Non-homogeneous random walks

Lyapunov function methods for near-critical stochastic systems

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Preface

What is this book about?

This book has two main goals:

• to give an up-to-date exposition of the ‘semimartingale’ or ‘Lyapunov function’ approach to the analysis of stochastic processes;

• to present applications of the methodology to fundamental models (classical and modern) in probability theory and related fields.

Our expository bridge between these dual aims, between methods and models, is the $d$-dimensional non-homogeneous random walk, which as a model is simple to describe, closely resembling the classical homogeneous random walk, but displays many interesting and subtle phenomena alien to the classical model. Non-homogeneous random walks cannot be studied by the techniques generally used for homogeneous random walks: new methods (and, just as importantly, new intuitions) are required.

Semimartingale and Lyapunov function ideas lead to a unified and powerful methodology in this context. As well as non-homogeneous random walks, we present applications of the methods to several other models from modern probability theory; while any of the models that we discuss can by studied by several probabilistic techniques, we believe that only the Lyapunov function method has something to say about all of them.

We emphasize that semimartingale methods are ‘robust’ in the sense that the underlying stochastic process need not satisfy simplifying assumptions such as the Markov property, reversibility, or time-homogeneity, for instance, and the state-space of the process need not be countable. In such a general setting, the semimartingale approach has few rivals. In particular, the methods presented work for non-reversible Markov chains. A general feeling is that, if a Markov chain is reversible, then things can be done in many possible ways: there are methods from electrical networks, spectral
calculations, harmonic analysis, etc. On the other hand, the non-reversible case is usually much harder. Similarly, the Markovian setting is not essential to the methods. In the semimartingale approach, the Markov property is a side issue and non-Markovian processes can be treated equally well.

The Lyapunov function approach for analysis of Markov processes originated with classical work of Foster, and the theory has since expanded greatly and proved very successful in analysis of numerous Markov models. Aspects of Foster–Lyapunov theory are presented in the books [96, 6, 239] (the presentation in [96] being the closest to our perspective). However, almost all of these existing presentations are concerned with the situation in which the process under consideration is not too close to a phase boundary in terms of its asymptotic behaviour. This book deals with analysis of near-critical systems, which exhibit fundamental phase transitions such as that between recurrence and transience.

Near-critical systems are exactly that: even if transient, they are not ballistic; even if positive-recurrent, they do not exhibit geometric ergodicity; random quantities associated with the system typically have heavy (power-law) tails. Heavy tails have become increasingly prevalent in applications across many fields over the last few years, including queueing theory and finance; physicists associate heavy-tails with the presence of “self-organized criticality”, a very fashionable topic at the moment. Naturally, the analysis of near-critical systems is more challenging and delicate than that for systems that are far from criticality.

The fundamental contributions to the near-critical situation come from classical work of Lamperti, and almost none of this has previously appeared in any book, despite its age and importance. Lamperti’s basic problem concerned the asymptotic analysis of a stochastic process on the half-line with mean drift at $x$ of order $1/x$: this is exactly the critical situation in respect of the recurrence classification of the process. Importantly, Lamperti’s techniques are based on semimartingale ideas, and so the Markov property is not essential. Building on Lamperti’s ideas, over the last 15 years much work has appeared in academic journals on semimartingale methods and near-critical probabilistic systems, and this is the theory that we present in this book.

We want to emphasize the importance of applications of the theory. The Lyapunov function ideology often enables one to reduce a question about a complex model arising in applications to a question about a simpler one-dimensional model by considering a function of the original process whose image is one-dimensional. If the original model is interesting, in that its behaviour is near-critical in some sense, then the image process (for a suitably
chosen Lyapunov function) will be near-critical. To give a concrete example, the classical and fundamental model of symmetric simple random walk on $\mathbb{Z}^d$ ($X_n$, say) can be analysed through the Lyapunov function $f(x) = \|x\|$ (the Euclidean norm). Then $f(X_n)$ is a stochastic process on the half-line with mean drift of order $1/x$ at $x$, i.e., precisely a critical Lamperti-type process. Moreover, $f(X_n)$ is not a Markov process, demonstrating the importance of the generality of the semimartingale approach.

Much more complicated and non-classical examples can be studied by the same methods, and we present several such examples to give a flavour of the power and utility of the techniques. We mention, for example, models from queueing theory, interacting particle systems, random walks in random environments, and so on. As mentioned above, our canonical example will be the non-homogeneous random walk. This model is a natural generalization of the very classical and extremely well-studied homogeneous random walk, but whose analysis requires entirely new methods. The semimartingale approach gives a systematic and intuitive way to analyse these processes.

So, to summarize: This book is about the analysis of Markov processes (such as random walks) via the method of Lyapunov functions; a correctly applied Lyapunov function of a process gives rise to a semimartingale. Our terminology here is neither particularly standard nor particularly precise; we discuss briefly here our usage.

What is a random walk?

Many random walks are sums of i.i.d. random variables (or vectors); this usage is too narrow for us. Many random walks take place on graphs or groups; combinatorial or algebraic considerations are not the focus of this book. Our random walks are (usually) Markov chains, on state-spaces that are embedded in Euclidean space, with transitions that are in some sense local (so that it is natural to speak of ‘jumps’ or ‘increments’). These are the models that are best suited to the probabilistic approach of this book, and include broad classes of models of interest in applications, such as queueing theory or ecology.

This book is not just about random walks; we discuss other Markov processes, including interacting particle systems and a stochastic billiards model, for example, but random walks provide a rich set of models on which to demonstrate some aspects of the Lyapunov function method.
What is a Lyapunov function?

The phrase ‘Lyapunov function’ has a quite precise technical meaning in the theory of stability of differential equations. For us, it has a much looser meaning: a Lyapunov function earns the name if the image under the function of a stochastic process is a process satisfying some conditions that enable one to deduce some property of the original process. For instance, if \((\xi_n, n \geq 0)\) is a time-homogeneous Markov chain on \(\mathbb{Z}^d\), and \(f : \mathbb{R}^d \to \mathbb{R}\) is a judicious choice of function such that there is a set \(A \subset \mathbb{Z}^d\) for which

\[
E[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = x] \leq 0, \text{ for all } x \notin A,
\]  

then one can deduce that \(\xi_n\) is recurrent, or transient, if \(f\) and \(A\) satisfy certain simple conditions. Results of this kind, which have their origins in work of F.G. Foster in the 1950s, are known as Foster–Lyapunov conditions.

In all but the simplest cases, one cannot compute the left-hand side of (\(\star\)) exactly, but often one can estimate (\(\star\)) using Taylor’s formula and coarse properties of the increments \(\xi_{n+1} - \xi_n\); usually one or two moments suffice. Truncation arguments may be needed to control unusually large increments, where Taylor’s formula will break down.

The language and tools of stopping times and martingale theory are close at hand. If \(\tau = \min\{n \geq 0 : \xi_n \in A\}\) denotes the first hitting time of \(A\), the drift condition (\(\star\)) above can be interpreted as saying that \(f(\xi_{n \wedge \tau})\) is a supermartingale adapted to the natural filtration. Since recurrence and transience properties of \(\xi_n\) can be related to properties of the stopping time \(\tau\), it is natural to try to examine \(\tau\) using the technology of martingale theory, such as the optional stopping or martingale convergence theorems. This is the basic ideology of the semimartingale method. This approach came after Foster; fundamental work was done by J. Lamperti in the 1960s.

What is a semimartingale?

