On the internal distance in the interlacement set

Jiří Černý Serguei Popov

Definition of the interlacement set $\mathcal{I}^u$

Graph distance within the interlacement set

- $\mathbb{Z}^d$, $d \geq 3$, so that SRW is transient
- informally speaking, *random interlacements* $=$ stationary soup of doubly infinite SRW's trajectories
- u is the "intensity" of the interlacement set, so $\mathcal{I}^{u_1} \succeq \mathcal{I}^{u_2}$ for $u_1 > u_2$
- see the recent papers of Sznitman

Construction of $\mathcal{I}^u$ on a *finite* set $A \subset \mathbb{Z}^d$:

- $e_A(x) := P_x[\text{SRW escapes from } A]\mathbf{1}_A(x)$
- $\text{cap}(A) := \sum_{x \in A} e_A(x)$
- place $Poisson(ue_A(x))$ particles to x, independently for $x \in A$
- each particle performs a SRW
- (so that the total number of particles walking on A is $Poisson(u\,\text{cap}(A))$)

For example,

- $A = S_n = \{x \in \mathbb{Z}^d : \|x\| \leq n\}$
- $e_{S_n}(x) = O(n^{-1})$ for $x \in \partial S_n$
- total number of particles on ∂S_n is $Poisson(u \,\mathrm{cap}(S_n)) = O(un^{d-2})$
- observe that $P_x[\text{SRW hits } y] \simeq \|x - y\|^{-(d-2)}$
- so, we have "just enough" particles (i.e., $0 < \mathbb{P}^u[0 \in \mathcal{I}^u_{S_n}] < 1$ uniformly)

In fact, on the previous page we have the *exact* definition of $\mathcal{I}^u$ on any given finite set (i.e., no need to take the limit $n \to \infty$ here)!

In particular, $\mathbb{P}^u[0 \notin \mathcal{I}^u_{S_n}] = \exp(-\frac{u}{g(0,0)})$

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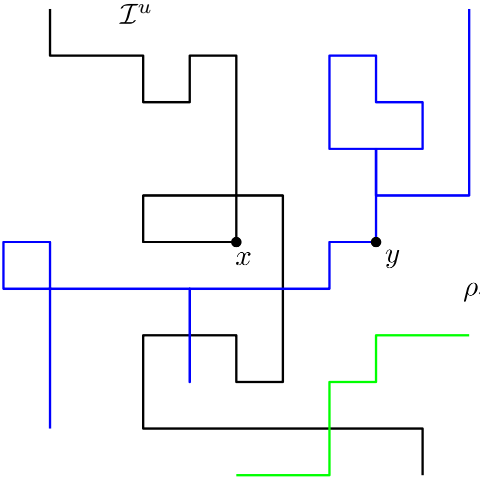

$$\rho_u(x, y) = 7$$

- let $\mathbb{P}_0^u = \mathbb{P}[\,\cdot \mid 0 \in \mathcal{I}^u]$ be the conditional law given that $0 \in \mathcal{I}^u$
- for $x, y \in \mathcal{I}^u$ we define $\rho_u(x, y)$ to be the internal distance between x and y within the interlacement set $\mathcal{I}^u$
- let $\Lambda^u(n) = \{y \in \mathcal{I}^u : \rho^u(0, y) \leq n\}$ be the ball of radius n in the internal distance

Theorem

For every $u > 0$ and $d \geq 3$ there exists $D_u \subset \mathbb{R}^d$ such that for any $\varepsilon > 0$

$$\big((1-\varepsilon)nD_u \cap \mathcal{I}^u\big) \subset \Lambda^u(n) \subset (1+\varepsilon)nD_u$$

eventually.

- the set D_u is symmetric under rotations and reflections of $\mathbb{Z}^d$
- $D_u \subset \{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}$ for all u
- it is straightforward to show that $D_u \to \{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}$ as $u \to \infty$
- it would be interesting, however, to be able to say something about the behaviour of D_u when $u \to 0$ (e.g., does the shape become close to the Euclidean ball, and what can be said about the size of D_u as $u \to 0$?)

Main tool: we prove that, for large enough C

$$\mathbb{P}_0^u[\text{for all } x, y \in S_n \cap \mathcal{I}^u, \rho^u(x, y) > Cn^2] < e^{-n^\delta}$$

(in fact, this also implies that $\mathcal{I}^u$ is connected *simultaneously* for all u)

Sketch of the proof (for $d = 4$):