General many-dimensional excited random walks

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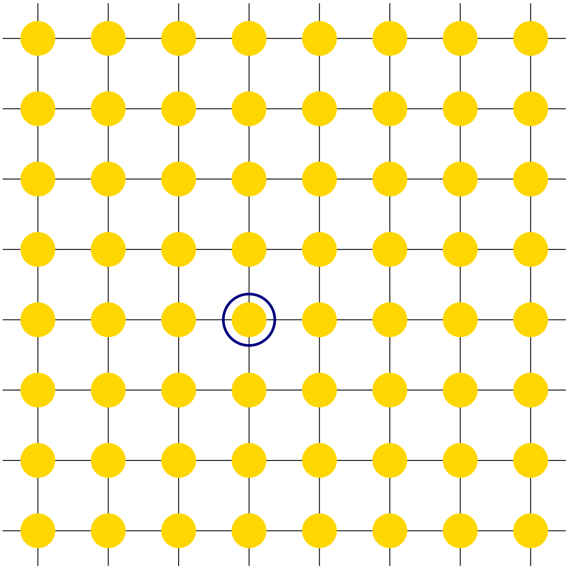
Excited Random Walk (ERW) was introduced by Benjamini and Wilson (2003).

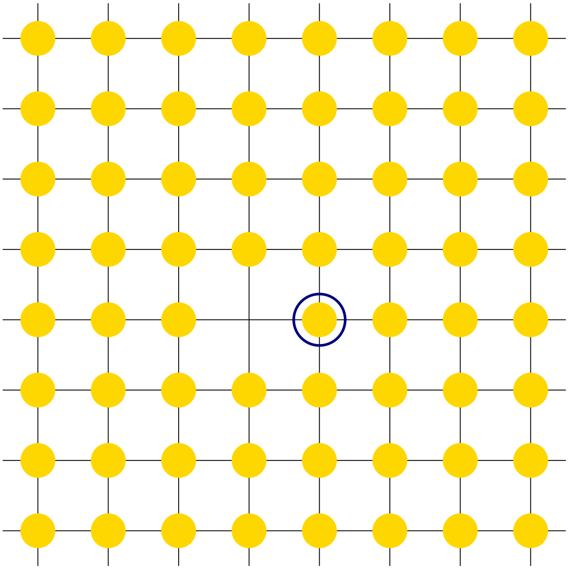
It is a discrete-time process that lives in $\mathbb{Z}^d$, and can be informally described as follows:

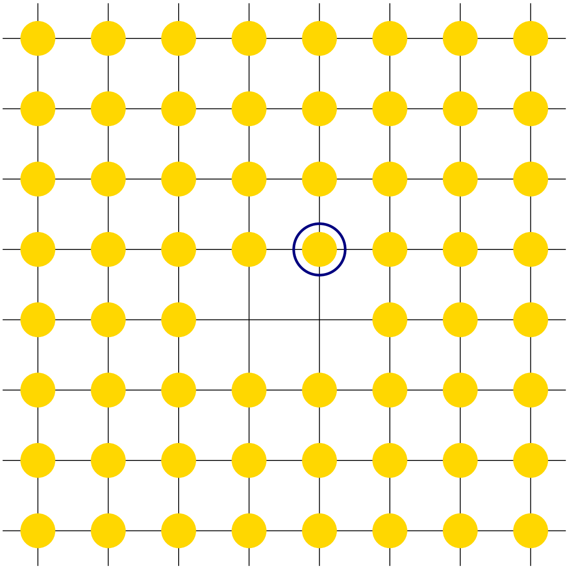
- fix a parameter $p \in (\frac{1}{2}, 1]$
- if the walk is at a site x which was already visited, it jumps with probabilities $1/(2d)$ to the nearest neighbor sites of x
- if the process visits a site x for the first time, it jumps to the right (i.e., in the direction of the first coordinate vector e_1) with probability p/d, to the left with probability $(1-p)/d$ and to the other nearest neighbor sites of x with probability $1/(2d)$.