The phrase ‘semimartingale’ has a quite precise technical meaning in stochastic analysis, as well as being an obsolete term for submartingale. For us, it has a much looser meaning: a semimartingale is a process that satisfies some drift condition like (\(\star\)), typically only locally. Often, with the aid of a stopping time, one can convert such a process into a true supermartingale or submartingale, as in the case of (\(\star\)), but we do not make this demand on all our semimartingales.
Features of this book

As mentioned above, we use the terminology Lyapunov function in a general sense, and that in our usage the term does not presuppose any particular property (stability or otherwise) for the transformed process. When presented with a Foster–Lyapunov result demanding verification of a drift condition such as \((\star)\), one immediately is faced with the problem of how to choose the Lyapunov function \(f\). Usually there is no fixed rule about how to discover the ‘right’ Lyapunov function, which must somehow encapsulate one’s intuition about the process \(\xi_n\) under consideration. For this reason, in this book we not only present the ‘right’ answers, but also do our best to explain the intuition behind the choice of Lyapunov function in the context of diverse applications and examples.

Finding a suitable Lyapunov function is not always easy, and there is no exact algorithm for that. Nevertheless, there are several intuitive rules and non-rigorous ideas that may help; in the text, we emphasize this kind of heuristics in the following way:

> Just try a good-looking Lyapunov function, work hard to perform all the computations, and hope for the best.

We usually avoid adorning the main text with citations to the literature, and instead collect bibliographical notes at the end of each chapter. We have endeavoured to track down original references where possible, and have uncovered several important works of which we were previously unaware; we apologise in advance for any egregious omissions that remain.

Overview of content

The material is presented in logical order, but the book has several entry points for the reader. Chapter 1 serves as a gentle introduction to the main theme of the book (non-homogeneous random walks). Chapters 2 and 3 introduce the technical apparatus of the semimartingale approach and describe its application to near-critical processes on the half-line. Our intention is that these two chapters will serve as a useful reference for researchers who wish to use these tools, and so we have tried to give relatively strong versions of the results in some generality. As an antidote to the technical demands of these two chapters, we have included many examples, as our intention is also that Chapters 2 and 3 should prove instructive to the student who wishes to develop an intuition for the method. Chapters 4–6
present applications of the Lyapunov function method to some near-critical stochastic processes. Thus while these chapters frequently refer to results from Chapter 2 (and Chapter 4 also relies heavily on Chapter 3), our intention is that the reader who is so inclined can take the technical tools for granted and read each of these later chapters as a stand-alone exposition. The final chapter, Chapter 7, switches focus to continuous-time; while the development parallels some of the ideas in Chapter 2, this chapter is essentially self-contained.

The section headings in the table of contents provide an indication of the subject matter of each chapter; here we outline briefly what each chapter contains.

Chapter 1: Introduction.

This chapter motivates the developments that follow by way of a classical and fundamental model in probability theory: the $d$-dimensional random walk. We describe the transition from (classical) homogeneous random walk to spatially non-homogeneous random walks, and how the investigation of such models is motivated by theoretical questions arising from trying to go beyond the classical setting and to probe the recurrence/transience phase transition. Immediately the relaxation of spatial homogeneity requires a significant re-adjustment of random walk intuition: one can readily construct two-dimensional, zero-drift, bounded-jump random walks that are transient, for example, provided spatial homogeneity is not enforced, completely contrary to classical behaviour.

Chapter 2: Semimartingale approach and Markov chains.

This section presents the basic technical apparatus that we rely on for the rest of the book. We review some basic martingale ideas (Doob’s inequality, martingale convergence, and the optional stopping theorem) and present a variety of semimartingale tools, including maximal inequalities, results on finiteness of hitting times, and existence and non-existence of moments for hitting times. These results include Foster–Lyapunov criteria for Markov chains, whereby a suitable Lyapunov function enables one to conclude that the process is transient, recurrent, positive recurrent, etc. We provide many examples of the application of these results.
Chapter 3: Lamperti’s problem.

This chapter presents applications of the semimartingale tools of Chapter 2 in the context of one-dimensional adapted processes with asymptotically vanishing drift, the so-called Lamperti’s problem. Lamperti’s problem serves as a first important example of a near-critical stochastic process, and is also motivated by its ubiquity arising from the application of the Lyapunov function method to near-critical processes in higher dimensions. This chapter studies in turn various aspects of the asymptotic behaviour of Lamperti processes, including the recurrence classification for processes with asymptotically zero drifts, results on existence and non-existence of passage-time moments, Gamma-type weak convergence results, and almost-sure bounds on the trajectory of the process.

Chapter 4: Many-dimensional random walks.

This chapter presents applications of the results of Chapter 3 to many-dimensional random walks. The applications in this chapter (and later on) proceed by the Lyapunov function ideology: use a suitably chosen Lyapunov function of the many-dimensional Markov process to obtain a (probably non-Markov) stochastic process in one dimension, which fits into the framework of Chapter 3. We consider in detail the recurrence classification problem for non-homogeneous random walks, with emphasis on the possibility of anomalous recurrence behaviour. We also give results on angular asymptotics and on the range of many-dimensional martingales.

Chapter 5: Heavy tails.

The processes considered in Chapters 3 and 4 are all assumed to have at least one or two moments for their increments. This chapter turns to the heavy-tailed case when the first or second increment moment is infinite. We present results for real-valued Markov chains with heavy-tailed jumps, focusing on the recurrence classification; we demonstrate how the Lyapunov function approach is equally effective in this heavy-tailed setting.

Chapter 6: Further applications.

This chapter presents a selection of applications of the Lyapunov function method to some near-critical stochastic systems. We consider Markov processes in random environments, some models of interacting particle systems, and a stochastic billiards process. This chapter focuses on recurrence and
transience results, obtained by applications of the Foster–Lyapunov criteria from Chapter 2.

**Chapter 7: Markov chains in continuous time.**

For the final chapter of the book, we switch from discrete to continuous time. This chapter presents recent developments on semimartingale techniques for continuous-time discrete-space Markov chains, non-homogeneous both in space and in jump rates. For example, the (embedded) jump process might be of Lamperti-type, while the rates are not uniformly bounded away from 0 and ∞. This gives rise to additional phenomena over the discrete-time setting. We present conditions for existence of moments of hitting times in this continuous-time setting, and give criteria for explosion and implosion for such processes, again using semimartingale techniques.

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Summary of notation

Miscellaneous

We write \( a := \cdots \) to indicate the definition of \( a \). Occasionally we also use \( \cdots =: a \). We use the standard abbreviation ‘a.s.’ for ‘almost surely’ (with probability 1).

Sets, probabilities and events

The set of integers is \( \mathbb{Z} \). The natural numbers are \( \mathbb{N} := \{1, 2, 3, \ldots\} \). The non-negative integers are \( \mathbb{Z}^+ := \{0\} \cup \mathbb{N} \). The real numbers are \( \mathbb{R} \). The non-negative half-line is \( \mathbb{R}^+ := [0, \infty) \). It is often convenient to extend these sets to include infinities, so we set \( \mathbb{Z}^+ := \mathbb{Z}^+ \cup \{+\infty\}, \mathbb{R} := \mathbb{R} \cup \{\pm\infty\}, \) and \( \mathbb{R}^+ := \mathbb{R}^+ \cup \{+\infty\} \). For a set \( S \), we write \#\( S \) for its cardinality (number of elements, when finite). For a measurable subset \( A \) of \( \mathbb{R}^d \), we write \( |A| \) for its \( d \)-dimensional Lebesgue measure (volume).

