Conditional quenched CLTs for random walks among random conductances

Christophe Gallesco Nina Gantert Serguei Popov
Marina Vachkovskaia

One-dimensional random walks with unbounded jumps

Many-dimensional random walks (nearest-neighbor and uniformly elliptic)

Initial motivation: gas of particles in a finite random tube (Comets, Popov, Schütz, Vachkovskaia, JSP–2010):

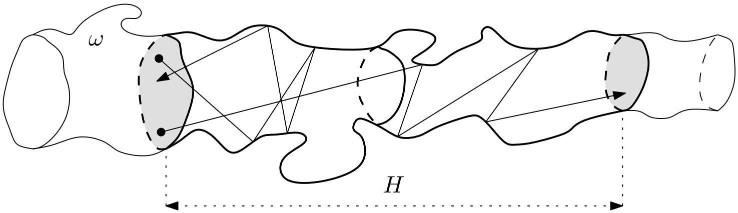


Figure: Particles are injected at the left boundary, and killed at both boundaries

Technical difficulty: prove that $P_\omega[\text{time} \le \varepsilon H^2 \mid \text{cross the tube}]$ is small.
This would be a concequence of a *conditional* CLT!

The model:

- in $\mathbb{Z}$, to any pair (x, y) attach a positive number $\omega_{x,y}$ (conductance between x and y).
- $\mathbb{P}$ stands for the law of this field of conductances. We assume that $\mathbb{P}$ is stationary and ergodic.
- define $\pi_x = \sum_y \omega_{x,y}$, and let the transition probabilities be

$$q_\omega(x, y) = \frac{\omega_{x,y}}{\pi_x}$$

- P^x_ω is the quenched law of the random walk starting from x, so that

$$P^x_\omega[X(0) = x] = 1, \quad P^x_\omega[X(k+1) = z \mid X(k) = y] = q_\omega(y, z).$$

We assume “local uniform ellipticity” and polynomial tails of jumps:

Condition E.

(i) There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $q_\omega(0, \pm 1) \geq \kappa$.

(ii) Also, there exists $\hat{\kappa} > 0$ such that $\hat{\kappa} \leq \sum_{y \in \mathbb{Z}} \omega_{0,y} \leq \hat{\kappa}^{-1}$, $\mathbb{P}$-a.s.

Condition K. There exist constants $K, \beta > 0$ such that $\mathbb{P}$-a.s., $\omega_{0,y} \leq K|y|^{-(3+\beta)}$, for all $y \in \mathbb{Z} \setminus \{0\}$.

(observe that this implies that the second moment of the jump is uniformly bounded)

Brownian Meander:

Let W be the Brownian Motion starting from 0, and define $\tau_1 = \sup\{s \in [0,1] : W(s) = 0\}$ and $\Delta_1 = 1 - \tau_1$.

Then, the Brownian Meander W^+ is defined in this way:

$$W^+(s) := \Delta_1^{-1/2} |W_1(\tau_1 + s\Delta_1)|, \qquad 0 \le s \le 1.$$

Informally, the Brownian Meander is the Brownian Motion conditioned on staying positive on the time interval $(0, 1]$.

Example: simple random walk S, conditioned on $\{S_1 > 0, \dots, S_n > 0\}$, after usual scaling converges to the Brownian Meander.

Let

$$\Lambda_n := \{X(k) > 0 \text{ for all } k = 1, \dots, n\}$$

Consider the conditional quenched probability measure $Q^n_\omega[\,\cdot\,] := P_\omega[\,\cdot \mid \Lambda_n]$.

Define the continuous map $Z^n(t), t \in [0,1])$ as the natural polygonal interpolation of the map $k/n \mapsto \sigma^{-1} n^{-1/2} X(k)$ (with σ from the quenched CLT).