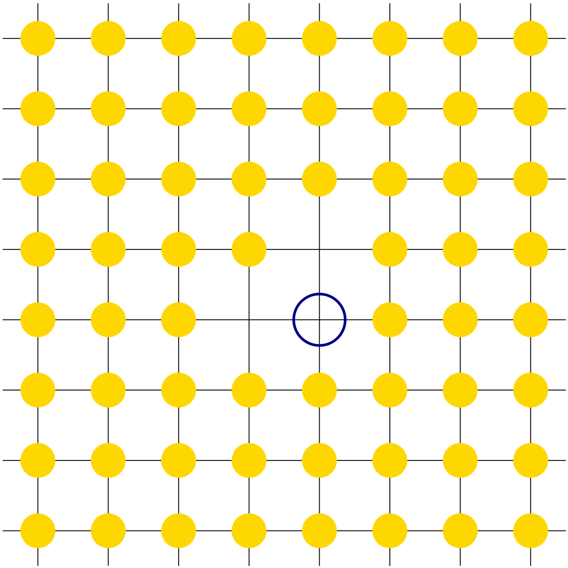
Informal interpretation:

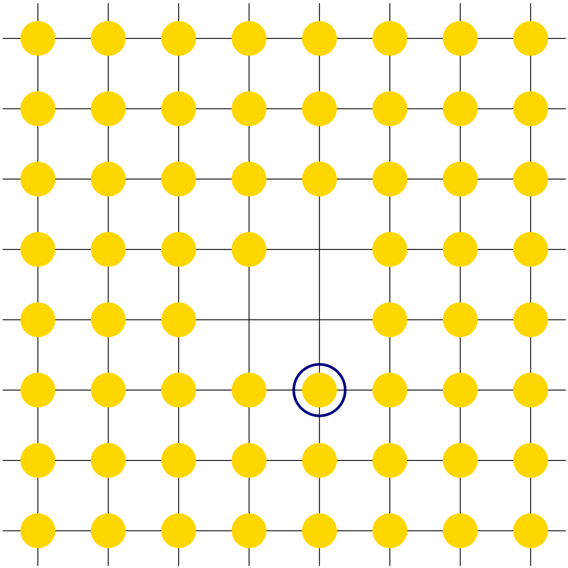
- initially each site contains one cookie
- the particle eats all cookies it finds
- immediately after eating a cookie, the particle gets a "bias" to the right
- no cookie $=$ no bias

