^{x_0}[S_0 = x_0] = 1,$$

so that $\mathtt{P}_\omega^{x_0}$ is the *quenched* law of the Markov chain starting from x_0 in the environment ω.

The environment is chosen at random from the space Ω according to a law $\mathbb{P}$ before the random walk starts; we assume that the sequence of random vectors $(\omega_{x\cdot}, x \in \mathbb{Z})$ is stationary and ergodic.

The random walk S is supposed to satisfy the following conditions:

Condition E. There exists $\tilde{\varepsilon}$ such that $\mathbb{P}[\omega_{01} \geq \tilde{\varepsilon}] = 1$.

Condition C. There exist $\gamma_1 > 0$ and $\alpha > 1$ such that for all $s \geq 1$ we have

$$\sum_{y:|y|\geq s} \omega_{0y} \leq \gamma_1 s^{-\alpha}, \qquad \mathbb{P}\text{-a.s.}$$

Denote by S^{ϱ} the random walk in the truncated environment (jumps that are larger than ϱ are rejected).

Let $N_{\infty}^{\varrho}(x)$ be the total number of visits of S^{ϱ} to x.

Condition D. There is a function $g_1 \geq 0$ with the property $\sum_{k=1}^{\infty} kg_1(k) < \infty$ and a finite ϱ_0, such that for all $x \leq 0$ and all $\varrho \geq \varrho_0$, $\mathbb{P}$-almost surely it holds that $\mathrm{E}_{\omega}^{0} N_{\infty}^{\varrho}(x) \leq g_1(|x|)$.

With these assumptions, we can prove that the speed of the random walk is well-defined and positive:

Theorem

For all $\varrho \in [\varrho_0, \infty]$ there exists $v_\varrho > 0$ such that for $\mathbb{P}$-a.a. ω we have

$$\frac{S_n^\varrho}{n} \to v_\varrho \quad \text{as } n \to \infty, \ \mathrm{P}_\omega\text{-a.s.}$$

Using this result, one can also prove the LLN for the random billiard with drift by a discretization/coupling argument.

Let us write an outline of the proof of the LLN for RWRE.

Denote by $T_z^\varrho = \min\{k \geq 0 : S_k^\varrho \geq z\}$.

Let

$$r_x^\varrho(z) = \mathrm{P}_\omega^x[S_{T_z^\varrho}^\varrho = z]$$

be the probability that, at moment T_z^ϱ, the (truncated) random walk is located *exactly* at z.

Lemma

Assume Conditions E, C, D. Then, there exists $\varepsilon_1 > 0$ such that, $\mathbb{P}$-a.s.,

$$r_x^{\varrho}(0) \geq 2\varepsilon_1$$

for all $x \leq 0$ and for all $\varrho \in [\varrho_0, \infty]$.

Proof:

- define $u_k = \operatorname*{ess\,inf}_{\mathbb{P}} \min_{y\in[-k,0]} r_y^{\varrho}(0)$
- using Condition C, prove that $u_{s^\beta} \geq (1 - C_3 s^{-\varphi})u_s$ for fixed constants β and φ
- Iterating, we obtain that $u_m \geq 2\varepsilon_1 > 0$ for all $m \geq 2$, where

$$\varepsilon_1 = \frac{1}{2}u_2(1 - C_3 2^{-\varphi})(1 - C_3 2^{-\beta\varphi})(1 - C_3 2^{-\beta^2\varphi}) \ldots > 0$$

Proof of the theorem (sketch):

- fix some integer $\varrho \in [\varrho_0, \infty)$, and consider a sequence of i.i.d. random variables $\zeta_1, \zeta_2, \zeta_3, \dots$ with $P[\zeta_j = 1] = 1 - P[\zeta_j = 0] = \varepsilon_1$
- for all $j \geq 1$, Lemma implies that $r_x^\varrho(j\varrho) \geq 2\varepsilon_1$ for all $x \in [(j-1)\varrho, j\varrho - 1]$
- we couple the sequence $\zeta = (\zeta_1, \zeta_2, \zeta_3, \dots)$ with the random walk S^ϱ in such a way that $\zeta_j = 1$ implies that $S^\varrho_{T^\varrho_{j\varrho}} = j\varrho$
- denote $\ell_1 = \min\{j : \zeta_j = 1\}$

- then, since ζ (and therefore ℓ_1) is independent of ω, $\theta_{\ell_1 \varrho}\omega$ has the same law $\mathbb{P}$ as ω
- this allows us to break the trajectory of the random walk into stationary ergodic (after suitable shift) sequence of pieces, and then apply the ergodic theorem to obtain the law of large numbers
- the stationary measure of the environment seen from the particle (for the truncated random walk) can also be obtained from this construction by averaging along the cycle
- finally, we pass to the limit as $\varrho \to \infty$