We will always assume an underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\); expectation with respect to \( \mathbb{P} \) will be denoted \( \mathbb{E} \). For \( A \subseteq \Omega \), we denote its complement by \( A^c = \Omega \setminus A \).

If \((A_n, n \in \mathbb{Z}^+)\) is a sequence of events, we use the standard notation

\[
\begin{align*}
\{A_n \text{ i.o.}\} &= \{A_n \text{ infinitely often}\} \\
&= \limsup A_n = \cap_{m \geq 0} \cup_{n \geq m} A_n; \\
\{A_n \text{ ev.}\} &= \{A_n \text{ eventually}\} = \{A_n \text{ all but finitely often}\} \\
&= \liminf A_n = \cup_{m \geq 0} \cap_{n \geq m} A_n; \\
\{A_n \text{ f.o.}\} &= \{A_n \text{ finitely often}\} = \{A_n \text{ i.o.}\}^c = \{A_n^c \text{ ev.}\} \\
&= \cup_{m \geq 0} \cap_{n \geq m} A_n^c.
\end{align*}
\]
Conventions and empty evaluations

Unless otherwise stated, the following conventions are in force throughout. An empty sum is zero, and an empty product is one. Also $\inf\emptyset := +\infty$, and $\sup\emptyset := 0$.

Real numbers, vectors, and matrices

For $x$ a real number we set

$$x^+ := \max\{0, x\}, \quad \text{and} \quad x^- := -\min\{0, x\},$$

so $x = x^+ - x^-$ and $|x| = x^+ + x^-$. For real numbers $x$ and $y$, we set

$$x \wedge y := \min\{x, y\}, \quad \text{and} \quad x \vee y := \max\{x, y\}.$$

For $x \in \mathbb{R}$ we write $\lfloor x \rfloor$ for the largest integer not exceeding $x$, and $\lceil x \rceil$ for the smallest integer no less than $x$; so $\lceil x \rceil - \lfloor x \rfloor \in \{0, 1\}$, and is 0 if and only if $x \in \mathbb{Z}$.

For emphasis, we sometimes denote monotone convergence by ‘↑’ and ‘↓’, so for a sequence $a_x \in \mathbb{R}$, $a_x \uparrow a$ as $x \to \infty$ means that $\lim_{x \to \infty} a_x = a$ and $a_x \leq a_y$ for all $x \leq y$.

For a matrix $M$ with real-valued entries we write $M^\top$ for its transpose, and $\lambda_{\max}(M)$ for its maximum eigenvalue. We usually use boldface letters for vectors in $\mathbb{R}^d$, and write, for example, $x = (x_1, \ldots, x_d)^\top$ in Cartesian components; for definiteness, vectors $x \in \mathbb{R}^d$ are viewed as column vectors throughout. The origin in $\mathbb{R}^d$ is denoted by $0$. We write $\| \cdot \|$ for the Euclidean norm on $\mathbb{R}^d$. We write $e_1, \ldots, e_d$ for the standard orthonormal basis of $\mathbb{R}^d$, and for vectors $u, v \in \mathbb{R}^d$ we denote their scalar product by $u \cdot v$, $u \cdot v$, or $\langle u, v \rangle$. For a non-zero vector $x \in \mathbb{R}^d$ we write $\hat{x} := x/\|x\|$ for the corresponding unit vector, and we adopt the convention $\hat{0} := 0$. The unit-radius sphere in $\mathbb{R}^d$ is

$$S^{d-1} := \{ u \in \mathbb{R}^d : \|u\| = 1 \}.$$

The (closed) Euclidean $d$-ball centred at $x \in \mathbb{R}^d$ with radius $r \in \mathbb{R}_+$ is

$$B(x; r) := \{ y \in \mathbb{R}^d : \|x - y\| \leq r \}.$$

We denote the $d$ by $d$ identity matrix by $I_d$. For a square matrix $M$ with real-valued entries, we write $\text{tr} M$ for its trace. A $d$ by $d$ real matrix $M$ acts on column vectors $x \in \mathbb{R}^d$ as an affine function from $\mathbb{R}^d$ to $\mathbb{R}^d$ via
x ↦ Mx. The associated matrix (operator) norm ∥ · ∥_{op} induced by the Euclidean norm on \( \mathbb{R}^d \) is

\[
\|M\|_{op} := \sup_{u \in S^{d-1}} \|Mu\|.
\]

Using the variational characterization of the largest eigenvalue as \( \lambda_{\text{max}}(M) = \sup_{u \in S^{d-1}} (u^\top Mu) \), we note that

\[
\sup_{u \in S^{d-1}} \|Mu\|^2 = \sup_{u \in S^{d-1}} (u^\top M^\top Mu) = \lambda_{\text{max}}(M^\top M),
\]

so that an alternative expression for the operator norm is

\[
\|M\|_{op} = \left( \lambda_{\text{max}}(M^\top M) \right)^{1/2}.
\]

Functions

The natural logarithm of \( x \) is \( \log x \). For \( r \in \mathbb{R} \), we write \( \log^r x \) for \( (\log x)^r \).

We also write \( \log_1 x := \log x \), and for \( k \geq 2 \) set \( \log_k x := \log \log_{k-1} x \), so that \( \log_k x \) is the \( k \)-fold iterated logarithm of \( x \).

Random variables

We use \( 1 \) for the indicator function of an event, indicated either in curly braces as \( 1\{ \cdot \} \) or via a previously assigned symbol such as \( 1(A) \).

For a sub-\( \sigma \)-field \( \mathcal{G} \) of \( \mathcal{F} \), \( \mathcal{G} \)-measurable random variables \( X \) and \( Y \), and an event \( A \in \mathcal{G} \), the statement “\( X = Y \) on \( A \)” is equivalent to “\( X1(A) = Y1(A) \)”; similarly for inequalities.

We use the standard notation for essential supremum and infimum: for a real-valued random variable \( X \),

\[
\text{ess inf } X := \sup\{x \in \mathbb{R} : P[X \geq x] = 1\};
\]
\[
\text{ess sup } X := \inf\{x \in \mathbb{R} : P[X > x] = 0\}.
\]

For \( \mathbb{R}^d \)-valued random variables \( X, X_1, X_2, \ldots \), we denote convergence by \( X_n \xrightarrow{\text{a.s.}} X \) qualified by the mode of convergence in the text; for example, \( X_n \xrightarrow{\text{a.s.}} X \) a.s. means that \( P[X_n \to X] = 1 \). Sometimes for compactness we write \( \xrightarrow{\text{a.s.}}, \xrightarrow{\text{p}}, \) and \( \xrightarrow{\text{d}} \) for convergence almost surely, in probability, and in distribution, respectively.
Summary of notation

Asymptotics

We reserve unadorned Landau $O(\cdot)$ and $o(\cdot)$ symbols for the case where implicit constants are non-random. Thus for a real-valued function $f$ and a $\mathbb{R}_+^*$-valued function $g$, the expression $f(x) = O(g(x))$ means that there exist finite deterministic constants $C$ and $x_0$ such that $|f(x)| \leq Cg(x)$ for all $x \geq x_0$. Similarly, $f(x) = o(g(x))$ means that for any $\varepsilon > 0$ there exists a finite deterministic $x_\varepsilon$ such that $|f(x)| \leq \varepsilon g(x)$ for all $x \geq x_\varepsilon$.