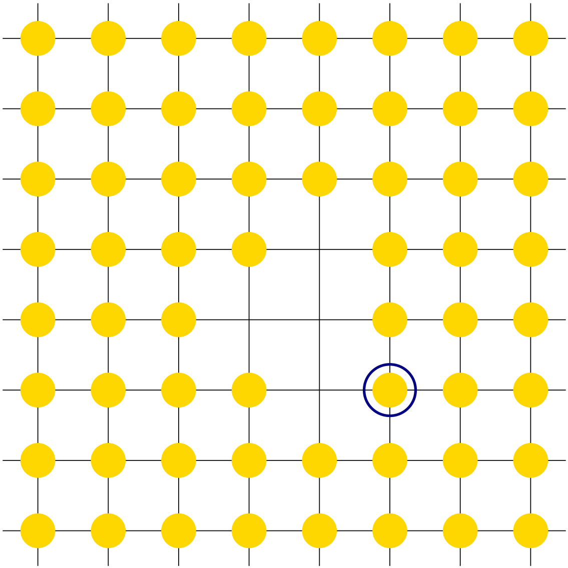


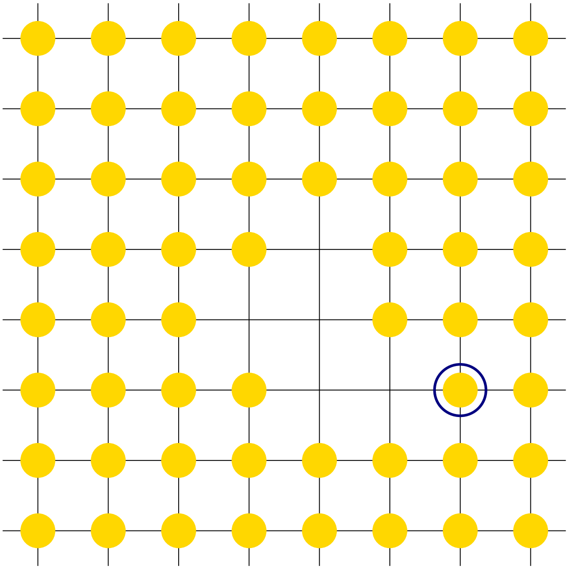


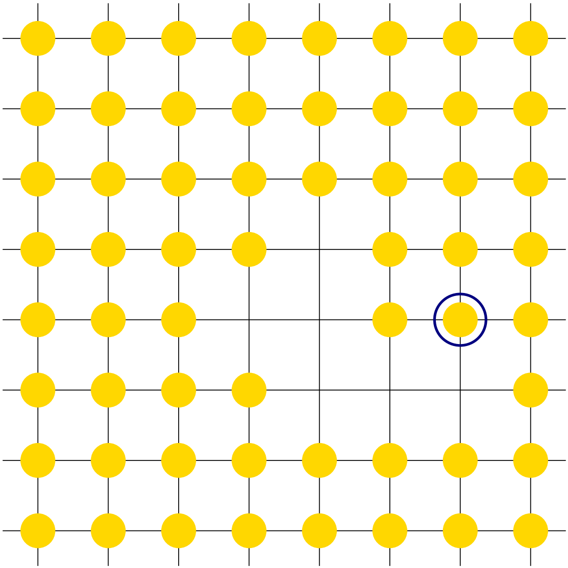


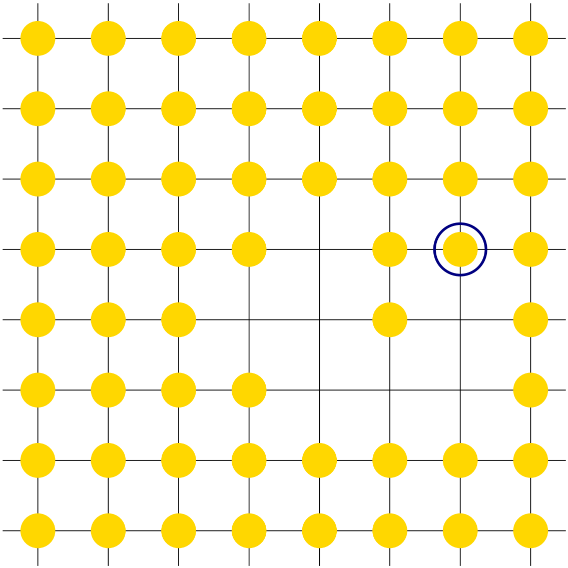


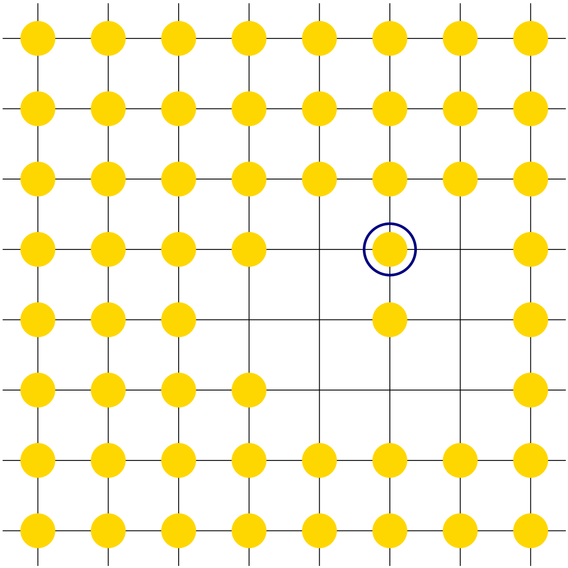


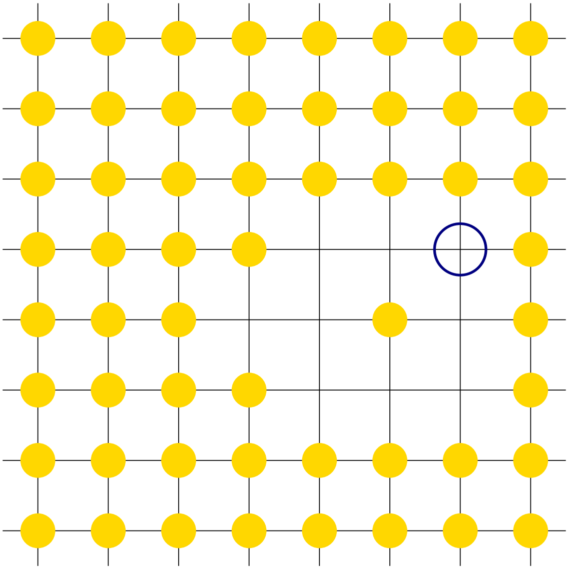


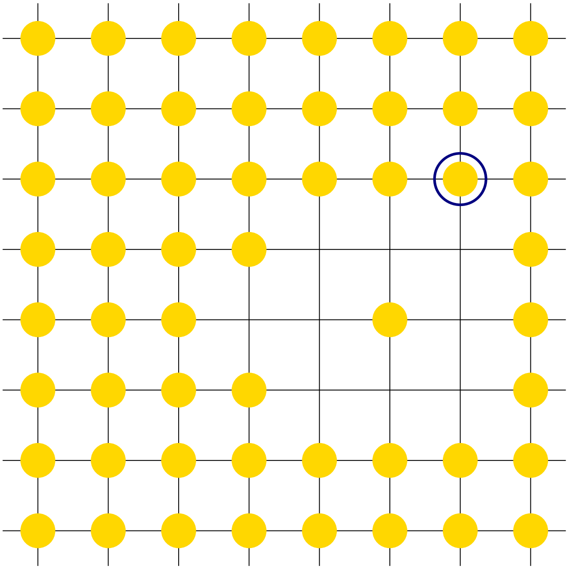


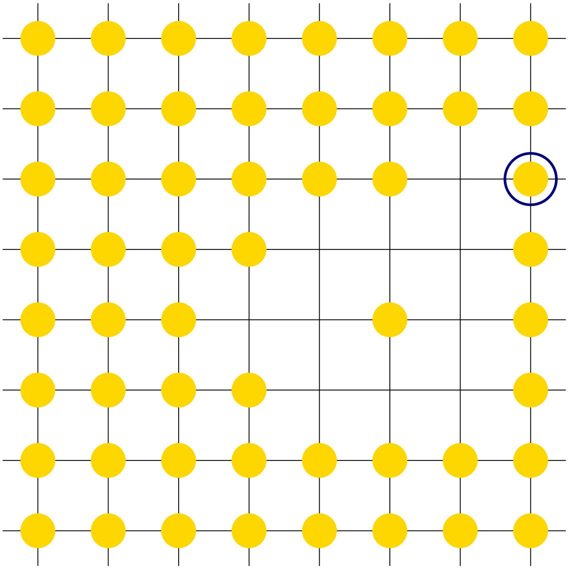












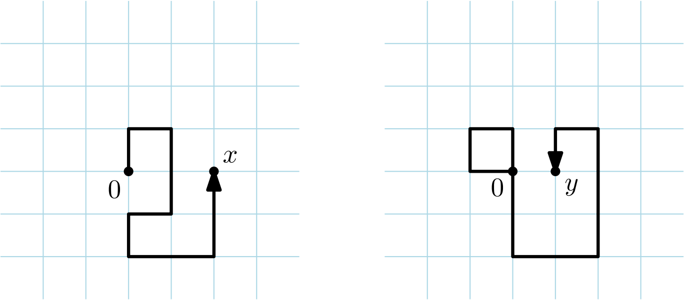
Known results (Benjamini, Wilson, Kozma, Bérard, Ramirez, van ver Hofstad, Holmes), case $d \geq 2$:

- ERW is transient to the right
- ERW is ballistic to the right
- LLN
- CLT
- monotonicity in p in high dimensions

Proof (main ideas, $d = 2$):

- coupling of ERW X with SRW Y, such that $(X_n - Y_n) \cdot e_1$ is nondecreasing in n and $(X_n - Y_n) \cdot e_2 = 0$
- tan points for SRW.

