On uniform closeness of local times of Markov chains and i.i.d. sequences

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Outline

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Motivating example

Let X be a Markov chain on the state space $\Sigma = \{0, 1\}$, with transition probabilities:

$\mathbb{P}[X_{n+1} = k \mid X_n = k] = 1 - \mathbb{P}[X_{n+1} = 1 - k \mid X_n = k] = \frac{1}{2}(1 + \varepsilon)$ for $k = 0, 1$, where $\varepsilon > 0$ is small.

By symmetry, $\pi = (\frac{1}{2}, \frac{1}{2})$ is the stationary distribution of this Markov chain.

Next, let Y be a sequence of i.i.d. Bernoulli random variables with success probability $\frac{1}{2}$.

Introductory example

Consider in this case,

$$L_n^Z(0) = \sum_{j=1}^{n} \mathbf{1}_{\{Z_j=0\}}$$

for $Z = X$ or Y. What can we say about $d_{TV}(L_n^X(0), L_n^Y(0))$?

Introductory example

Well, the random variable $L_n^Y(0)$ has the Binomial distribution with parameters n and $\frac{1}{2}$, so it is approximately Normal with mean $\frac{n}{2}$ and standard deviation $\frac{\sqrt{n}}{2}$.

As for $L_n^X(0)$, it is approximately Normal with mean $\frac{n}{2}$ and standard deviation $\sqrt{n}\big(\frac{1}{2} + O(\varepsilon)\big)$.

It is not difficult obtain that the total variation distance between these two Normals is $O(\varepsilon)$, uniformly in n.

This *suggests* that the total variation distance between $L_n^X(0)$ and $L_n^Y(0)$ should be also of order ε uniformly in n.

Why bother with local times?

Many quantities of interest that can be expressed in terms of local times only, such as,

- hitting time of a site x: $\tau(x) = \min\{n : L_n(x) > 0\}$;
- cover time: $\min\{n : L_n(x) > 0 \text{ for all } x \in \Sigma\}$;
- blanket time : $\min\{n \geq 1 : L_n(x) \geq \delta n \pi(x)\}$, where $\delta \in (0, 1)$;
- disconnection time;
- the set of favorite (most visited) sites;
- and so on...

Outline

Main results

Let (Σ, d) be a compact metric space, with $\mathcal{B}(\Sigma)$ representing its Borel σ-algebra.

Assumption (A1)

We assume that (Σ, d) is of polynomial class: there exist some $\beta \geq 0$ and $\varphi \geq 1$ such that for all $r \in (0, 1]$, the number of open balls of radius r needed to cover Σ is smaller than or equal to $\varphi r^{-\beta}$.

Example: Σ finite, endowed with the discrete metric

$$d(x, y) = \mathbf{1}_{\{x \neq y\}}, \text{ for } x, y \in \Sigma.$$

We can choose $\beta = 0$ and $\varphi = |\Sigma|$.

Main results

Consider a Markov chain $X = (X_i)_{i\geq 1}$ with transition kernel $\mathfrak{P}(x, dy)$ on $(\Sigma, \mathcal{B}(\Sigma))$, and unique invariant probability measure π such that $\mathfrak{P}(x, \cdot) \ll \pi(\cdot)$ for all $x \in \Sigma$.

Denote by $p(x, \cdot)$ the *density* of $\mathfrak{P}(x, \cdot)$ with respect to π: for $x \in \Sigma$,

$$\mathfrak{P}(x, A) = \int_A p(x, y)\pi(dy), \text{ for all } A \in \mathcal{B}(\Sigma).$$

We also consider

Assumption (A2)

Assume that the density $p(x, \cdot)$ is uniformly Hölder continuous, that is, there exist constants $\kappa > 0$ and $\gamma \in (0, 1]$ such that for all $x, z, z' \in \Sigma$,

$$|p(x, z) - p(x, z')| \leq \kappa d^{\gamma}(z, z').$$

Main results

We assume that the chain X starts with some probability law absolutely continuous with respect to π and we denote by ν its density.

Assumption (A3)

There exists $\varepsilon \in (0, 1/2)$ *such that*

$$\sup_{x,y\in\Sigma} |p(x,y) - 1| \vee \sup_{x\in\Sigma} |\nu(x) - 1| \leq \varepsilon.$$

Main results

Let us denote also by $Y = (Y_n)_{n \geq 1}$ a sequence of i.i.d. random variable with law π.

We have the following

Theorem 1

Under Assumptions 1-3, there exists a universal constant $K > 0$ such that, for all $n \geq 1$, it holds that

$$d_{\mathrm{TV}}(L_n^X, L_n^Y) \leq K\varepsilon \sqrt{1 + \ln(\varphi 2^\beta) + \frac{\beta}{\gamma} \ln\left(\frac{\kappa \vee (2\varepsilon)}{\varepsilon}\right)}.$$

Main results

Case of a finite state space Σ, endowed with the discrete metric:

Choosing $\beta = 0$ and $\varphi = |\Sigma|$, Theorem 1 boils down to

$$\mathrm{d}_{\mathrm{TV}}(L_n^X, L_n^Y) \leq K\varepsilon\sqrt{1 + \ln|\Sigma|},$$

for all $n \geq 1$.

Recall motivating example!

Main results

We also have

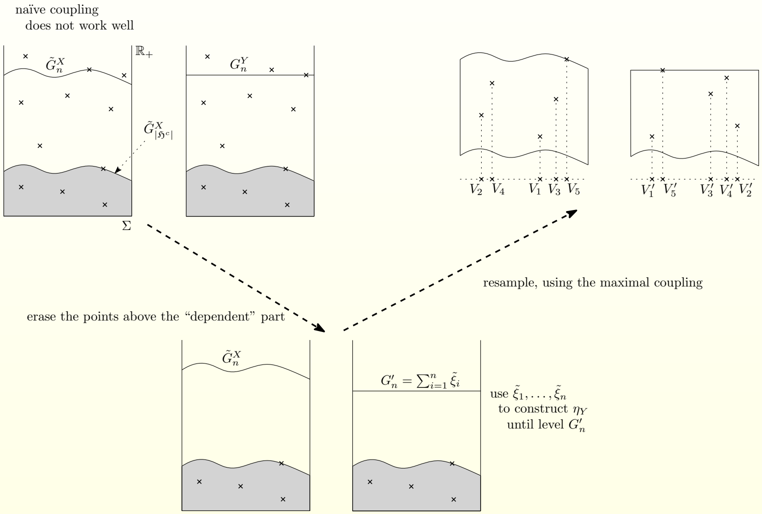
Theorem 2

Under Assumptions 1-3, there exists a positive constant $K' = K'(\beta, \varphi, \kappa, \gamma, \varepsilon)$, such that, for all $n \geq 1$, it holds that

$$\mathrm{d_{TV}}(L_n^X, L_n^Y) \leq 1 - K'.$$

Outline

Idea of the proof: the soft local times method



Thanks!