It is convenient to extend the $O$, $o$ notation to permit random variables, but it is important to do this carefully to avoid ambiguous formulas. Given a $\sigma$-field $\mathcal{F}$ and $\mathcal{F}$-measurable random variables $X$ and $Y$, we write $O^\mathcal{F}_X(Y)$ to represent an $\mathcal{F}$-measurable random variable such that there exist finite deterministic constants $C$ and $x_0$ such that $|O^\mathcal{F}_X(Y)| \leq CY$ on the event $\{X \geq x_0\}$. Although this notation is a little cumbersome, we feel the extra clarity is worthwhile, as an ambiguous $O(\cdot)$ can hide a multitude of sins.
Chapter 1

Introduction

1.1 Random walks

Random walks are fundamental models in probability theory that exhibit deep mathematical properties and enjoy broad application across the sciences and beyond. Generally speaking, a random walk is a stochastic process modelling the random motion of a particle (or random walker) in space. The particle’s trajectory is described by a series of random increments or jumps at discrete instants in time. Central questions for these models involve the long-time asymptotic behaviour of the walker.

Random walks have a rich history involving several disciplines. Classical one-dimensional random walks were first studied several hundred years ago as models for games of chance, such as the so-called “gambler’s ruin” problem. Similar reasoning led to random walk models of stock prices described by Jules Regnault in his 1863 book [265] and Louis Bachelier in his 1900 thesis [14]. Many-dimensional random walks were first studied at around the same time, arising from work of pioneers of science in diverse applications such as acoustics (Lord Rayleigh’s theory of sound developed from about 1880 [264]), biology (Karl Pearson’s 1906 [254] theory of random migration of species), and statistical physics (Einstein’s theory of Brownian motion developed during 1905–08 [86]). The mathematical importance of the random walk problem became clear after Pólya’s work in the 1920s, and over the last 60 years or so there have emerged beautiful connections linking random walk theory and other influential areas of mathematics, such as harmonic analysis, potential theory, combinatorics, and spectral theory. Random walk models have continued to find new and important applications in many highly active domains of modern science: see for example the wide
range of articles in [287]. Specific recent developments include modelling of microbe locomotion in microbiology [245, 23], polymer conformation in molecular chemistry [15, 202], and financial systems in economics.

Spatially homogeneous random walks are the subject of a substantial literature, including numerous books, such as [293, 269, 195, 139]. In many modelling applications, the classical assumption of spatial homogeneity is not realistic: the behaviour of the random walker may depend on the present location in space. Applications thus motivate the study of non-homogeneous random walks. These models are also motivated naturally from a mathematical perspective: non-homogeneous random walks are the natural setting in which to probe near-critical behaviour and obtain a finer understanding of phase transitions present in the classical random walk models.

The main theme of this book is the analysis of near-critical stochastic systems using the method of Lyapunov functions. The non-homogeneous random walk serves as a prototypical near-critical system; the Lyapunov function methodology is robust and powerful, and can be applied to many other near-critical models, including those with applications across modern probability and beyond, to areas such as queueing theory, interacting particle systems, and random media. In this chapter we give an informal introduction to non-homogeneous random walks, and how their behaviour differs from classical random walks; we also describe some fundamental ideas of the Lyapunov function technique. We state some theorems, but we often omit technical details and generally omit proofs. All of the results that we mention will be stated more precisely (and proved) later in the book, and also applied to a wide variety of near-critical stochastic systems: the non-homogeneous random walk serves as an expository bridge between well-known classical results and the near-critical behaviour that is the subject of this book.

1.2 Simple random walk

The most intensively studied random walk model is symmetric simple random walk. Simple random walk is a discrete-time Markov process \( (S_n, n \geq 0) \) on the \( d \)-dimensional integer lattice \( \mathbb{Z}^d \): \( S_n \) can be thought of as the location (in the state-space \( \mathbb{Z}^d \)) of the random walker at time \( n \) (or after \( n \) steps). The stochastic evolution of the process is as follows. Given \( S_n \) in \( \mathbb{Z}^d \), the next point \( S_{n+1} \) is chosen uniformly at random from among the \( 2d \) lattice points adjacent to \( S_n \), i.e., those points that differ from \( S_n \) by exactly \( \pm 1 \) in a single coordinate. In other words, the transition probabilities of the
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Markov chain for are given for \( x, y \in \mathbb{Z}^d \) by

\[
P[S_{n+1} = y \mid S_n = x] = \begin{cases} \frac{1}{2^d} & \text{if } \|x - y\| = 1; \\ 0 & \text{otherwise}; \end{cases}
\]

where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^d \).

For example, when \( d = 1 \) one-dimensional simple random walk jumps one unit to the left or right, each with probability \( 1/2 \), while when \( d = 2 \) two-dimensional simple random walk jumps to one of its four neighbours, with probability \( 1/4 \) of each.

For definiteness, suppose that the walk starts at the origin of \( \mathbb{Z}^d \): \( S_0 = 0 \). By the Markov property and spatial homogeneity, the increments (or jumps) \( S_{n+1} - S_n \) of the walk are independent and identically distributed (i.i.d.) random vectors. If \( \{e_1, \ldots, e_d\} \) is the standard orthonormal basis on \( \mathbb{R}^d \), let

\[U_d := \{\pm e_1, \ldots, \pm e_d\}\]

for the possible values of the increments of the walk. Then we can write \( S_n = \sum_{k=1}^n Z_k \), where \( Z_1, Z_2, \ldots \) are i.i.d. with

\[
P[Z_1 = e] = \frac{1}{2^d} \quad \text{for } e \in U_d.
\]

(With the usual convention that an empty sum is zero, \( S_0 = 0 \).) Thus we may represent the random walk \( S_n \) via a sequence of partial sums of the i.i.d. random increments \( Z_n \).

A fundamental question, addressed by Pólya [259], concerns the recurrence or transience of the random walk: What is the probability that the walk eventually returns to \( 0 \)? If we write \( \tau_d := \min\{n \geq 1 : S_n = 0\} \) for the time of the first return to \( 0 \) (with the usual convention that \( \min\emptyset := \infty \)), the recurrence question concerns

\[p_d := P[\tau_d < \infty].\]

The random walk is recurrent if \( p_d = 1 \), in which case with probability one the random walk will visit \( 0 \) infinitely often. On the other hand, if \( p_d < 1 \) the random walk is transient, and will, with probability one, visit \( 0 \) only finitely many times, before eventually leaving, never to return.

The following fundamental result is due to Pólya [259].

**Theorem 1.2.1.** Simple random walk is recurrent in 1 or 2 dimensions, but transient in 3 or more; i.e., \( p_1 = p_2 = 1 \) but \( p_d < 1 \) for all \( d \geq 3 \).
The content of the theorem is nicely captured by an aphorism attributed to Shizuo Kakutani: “A drunk man will eventually find his way home, but a drunk bird may get lost forever” (see [83, p. 191]).

Pólya’s theorem (Theorem 1.2.1) tells us that the walk returns to 0 eventually when \( d = 1 \) or \( d = 2 \). But how long might we have to wait? The answer is, potentially, a very long time, since \( \tau_d \) has very heavy tails:

\[
\mathbb{P}[\tau_1 > n] \sim \frac{1}{\sqrt{\pi n}}, \quad \text{and} \quad \mathbb{P}[\tau_2 > n] \sim \frac{\pi}{\log n}, \quad \text{as} \quad n \to \infty;
\]

see e.g. [259, p. 159] for the first expression and [84, pp. 356–357] for the second. So in \( d = 1 \), \( \mathbb{E}[\tau_1^{1/2}] = \infty \), while in \( d = 2 \), \( \tau_2 \) has no moments at all. According to Hughes [139, p. 42],

the failure of certain moments of distributions or densities to converge . . . is pregnant with physical meaning, and indicative of connections to scaling laws, renormalization group methods, and fractals.