A tan point in dimension $d = 2$ is defined as any site $x \in \mathbb{Z}^2$ with the property that the ray $\{x + ke_1 : k \geq 0\}$ is visited by the SRW for the first time at site x.



x is a tan point, y is not a tan point

It is known (Bousquet-Mélou, Schaeffer (2002)) that with "large" probability, the number of tan points up to time n is at least $n^{\frac{3}{4}-\varepsilon}$.

Let $\mathcal{R}_n$ be the set of sites visited up to time n.

$|\mathcal{R}_n| \geq$ the number of tan points of Y by time n
(using the coupling with SRW)

So (taking $\varepsilon < \frac{1}{4}$)

$$|\mathcal{R}_n| > n^{\frac{3}{4}-\varepsilon} \gg n^{\frac{1}{2}}$$

with "large" probability.

Proof of transience:

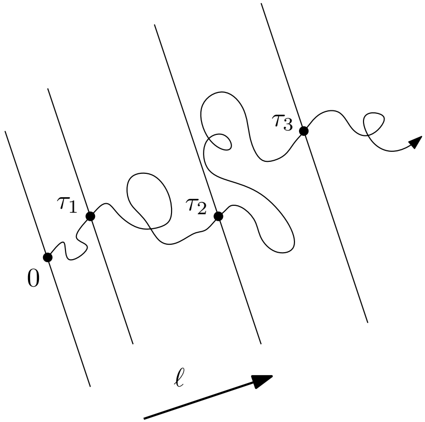
$M_n = X_n - \frac{2p-1}{2}|\mathcal{R}_n|$ is a martingale.

Azuma inequality: if $\{Z_n\}_{n\in\mathbb{N}}$ is a martingale with respect to some filtration, and such that $|Z_k - Z_{k-1}| < c$ a.s., then

$$\mathbb{P}[|Z_n - Z_0| \geq a] \leq 2\exp\Big(-\frac{a^2}{2nc^2}\Big).$$

So (take $a = n^{\frac{1}{2}+\delta}$), since we should have $|\mathcal{R}_n| \gg n^{\frac{1}{2}}$, it holds also that (again, with "large" probability) $X_n \gg n^{\frac{1}{2}}$, and then one can use Borel-Cantelli to obtain transience to the right.

Proofs of LLN and CLT (here $\ell = e_1$):



regeneration structure + estimates on tails of $\tau_{k+1} - \tau_k$

What if we modify the model?

- drift in cookies not parallel to e_1
- different drifts in different cookies
- SRW $\to$ some RW with zero drift and bounded jumps
- etc.

— there are difficulties, because we cannot use the coupling with SRW and tan points!

"Simple" ERW in dimension 2

Generalized ERW

Proofs

Generalized ERW is a discrete-time process X in $\mathbb{Z}^d$, $d \geq 2$, satisfying the following conditions:

Condition B. There exists a constant $K > 0$ such that $\sup_{n\geq 0} \|X_{n+1} - X_n\| \leq K$ a.s.

Condition E. Let $\ell \in \mathbb{S}^{d-1}$. We say that Condition E is satisfied with respect to ℓ if there exist $h, r > 0$ such that for all n

$$\mathbb{P}[(X_{n+1} - X_n) \cdot \ell > r \mid \mathcal{F}_n] \geq h$$

and for all ℓ' with $\|\ell'\| = 1$, on $\{\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0\}$

$$\mathbb{P}[(X_{n+1} - X_n) \cdot \ell' > r \mid \mathcal{F}_n] \geq h.$$

Condition C$^+$. Let $\ell \in \mathbb{S}^{d-1}$. We say that Condition C$^+$ is satisfied with respect to ℓ if there exist a $\lambda > 0$ such that

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) = 0 \text{ on } \{\exists\, k < n \text{ such that } X_k = X_n\},$$

and

$$\mathbb{E}(X_{n+1} - X_n \mid \mathcal{F}_n) \cdot \ell \geq \lambda \text{ on } \{X_k \neq X_n \text{ for all } k < n\}.$$

Results:

Theorem

Let $d \geq 2$ and $\ell \in \mathbb{S}^{d-1}$. Assume that X is a generalized excited random walk in direction ℓ. Then, there exists $v = v(d, K, h, r, \lambda) > 0$ such that

$$\liminf_{n \to \infty} \frac{X_n \cdot \ell}{n} \geq v \qquad a.s.$$

Also, for "homogeneous" ERW and ERW in i.i.d. random environment we prove LLN and (averaged) CLT.
They follow from the regeneration times argument, using the estimates obtained in the course of the proof of the above result.