For each n, the random map Z^n induces a probability measure μ^n_ω on $(C[0,1], \mathcal{B}_1)$: for any $A \in \mathcal{B}_1$,

$$\mu^n_\omega(A) := Q^n_\omega[Z^n \in A].$$

Main result:

Theorem

Under Conditions E and K, we have that, $\mathbb{P}$-a.s., μ_ω^n tends weakly to P_{W^+} as $n \to \infty$, where P_{W^+} is the law of the Brownian meander W^+ on $C[0,1]$.

As a corollary of Theorem 1.1, we obtain a limit theorem for the process conditioned on crossing a large interval. Define

$$\hat{\tau}_n = \inf\{k \geq 0 : X_k \in [n, \infty)\} \quad \text{and} \quad \Lambda_n' = \{\hat{\tau}_n < \hat{\tau}\}.$$

Corollary

Assume Conditions E and K. Then, conditioned on Λ_n', the process converges to the "Brownian crossing".

- strategy of the proof: force the walk a bit away from the origin, and use the (unconditional) quenched invariance principle.
- in fact, one needs even the “uniform” version of the quenched invariance principle (i.e., at time t the rescaled RW is “close” to BM *uniformly* with respect to the starting point chosen in the interval of length $O(\sqrt{t})$ around the origin)
- the main difficulty: control the (both conditional and unconditional) exit measure from large intervals
- (observe that is ξ has only polynomial tail, then $\frac{P[x<\xi\leq x+a]}{P[\xi>x]} \to 0$ as $x \to \infty$)

One-dimensional random walks with unbounded jumps

Many-dimensional random walks (nearest-neighbor and uniformly elliptic)

The model:

- in $\mathbb{Z}^d$, to any unordered pair of neighbors attach a positive number $\omega_{x,y}$ (conductance between x and y).
- $\mathbb{P}$ stands for the law of this field of conductances. We assume that $\mathbb{P}$ is stationary and ergodic.
- define $\pi_x = \sum_{y \sim x} \omega_{x,y}$, and let the transition probabilities be

$$q_\omega(x,y) = \begin{cases} \frac{\omega_{x,y}}{\pi_x}, & \text{if } y \sim x, \\ 0, & \text{otherwise,} \end{cases}$$

- P_ω^x is the quenched law of the random walk starting from x, so that

$$P_\omega^x[X(0) = x] = 1, \quad P_\omega^x[X(k+1) = z \mid X(k) = y] = q_\omega(y,z).$$

(many recent papers) $\Longrightarrow$ under mild conditions on the law of ω-s, the Quenched Invariance Principle holds:

For almost every environment ω, suitably rescaled trajectories of the random walk converge to the Brownian Motion (with nonrandom diffusion constant σ) in a suitable sense.

Main method of the proof: the “corrector approach”, i.e., find a “stationary deformation” of the lattice such that the random walk becomes martingale.

The corrector is shown to exist, but usually no explicit formula is known for it.

Let

$$\Lambda_n := \{X_1(k) > 0 \text{ for all } k = 1, \dots, n\}$$

(X_1 is the first coordinate of X).

Consider the conditional quenched probability measure $Q^n_\omega[\,\cdot\,] := P_\omega[\,\cdot \mid \Lambda_n]$.

Define the continuous map $Z^n(t), t \in [0,1])$ as the natural polygonal interpolation of the map $k/n \mapsto \sigma^{-1} n^{-1/2} X(k)$ (with σ from the quenched CLT).

For each n, the random map Z^n induces a probability measure μ^n_ω on $(C[0,1], \mathcal{B}_1)$: for any $A \in \mathcal{B}_1$,

$$\mu^n_\omega(A) := Q^n_\omega[Z^n \in A].$$

Condition E'. There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $\kappa < \omega_{0,x} < \kappa^{-1}$ for $x \sim 0$.

Denote by $P_{W^+} \otimes P_{W^{(d-1)}}$ the product law of Brownian meander and $(d-1)$-dimensional standard Brownian motion on the time interval $[0, 1]$.