The recurrence exhibited by the simple random walk for \( d \in \{1, 2\} \) is null-recurrence, meaning that \( \mathbb{E} \tau_d = \infty \); more stable processes may exhibit positive-recurrence, meaning that the analogue of \( \tau_d \) is integrable.

### 1.3 Lamperti’s problem

There are several proofs of Theorem 1.2.1 in the literature, the most popular being those that are largely combinatorial (such as Pólya’s original argument [259]) and those based on potential theory and electrical networks (see e.g. [81]). A drawback of each of these approaches is that they rapidly break down when one tries to generalize Pólya’s theorem to other random walks. In this section we describe a robust approach to proving Pólya’s theorem, due to Lamperti, which enables very broad generalization. This approach is based on the methodology of Lyapunov functions.

Again let \( S_n \) be the symmetric simple random walk on \( \mathbb{Z}^d \), starting at 0. In the context of Pólya’s recurrence theorem, we are interested in the events \( \{S_n = 0\} \). We can reduce this \( d \)-dimensional problem to a one-dimensional problem by considering a transformation of the process (a Lyapunov function) given by

\[
X_n := \|S_n\|, \quad (1.1)
\]

i.e., \( X_n \) is the distance from the origin of the walker at time \( n \). The stochastic process \((X_n, n \geq 0)\) takes values in the countable set \( \mathcal{S} = \{\|\mathbf{x}\| : \mathbf{x} \in \mathbb{Z}^d\} \),
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Figure 1.1: Illustration of the non-Markovian nature of $X_n$ in $d = 2$.

a subset of the half-line $\mathbb{R}_+$, and $X_n = 0$ if and only if $S_n = 0$. So we can study the recurrence or transience of $S_n$ via the recurrence or transience of $X_n$, a one-dimensional process.

This reduction in dimensionality of the problem comes at a price: $X_n$ is not in general a Markov process. For instance, when $d = 2$, given one of the two events $\{S_n = (5, 0)\}$ and $\{S_n = (3, 4)\}$ we have $X_n = 5$ in each case but $X_{n+1}$ has two different distributions; $X_{n+1}$ can take the value 6 (with probability 1/4) in the first case, but this is impossible in the second case. See Figure 1.3. Thus the tools that we use to study $X_n$ must not rely too heavily on the Markov property.

First we compute the expected increment of $X_n$ given $S_n = x$, namely

$$E[X_{n+1} - X_n \mid S_n = x] = \frac{1}{2d} \sum_{i=1}^{d} \left( \|x + e_i\| + \|x - e_i\| - 2\|x\| \right). \quad (1.2)$$

To proceed we apply Taylor’s theorem in an elementary way. Using the Taylor expansion

$$(1 + y)^{1/2} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + O(y^3), \text{ as } y \to 0,$$

we obtain that, for any $e \in \mathbb{S}^{d-1}$,

$$\|x + e\| - \|x\| = \sqrt{(x + e) \cdot (x + e)} - \|x\|$$

$$= \|x\| \left[ \left( 1 + \frac{2e \cdot x + 1}{\|x\|^2} \right)^{1/2} - 1 \right].$$
\[ \|x\| \left( \frac{2e \cdot x + 1}{2\|x\|^2} - \frac{(e \cdot x)^2}{2\|x\|^4} + O(\|x\|^{-3}) \right). \]  \hfill (1.3)

It follows that
\[ \|x + e\| + \|x - e\| - 2\|x\| = \frac{1}{\|x\|} - \frac{(e \cdot x)^2}{\|x\|^3} + O(\|x\|^{-2}). \]

Thus from (1.2), using the fact that \( \sum_{i=1}^d (e_i \cdot x)^2 = \|x\|^2 \), we obtain
\[ \mathbb{E}[X_{n+1} - X_n \mid S_n = x] = \frac{1}{2d} \sum_{i=1}^d (\|x + e_i\| + \|x - e_i\| - 2\|x\|) \]
\[ = \left( \frac{d - 1}{2d} \right) \frac{1}{\|x\|} + O(\|x\|^{-2}). \]  \hfill (1.4)

Similarly
\[ \mathbb{E}[X_{n+1}^2 - X_n^2 \mid S_n = x] = \frac{1}{2d} \sum_{i=1}^d (\|x + e_i\|^2 + \|x - e_i\|^2 - 2\|x\|^2) = 1. \]

Then since \((X_{n+1} - X_n)^2 = X_{n+1}^2 - X_n^2 - 2X_n(X_{n+1} - X_n)\) we obtain
\[ \mathbb{E}[(X_{n+1} - X_n)^2 \mid S_n = x] = \frac{1}{d} + O(\|x\|^{-1}). \]  \hfill (1.5)

Informally speaking, (1.4) says that the mean increment of \(X_n\) at \(x \in S\) is \(\frac{1}{2d}(1 - \frac{1}{d}) + O(x^{-2})\), and similarly (1.5) says that the second moment of the increment at \(x\) is \(\frac{1}{d} + O(x^{-1})\); however, to formalize these statements we need to be careful since \(X_n\) is not a Markov process, and we have to clarify what we mean by the increment moments “at \(x\)”. We deal with these technicalities later, since they complicate the notation (although they will not complicate the proofs, which are based on martingale arguments). For now, for the purposes of exposition, we switch to the case where \(X_n\) is a Markov process.

Suppose now that \((X_n, n \geq 0)\) is a time-homogeneous Markov process on an unbounded subset \(S\) of \(\mathbb{R}_+\). Consider the increment moment functions
\[ \mu_k(x) := \mathbb{E}[(X_{n+1} - X_n)^k \mid X_n = x]. \]

For simplicity, suppose that \(X_n\) has uniformly bounded increments, so that
\[ \mathbb{P}[\|X_{n+1} - X_n\| \leq B] = 1 \]  \hfill (1.6)
for some \( B \in \mathbb{R}_+ \); under condition (1.6), the \( \mu_k \) are well-defined functions of \( x \in S \). The first moment function, \( \mu_1(x) \), is the one-step mean drift of \( X_n \) at \( x \).

Lamperti [190, 191, 192] investigated the extent to which the asymptotic behaviour of such a process is determined by the \( \mu_k \); essentially, \( \mu_1 \) and \( \mu_2 \) turn out to govern the asymptotic behaviour. For example, the following result is a version Lamperti’s fundamental recurrence classification.

**Theorem 1.3.1.** Suppose that \( X_n \) is a Markov process on \( S \) satisfying (1.6). Under mild conditions, the following recurrence classification holds. Let \( \varepsilon > 0 \).

- If \( 2x\mu_1(x) + \mu_2(x) < -\varepsilon \), then \( X_n \) is positive-recurrent;
- If \( 2x|\mu_1(x)| \leq \mu_2(x) + O(x^{-\varepsilon}) \), then \( X_n \) is null-recurrent;
- If \( 2x\mu_1(x) - \mu_2(x) > \varepsilon \), then \( X_n \) is transient.

The mild conditions mentioned in the statement of Theorem 1.3.1 are related to issues of irreducibility; the exact nature of the conditions required depends on the state-space \( S \) and the notion of recurrence and transience desired. We leave the technical details until later in the book.