"Simple" ERW in dimension 2

Generalized ERW

Proofs

Define

$$H(a,b) = \{x \in \mathbb{Z}^d : x \cdot \ell \in [a,b]\},$$

and

$$L_n(m) := \sum_{j=0}^{n} \mathbf{1}\{X_j \cdot \ell \in [m, m+1)\}.$$

Lemma

Let X' be a submartingale in direction ℓ with uniformly bounded jumps and uniform ellipticity. Then, for any $\delta > 0$ there exists a constant γ_1' such that for all m we have

$$\mathbb{P}[L_n(m) \geq n^{\frac{1}{2}+2\delta}] \leq e^{-\gamma_1' n^{\delta}}.$$

Proof (for $m = -1$):

- Optional Stopping Theorem $\Rightarrow$ with probability $\geq n^{-\frac{1}{2}-\delta}$, X' hits $H(n^{\frac{1}{2}+\delta}, +\infty)$ before coming back to $H(-\infty, 0)$
- Azuma inequality $\Rightarrow$ no return to $H(-\infty, 0)$ after n additional steps
- so, $X'_k \cdot \ell > 0$ for all $k \leq n$ with probability at least $O(n^{-\frac{1}{2}-\delta})$
- analogously, each return to $H(-1, 0)$ will be the last one up to time n with probability at least $O(n^{-\frac{1}{2}-\delta})$
- thus, no more than $n^{\frac{1}{2}+2\delta}$ returns with large probability.

Lemma

Let Y be a d-dimensional martingale, with uniformly bounded jumps and uniform ellipticity. Then, there exist $\gamma > 0, b \in (0,1)$ such that, by time m, Y will visit at least $m^{1-\frac{b}{2}}$ different sites with probability $\geq 1 - e^{-m^\gamma}$.

Key fact ($d \geq 2$): there exist $b \in (0,1)$ close enough to 1 and $\gamma'_2 > 0$ (depending only on K, h, r) such that

$$\mathbb{E}(\|Y_{n+1}\|^b \mid \mathcal{F}_n) \geq \|Y_n\|^b \mathbf{1}\{\|Y_n\| > \gamma'_2\}.$$

Proof:

- Optional Stopping Theorem $\Rightarrow$ starting from 0, Y will reach N (without coming back to 0) with probability $\geq N^{-b}$
- Azuma inequality $\Rightarrow$ no return to 0 after additional N^2 jumps
- then (use $N = m^{1/2}$), the probability of not returning to 0 after m steps is at least $m^{-b/2}$
- so, up to time m no more than $m^{b/2}$ visits to 0 (and to any other point)
- thus, have to visit $\geq m^{1-b/2}$ different sites.

Recall that $\mathcal{R}_n$ is the set of sites visited up to time n.

Proposition

There exist positive constants $\alpha, \gamma_1, \gamma_2$ *which depend only on* d, K, h, r*, such that*

$$\mathbb{P}[|\mathcal{R}_n| < n^{\frac{1}{2}+\alpha}] < e^{-\gamma_1 n^{\gamma_2}}$$

for all $n \geq 1$*.*

Proof: fix $a \in (0, \frac{1}{2})$ and $\varepsilon > 0$ in such a way that

$$(1 - a + \varepsilon) \wedge \left(\frac{1}{2} + \frac{a}{2}(1 - b) - 4\varepsilon\right) > \frac{1}{2}.$$