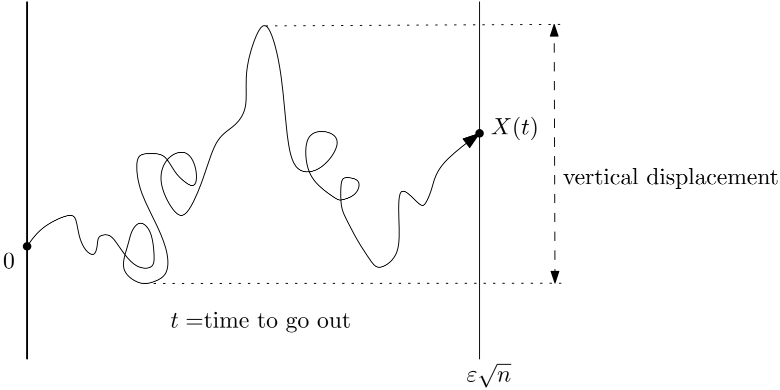
Now, we formulate our main result:

Theorem

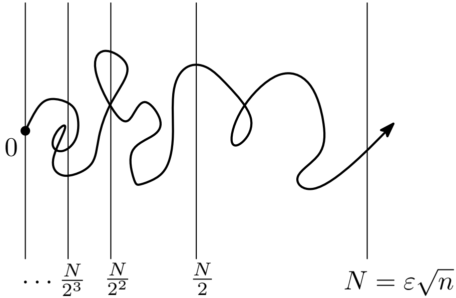
Under Condition E', we have that, $\mathbb{P}$-a.s., μ_ω^n tends weakly to $P_{W^+} \otimes P_{W^{(d-1)}}$ as $n \to \infty$ (as probability measures on $C[0, 1]$).

Strategy of the proof: “go avay a little bit from the forbidden area in a controlled way”

(we need to control the time and the vertical displacement), and then use unconditional CLT (in fact, again, the *uniform* version of the CLT makes life easier)



control of time:



$\alpha \in (\frac{1}{4}, 1)$

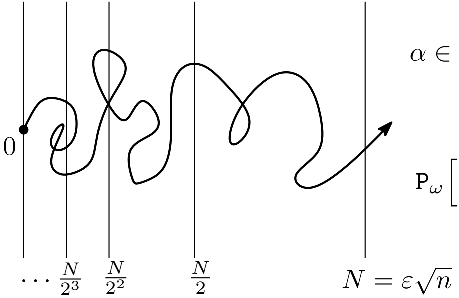
$\mathrm{P}_\omega[\tau_N > n \mid \Lambda_n] \approx \text{small}$

$$P_\omega[\tau_N > n \mid \Lambda_n] \leq P_\omega[\tau_{N/2} > \alpha n \mid \Lambda_n] + \text{something small},$$

then iterate:

$$P_\omega[\tau_{2^{-j}N} > \alpha^j n \mid \Lambda_n] \leq P_\omega[\tau_{2^{-(j+1)}N} > \alpha^{j+1} n \mid \Lambda_n] + \text{smth very small}$$

control of “vertical” displacement:



$\alpha \in (\frac{1}{2}, 1)$

$$\mathrm{P}_\omega\Big[\sup_{j\le\tau_N} |X_2(j)| > \varepsilon' N \mid \Lambda_n\Big] \approx \text{small}$$

$$G_k = \Big\{ \sup_{j\in(\tau_{2^{-k}N}, \tau_{2^{-k+1}N}]} |X_2(j) - X_2(\tau_{2^{-k}N})| \le \varepsilon'' \alpha^k N \Big\}$$

observe that, for G_k, $\frac{\text{vertical size}}{\text{horizontal size}} \simeq (2\alpha)^k$

Open questions:

- not uniformly bounded conductances, RWs on percolation clusters, ...?
- other types of conditioning?
- $P_\omega[\Lambda_n] \simeq ?$
- in particular, can one prove that $\frac{C_1}{n} \leq P_\omega[\text{cross the strip of width } n] \leq \frac{C_2}{n}$?