A version of Theorem 1.3.1 applies to non-Markov processes \( X_n \) when formulated correctly, using appropriate versions of the \( \mu_k \). In particular, we can use such results to study the process \( X_n \) defined by (1.1): in this case, the analogue of \( 2x\mu_1(x) \) is, by (1.4),

\[
1 - \frac{1}{d} + O(x^{-1})
\]

and the analogue of \( \mu_2(x) \) is, by (1.5),

\[
\frac{1}{d} + O(x^{-1}).
\]

An application of the generalized version of Theorem 1.3.1 shows that \( X_n \) is transient if and only if

\[
1 - \frac{1}{d} > \frac{1}{d'},
\]

or, in other words, \( d > 2 \). This gives a very robust strategy for proving Pólya’s theorem (Theorem 1.2.1) and its generalizations, based on computations of increment moments for \( X_n \) defined by (1.1). These computations use elementary Taylor’s theorem ideas, and do not rely at all on special structure of the original process \( S_n \).
1.4 General random walk

Simple random walk is an attractive model and can be studied using combinatorial methods based on counting sample paths, for example; it is, however, a very specific model. Naturally, it is of interest to study a much broader class of random walks. In particular, for what class of models does a result similar to Theorem 1.2.1 hold? To put the question in another way: What are the essential properties possessed by simple random walk that imply Theorem 1.2.1? To answer such questions, we start by describing a much more general model of a random walk.

By a random walk we mean a discrete-time Markov process \((\xi_n, n \geq 0)\) on an unbounded state-space \(\Sigma \subseteq \mathbb{R}^d\). We assume that the random walk is time-homogeneous. This means that the distribution of \(\xi_{n+1}\) given \((\xi_0, \xi_1, \ldots, \xi_n)\) depends only on \(\xi_n\) (and not on \(n\)). We also need some form of irreducibility to ensure that the random walk cannot get ‘trapped’ in some part of the state-space: it is simplest to take \(\Sigma\) to be a locally finite set (such as \(\mathbb{Z}^d\)) to avoid technical issues at this point.

We are thus using the term random walk in a rather general sense, requiring that the Markov process inhabit Euclidean space. We also want to impose some regularity assumptions on the increments of the process, to rule out very long jumps for the random walk. For this chapter, for convenience we often suppose that the increments of the walk are uniformly bounded, so that there exists \(B \in \mathbb{R}_+\) for which, almost surely (a.s.),

\[
P[\|\xi_{n+1} - \xi_n\| \leq B \mid \xi_n = x] = 1, \quad \text{for all} \ x \in \Sigma. \tag{1.7}
\]

In many cases this assumption can be replaced by a weaker assumption on the existence of higher moments for the increments, without producing fundamentally new behaviour. On the other hand, the case of genuinely heavy-tailed increments leads to different phenomena, as discussed in Chapter 5 of this book.

Under the assumption (1.7), the random vectors \(\xi_{n+1} - \xi_n\) have well-defined moments, which may depend on \(\xi_n\). In particular, an important quantity is the one-step mean drift vector

\[
\mu(x) := \mathbb{E}[\xi_{n+1} - \xi_n \mid \xi_n = x],
\]

which is the average change in position in a single step starting from \(x \in \Sigma\). (Note that the definition via the conditional expectation on \(\{\xi_n = x\}\) is clear in the case of a countable state-space \(\Sigma\), and makes sense when correctly interpreted for uncountable \(\Sigma\) as well.)
1.5. Recurrence and transience

An important and well-studied sub-class of random walks are spatially homogeneous, for which the distribution of the increment \( \xi_{n+1} - \xi_n \) does not depend on the current location \( \xi_n \). Writing \( \theta_n := \xi_{n+1} - \xi_n \), spatial homogeneity implies that \( \theta_0, \theta_1, \ldots \) are i.i.d. random vectors. Then the representation

\[
\xi_n = \sum_{k=0}^{n-1} \theta_k
\]

as a sum of i.i.d. random vectors enables classical tools of probability theory, such as Fourier methods, to be brought to bear in the analysis of the random walk in the spatially homogeneous case.

**Example 1.4.1** (Simple random walk). Using \( e_1, \ldots, e_d \) to denote the standard orthonormal basis vectors of \( \mathbb{R}^d \), simple random walk is spatially homogeneous with \( \theta_n \) uniformly distributed on the set \( U_d = \{ \pm e_1, \ldots, \pm e_d \} \).

**Example 1.4.2** (Pearson–Rayleigh random walk). Take \( \theta_n \) to be uniformly distributed on the unit-radius sphere \( S^{d-1} \subset \mathbb{R}^d \). The corresponding \( d \)-dimensional random walk, that proceeds via a sequence of unit-length steps, each in an independent and uniformly random direction, is sometimes called a Pearson–Rayleigh random walk: see the bibliographical notes at the end of this chapter.

Spatially homogeneous random walks have been extensively studied in classical probability theory. Non-homogeneous random walks, on the other hand, require new techniques and, just as importantly, new intuitions.

### 1.5 Recurrence and transience

For our general random walks, we need a more general definition of recurrence and transience. In the absence of any structural assumptions, our basic use of the terminology is as follows. Note that, in general, the two behaviours are not a priori exhaustive.

**Definition 1.5.1.** A stochastic process \((\xi_n, n \geq 0)\) taking values in \( \Sigma \subseteq \mathbb{R}^d \) is transient if \( \lim_{n \to \infty} \|\xi_n\| = \infty \), a.s. The process is recurrent if, for some constant \( r_0 \in \mathbb{R}_+ \), \( \liminf_{n \to \infty} \|\xi_n\| \leq r_0 \), a.s.

If \( \xi_n \) is an irreducible time-homogeneous Markov chain whose state-space is a locally finite subset of \( \mathbb{R}^d \) (such as \( \mathbb{Z}^d \)), then recurrence and transience in the sense of Definition 1.5.1 coincide with the usual Markov chain definition.
in terms of returns to any given state. Definition 1.5.1 allows more general processes, and is the most convenient definition in the context of Lamperti’s problem outlined in Section 1.3.

First we return to the classical spatially homogeneous random walk described in Section 1.4, in which case the increments $\theta_n$ in (1.8) are i.i.d. In this case, when defined, $E[\xi_{n+1} - \xi_n \mid \xi_n = x] = E \theta_0 = \mu$ does not depend on $x$.

If $\mu \neq 0$, then the strong law of large numbers shows that the walk is transient, and $\lim_{n \to \infty} n^{-1} \xi_n = \mu$, a.s., so the walk escapes to infinity at positive speed. The most subtle case is that of zero drift when $\mu = 0$. Here, under mild conditions, Pólya’s theorem (Theorem 1.2.1) for simple symmetric random walk extends to the case of the general spatially homogeneous random walks with zero drift. Recall that we view $\theta_0$ and other $d$-dimensional vectors as column vectors throughout.

**Theorem 1.5.2.** For a spatially homogeneous random walk $\xi_n$ on $\mathbb{R}^d$, suppose that $E[\|\theta_0\|^2] < \infty$, $E \theta_0 = 0$, and $E[\theta_0 \theta_0^\top]$ is positive-definite. Then $\xi_n$ is recurrent for $d \in \{1, 2\}$ and transient for $d \geq 3$.

The positive-definite covariance condition ensures that the increments are not supported on a lower-dimensional subspace.

How does the situation change if the walk is allowed to be spatially non-homogeneous? In the general, non-homogeneous case $\mu(x) = E[\theta_n \mid \xi_n = x]$ will depend on $x$. Even if $\mu(x) = 0$ for all $x$, however, the non-homogeneous zero-drift walk can behave completely differently to the homogeneous zero-drift walk.

**Theorem 1.5.3.** Let $\xi_n$ be a spatially non-homogeneous random walk on $\mathbb{R}^d$ with zero drift, so that $\mu(x) = 0$ for all $x$.