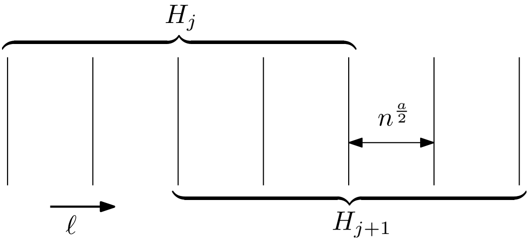
Consider (for fixed n) the event

$$G = \left\{|\mathcal{R}_n| \geq \frac{1}{2} n^{(1-a+\varepsilon)\wedge(\frac{1}{2}+\frac{a}{2}(1-b)-4\varepsilon)}\right\}.$$

We need to prove that

$$\mathbb{P}[G] \geq 1 - e^{-C_1 n^{\varepsilon/2}}. \tag{1}$$

Let $H_j := H\big(2(j-1)n^{\frac{a}{2}}, 2(j+1)n^{\frac{a}{2}}\big)$.



The strip H_j is a *trap* if $|\mathcal{R}_n \cap H_j| \geq n^{a(1-\frac{b}{2})-2\varepsilon}$.

If x is not in a trap, then the particle will hit al least one "new point" in time $n^{a-\varepsilon}$ with probability $\geq 1 - e^{-n^{\gamma''}}$ (when the particle walks on previously visited sites, it has zero drift).

Let us introduce the event

$$G_1 = \{L_n(k) \le n^{\frac{1}{2}+\varepsilon} \text{ for all } k \in [-Kn, Kn]\}.$$

Let $\hat{L}_j$ be the total number of visits to H_j.

On $\{|\mathcal{R}_n| < n^{\frac{1}{2}+\frac{a}{2}(1-b)-4\varepsilon}\}$ the number of traps is at most $2n^{\frac{1}{2}-\frac{a}{2}-2\varepsilon}$.

On the event G_1, we can write

$$\sum_j \hat{L}_j \mathbf{1}\{H_j \text{ is a trap}\} \le 4n^{\frac{a}{2}} \times 2n^{\frac{1}{2}-\frac{a}{2}-2\varepsilon} \times n^{\frac{1}{2}+\varepsilon} = 8n^{1-\varepsilon} < \frac{n}{2},$$

so the total time spent in non-traps is at least $\frac{n}{2}$.

Denote $\sigma_0 := 0$, and, inductively ($B(x, r)$ is a ball with center x and radius r)

$$\sigma_{k+1} = \min\left\{ j \geq \sigma_k + \lfloor n^{a-\varepsilon} \rfloor : |\mathcal{R}_j \cap B(X_j, n^{a/2})| \leq n^{a(1-\frac{b}{2})-2\varepsilon} \right\}.$$

Consider the event

$$G_2 = \Big\{ \text{at least one new point is hit on each} \\ \text{of the time intervals } [\sigma_{j-1}, \sigma_j), j = 1, \dots, \frac{1}{2} n^{1-a+\varepsilon} \Big\}.$$

The total time spent in non-traps is at least $\frac{n}{2}$.

On the other hand, up to the moment σ_k we can have at most $kn^{a-\varepsilon}$ instances j such that $|\mathcal{R}_j \cap B(X_j, n^{a/2})| \leq n^{a(1-\frac{b}{2})-2\varepsilon}$.

So, on the event

$$\Big\{ \sum_j \hat{L}_j \mathbf{1}\{H_j \text{ is a trap}\} \leq 8n^{1-\varepsilon} \Big\}$$

we have that $\sigma_{\frac{1}{2}n^{1-a+\varepsilon}} < n$.

But then, on the event G_2 we have that $|\mathcal{R}_n| \geq \frac{1}{2}n^{(1-a+\varepsilon)}$.

This means that $(G_1 \cap G_2) \subset G$, and (1) follows.

Then, after obtaining the last proposition, we can

- observe that $\ell \cdot \sum_{k=0}^{n-1} \mathbb{E}(X_{k+1} - X_k \mid \mathcal{F}_k) \geq \lambda |\mathcal{R}_n|$
- apply the Azuma inequality to the martingale

$$Z_n = X_n - \sum_{k=0}^{n-1} \mathbb{E}(X_{k+1} - X_k \mid \mathcal{F}_k)$$

  to obtain displacement estimates (and transience, with Borel-Cantelli)
- use regenerations times to obtain the ballisticity (and, eventually, LLN+CLT).