A conditional quenched CLT for random walks among random conductances on $\mathbb{Z}^d$

Christophe Gallesco Nina Gantert Serguei Popov
Marina Vachkovskaia

The model:

- In $\mathbb{Z}^d$, to any unordered pair of neighbors attach a positive number $\omega_{x,y}$ (conductance between x and y).
- $\mathbb{P}$ stands for the law of this field of conductances. We assume that $\mathbb{P}$ is stationary and ergodic.
- Define $\pi_x = \sum_{y \sim x} \omega_{x,y}$, and let the transition probabilities be

$$q_\omega(x,y) = \begin{cases} \frac{\omega_{x,y}}{\pi_x}, & \text{if } y \sim x, \\ 0, & \text{otherwise,} \end{cases}$$

- P_ω^x is the quenched law of the random walk starting from x, so that

$$P_\omega^x[X(0) = x] = 1, \quad P_\omega^x[X(k+1) = z \mid X(k) = y] = q_\omega(y,z).$$

(many recent papers) $\Longrightarrow$ under mild conditions on the law of ω-s, the Quenched Invariance Principle holds:

For almost every environment ω, suitably rescaled trajectories of the random walk converge to the Brownian Motion (with nonrandom diffusion constant σ) in a suitable sense.

Main method of the proof: the "corrector approach", i.e., find a "stationary deformation" of the lattice such that the random walk becomes martingale.

The corrector is shown to exist, but usually no explicit formula is known for it.

Brownian Meander:

Let W be the Brownian Motion starting from 0, and define $\tau_1 = \sup\{s \in [0,1] : W(s) = 0\}$ and $\Delta_1 = 1 - \tau_1$.

Then, the Brownian Meander W^+ is defined in this way:

$$W^+(s) := \Delta_1^{-1/2} |W_1(\tau_1 + s\Delta_1)|, \qquad 0 \le s \le 1.$$

Informally, the Brownian Meander is the Brownian Motion conditioned on staying positive on the time interval $(0,1]$.

Example: simple random walk S, conditioned on $\{S_1 > 0, \dots, S_n > 0\}$, after usual scaling converges to the Brownian Meander.

Let

$$\Lambda_n := \{X_1(k) > 0 \text{ for all } k = 1, \ldots, n\}$$

(X_1 is the first coordinate of X).

Consider the conditional quenched probability measure $Q^n_\omega[\,\cdot\,] := P_\omega[\,\cdot \mid \Lambda_n]$.

Define the continuous map $Z^n(t), t \in [0, 1])$ as the natural polygonal interpolation of the map $k/n \mapsto \sigma^{-1} n^{-1/2} X(k)$ (with σ from the quenched CLT).

For each n, the random map Z^n induces a probability measure μ^n_ω on $(C[0, 1], \mathcal{B}_1)$: for any $A \in \mathcal{B}_1$,

$$\mu^n_\omega(A) := Q^n_\omega[Z^n \in A].$$

Condition E. There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $\kappa < \omega_{0,x} < \kappa^{-1}$ for $x \sim 0$.

Denote by $P_{W^+} \otimes P_{W^{(d-1)}}$ the product law of Brownian meander and $(d-1)$-dimensional standard Brownian motion on the time interval $[0,1]$.

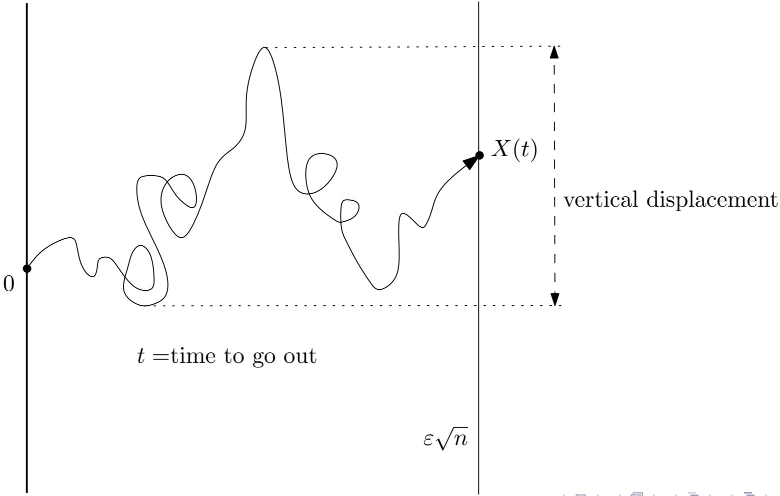
Now, we formulate our main result:

Theorem

Under Condition E, we have that, $\mathbb{P}$-a.s., μ_ω^n (after suitable linear transformation) tends weakly to $P_{W^+} \otimes P_{W^{(d-1)}}$ as $n \to \infty$ (as probability measures on $C[0,1]$).

Strategy of the proof: “go avay a little bit from the forbidden area in a controlled way”

(we need to control the time and the vertical displacement), and then use unconditional CLT.



Open questions:

- other types of conditioning;
- $P_\omega[\Lambda_n] \simeq ?$ (at least prove it is of order $n^{-1/2}$).