- If $d = 2$, then we can exhibit such a walk that is transient.
- If $d \geq 3$, then we can exhibit such a walk that is recurrent.

In all cases, we may take these examples to have uniformly bounded increments, as at (1.7).

We emphasize that, for example, in two dimensions, zero drift does not imply recurrence for a non-homogeneous random walk with bounded jumps. This fact is contrary to intuition built from homogeneous random walks, but should not be surprising to readers who have encountered random walks in random environments: for example, Zeitouni [319, pp. 90-91] discusses an
example of a transient walk in $d = 2$ with symmetric increments; see also the examples in Chapter 4.

So non-homogeneous random walks can show anomalous (non-classical) recurrence behaviour. We can reassert some control by imposing additional regularity structure on the second moments of the increments $\theta_n = \xi_{n+1} - \xi_n$.

Assuming (1.7), then the matrix function

$$M(x) = \mathbb{E}[\theta_n \theta_n^\top | \xi_n = x]$$

is well defined, since for any $e \in S^{d-1}$, $|e \cdot \theta_n \theta_n^\top e| = (e \cdot \theta_n)^2 \leq \|\theta_n\|^2$; for each $x$, $M(x)$ is a symmetric, nonnegative-definite matrix. We refer to $M(x)$ defined at (1.9) as the increment covariance matrix; this is a slight abuse of terminology, which more accurately should refer to

$$\mathbb{E}[(\theta_n - \mu(x))(\theta_n - \mu(x))^\top | \xi_n = x] = M(x) - \mu(x)\mu(x)^\top,$$

but in our calculations it is always $M(x)$ that appears.

Suppose that $M(x) = M$ is fixed, for a positive-definite matrix $M$. This condition is satisfied with $M = \sigma^2 I_d$ for some $\sigma^2 \in (0, \infty)$, where $I_d$ is the $d$ by $d$ identity matrix, by a walk with sufficient symmetry in its increments, such as simple symmetric random walk or the Pearson–Rayleigh walk. With this additional regularity structure, the wild behaviour shown in Theorem 1.5.3 is ruled out, and we recapture the result of Theorem 1.5.2.

**Theorem 1.5.4.** Let $\xi_n$ be a spatially non-homogeneous random walk on $\mathbb{R}^d$ for which (1.7) holds. Suppose that, for $x$, $\mu(x) = 0$ and $M(x) = M$ for some positive-definite matrix $M$. Then $\xi_n$ is recurrent for $d \in \{1, 2\}$ and transient for $d \geq 3$.

**Remark 1.5.5.** Given that $M(x) = M$ for some positive-definite (necessarily, symmetric) matrix $M$, there exists a (unique) positive definite, symmetric matrix square-root $M^{1/2}$ with matrix inverse $M^{-1/2}$, such that $M^{1/2}M^{1/2} = M$. The matrix $M^{-1/2}$ defines a linear transformation of $\mathbb{R}^d$ via $x \mapsto M^{-1/2}x$ ($x \in \mathbb{R}^d$). Define $\zeta_n := M^{-1/2}\xi_n$. Then $\varphi_n := \zeta_{n+1} - \zeta_n = M^{-1/2}\theta_n$, where $\theta_n = \xi_{n+1} - \xi_n$, and, by linearity,

$$\mathbb{E}[\varphi_n | \zeta_n = x] = M^{-1/2} \mathbb{E}[\theta_n | \xi_n = M^{1/2}x] = 0,$$

$$\mathbb{E}[\varphi_n \varphi_n^\top | \zeta_n = x] = M^{-1/2} \mathbb{E}[\theta_n \theta_n^\top | \xi_n = M^{1/2}x] M^{-1/2},$$

which is the identity. The transformed process $\zeta_n$ is recurrent if and only if the original process $\xi_n$ is recurrent. Thus it suffices to prove Theorem 1.5.4 in the case $M = I_d$, or indeed for any other fixed positive-definite matrix.
Chapter 1. Introduction

We describe the relevance of Lamperti's problem to Theorem 1.5.4. The basic idea is the same as in Section 1.3, but we use Taylor's theorem in a slightly more sophisticated way.

Consider a random walk \( \xi \) satisfying the conditions of Theorem 1.5.4. Let \( X_n := \|\xi_n\| \). Then

\[
E[X_{n+1} - X_n \mid \xi_n = x] = E[\|x + \theta_n\| - \|x\| \mid \xi_n = x],
\]

where, as usual, \( \theta_n = \xi_{n+1} - \xi_n \) is the increment of the walk. We now use Taylor's theorem to expand the expression inside the expectation. Note first that, by a similar argument to (1.3), for any \( y \in \mathbb{R}^d \) with \( \|y\| \leq B \),

\[
\|x + y\| - \|x\| = \|x\| \left( 1 + \frac{2y \cdot x + \|y\|^2}{\|x\|^2} \right)^{1/2} - 1
= \frac{y \cdot x}{\|x\|} + \frac{\|y\|^2}{2\|x\|} - \frac{(y \cdot x)^2}{2\|x\|^3} + O(\|x\|^{-2}),
\]

as \( \|x\| \to \infty \). We are going to apply this with \( y = \theta_n \). In other words,

\[
\|x + \theta_n\| - \|x\| = \hat{x} \cdot \theta_n + \frac{1}{2\|x\|} \left( \|\theta_n\|^2 - (\hat{x} \cdot \theta_n)^2 \right) + O(\|x\|^{-2}).
\]

It is important to note that this is legitimate because of the uniform jumps bound (1.7), and that the implicit constant in the \( O(\cdot) \) term is non-random, i.e., uniform in \( \theta_n \); we use this notation strictly for such circumstances. Thus we may take expectations to obtain

\[
E[X_{n+1} - X_n \mid \xi_n = x]
= E[\hat{x} \cdot \theta_n \mid \xi_n = x] + \frac{1}{2\|x\|} E[\|\theta_n\|^2 - (\hat{x} \cdot \theta_n)^2 \mid \xi_n = x] + O(\|x\|^{-2}).
\]

With the notation \( \mu(x) = E[\theta_n \mid \xi_n = x] \) and \( M(x) = E[\theta_n\theta_n^T \mid \xi_n = x] \), using linearity of expectation and a little linear algebra we can write this as

\[
E[X_{n+1} - X_n \mid \xi_n = x] = \hat{x} \cdot \mu(x) + \frac{1}{2\|x\|} (\text{tr} M(x) - \hat{x}^T M(x) \hat{x}) + O(\|x\|^{-2}).
\]

Suppose that the conditions of Theorem 1.5.4 hold, with \( M = \sigma^2 I_d \) for some \( \sigma^2 \in (0, \infty) \), which is no loss of generality by Remark 1.5.5. Then we have \( \mu(x) = 0 \), \( \text{tr} M(x) = d\sigma^2 \), and \( e^T M(x)e = \sigma^2 \) for any \( e \in \mathbb{S}^{d-1} \), so that

\[
E[X_{n+1} - X_n \mid \xi_n = x] = \frac{(d-1)\sigma^2}{2\|x\|} + O(\|x\|^{-2}).
\]
For symmetric simple random walk, $\sigma^2 = 1/d$, and we recover (1.4), but now we have used only the first two moments of the increments. Similarly,

$$\mathbb{E}[(X_{n+1} - X_n)^2 \mid \xi_n = x] = \sigma^2 + O(\|x\|^{-1}),$$

which generalizes (1.5). Again, Lamperti’s approach yields transience when $(d - 1)\sigma^2 > \sigma^2$, i.e., $d > 2$.

From this perspective, the proof of Theorem 1.5.4 is no more difficult than the proof of Theorem 1.5.2, at least if we assume the jumps bound (1.7); this robustness is a feature of the Lyapunov function method.

Later in the book, we shall see how to relax the jumps bound (1.7) in favour of a moments condition. The issue is clear: Taylor’s theorem for $\|x + \theta_n\|$ is only useful if $\|\theta_n\|$ is small compared to $\|x\|$. To overcome this obstacle, one uses a basic truncation idea: partition the increment as

$$X_{n+1} - X_n = (X_{n+1} - X_n) \mathbf{1}\{\|\theta_n\| \leq \|x\|\} + (X_{n+1} - X_n) \mathbf{1}\{\|\theta_n\| > \|x\|\}$$

for some constant $\gamma \in (0, 1)$; then the first term can be expanded using Taylor’s theorem, while the second term can be bounded in expectation using a version of Markov’s inequality and a bound on the moments of $\|\theta_n\|$.

We shall see these ideas carried through later in the book.

### 1.6 Angular asymptotics

The recurrence and transience properties described in the previous section are properties of the radial component of the process; it is also natural to investigate asymptotics of the angular component.

We say that a random walk $\xi_n$ has a limiting direction if $\lim_{n \to \infty} \hat{\xi}_n$ exists in $S^{d-1}$. Here and elsewhere $\hat{x}$ denotes the unit vector $x/\|x\|$ when $x \neq 0$; we adopt the convention $\hat{0} := 0$. A walk with a limiting direction eventually remains in an arbitrarily narrow cone with that direction as its axis; when is this the case?

Here we review briefly the situation for classical random walks. Suppose that $\xi_n$ is a spatially homogeneous random walk, so the increments $\theta_n$ in (1.8) are i.i.d.

**Theorem 1.6.1.** For a spatially homogeneous random walk $\xi_n$ on $\mathbb{R}^d$, suppose that $\mathbb{E}\|\theta_0\| < \infty$; let $\mu = \mathbb{E}\theta_0 \in \mathbb{R}^d$.

(i) If $\mu \neq 0$, then $\mathbb{P}[\lim_{n \to \infty} \hat{\xi}_n = \hat{\mu}] = 1$.

(ii) If $\mu = 0$ and $\mathbb{P}[^{\theta_0 = 0}] < 1$, then $\mathbb{P}[\lim_{n \to \infty} \hat{\xi}_n \text{ exists}] = 0$. 


A proof of Theorem 1.6.1 is presented in the bibliographical notes at the end of this chapter. Once spatially homogeneity is relaxed, the situation becomes much more interesting.

1.7 Centrally biased random walks

To probe more precisely the recurrence-transience phase transition, it is natural to consider the case in which \( \mu(x) \to 0 \) as \( \|x\| \to \infty \): this is known as the asymptotically zero drift regime.

For definiteness, consider a 2-dimensional non-homogeneous random walk whose increments have a fixed covariance structure. We have seen (Theorem 1.5.4) that if \( \mu(x) = 0 \) for all \( x \), the random walk is recurrent. On the other hand, if the walk has uniformly positive drift radially away from \( 0 \), i.e., \( \mu(x) = \varepsilon \hat{x} \) for any \( \varepsilon > 0 \), then it can be shown that the walk is transient with linear rate of escape.

The natural setting to look for finer behaviour is the case in which the magnitude of the drift tends to zero as the distance from the origin increases. We focus on a concrete example of the asymptotically zero drift regime, in which the drift is always in the radial direction. Consider a non-homogeneous random walk on \( \mathbb{R}^d \). Let \( r_0 \) be a finite positive constant. Suppose that, for \( \rho \in \mathbb{R} \),

\[
\mu(x) = \rho \frac{\hat{x}}{\|x\|}, \quad \text{for all } x \text{ with } \|x\| \geq r_0; \tag{1.10}
\]

in other words, the mean drift is radially towards or away from \( 0 \), vanishing in magnitude as the distance from \( 0 \) increases. Such centrally biased random walks were studied by Lamperti; one can view these processes as perturbations of zero-drift random walks. It is natural to investigate what magnitude of perturbation is needed to change the asymptotic behaviour of the process.

For simplicity, we take the state space of our random walk to be \( \mathbb{Z}^d \). We also need to impose an assumption to ensure that the walk cannot get trapped in a subset of the state space. For simplicity, at this point we suppose that the increments are uniformly elliptic:

\[
\inf_{u \in \mathbb{S}^{d-1}} \mathbb{P}[(\xi_{n+1} - \xi_n) \cdot u \geq \varepsilon \mid \xi_n = x] \geq \varepsilon, \quad \text{for all } x \in \mathbb{Z}^d. \tag{1.11}
\]

We will see later in the book (Example 3.3.5) that uniform ellipticity (1.11) implies \( \limsup_{n \to \infty} \|\xi_n\| = \infty \), a.s., so questions concerning the asymptotic behaviour of the walk are non-trivial.
The following result due to Lamperti can be viewed as a generalization of the one-dimensional Theorem 1.3.1. Recall the definition of the increment covariance matrix $M(x)$ from (1.9).

**Theorem 1.7.1.** Let $\xi_n$ be a spatially non-homogeneous random walk on $\mathbb{Z}^d$ for which (1.7) and (1.11) hold, with asymptotically zero drift of the form (1.10). Suppose that $M(x) = M$ for some positive-definite $M$. Then there exist constants $c_1 = c_1(d, M)$ and $c_2 = c_2(d, M)$, with $-\infty < c_1 < c_2 < \infty$, for which:

- If $\rho < c_1$, then $\xi_n$ is positive-recurrent;
- If $c_1 \leq \rho \leq c_2$, then $\xi_n$ is null-recurrent;
- If $\rho > c_2$, then $\xi_n$ is transient

In particular, for $d \in \{1, 2\}$, $c_2 \geq 0$, while for $d \geq 3$, $c_2 < 0$.

This result implies that by adding an appropriate drift of order $1/\|x\|$ away from 0 to a zero drift random walk in $d \in \{1, 2\}$, we can change the walk from recurrent to transient. On the other hand, by adding an appropriate drift of order $1/\|x\|$ towards 0 to a zero drift random walk in $d \geq 3$, we can change the walk from transient to recurrent (even positive-recurrent).

Theorem 1.7.1 shows that the case where $\|\mu(x)\| = O(\|x\|^{-1})$ is critical from the point of view of the recurrence classification of the non-homogeneous random walk. It also turns out that several other important characteristics of the process undergo phase transitions in this regime; in this sense, the centrally biased random walk is a *near-critical* stochastic system. We will see this near-critical behaviour exhibited by several quantitative characteristics later in the book, including passage-time distributions, stationary measures, and almost-sure scaling exponents; typical of near-criticality is that scaling is *polynomial* and distributions are *heavy-tailed*.

The regime $\|\mu(x)\| = O(\|x\|^{-1})$ also turns out to be critical from the point of view of the angular asymptotics in the sense of Section 1.6. Here, where the walk may be transient or recurrent, by Theorem 1.7.1, the following result shows that the walk has no limiting direction.

**Theorem 1.7.2.** Let $\xi_n$ be a spatially non-homogeneous random walk on $\mathbb{Z}^d$, $d \geq 2$, for which (1.7) and (1.11) hold. Suppose that $M(x) = M$ for some positive-definite $M$ and that $\|\mu(x)\| = O(\|x\|^{-1})$. Then $\mathbb{P}[\lim_{n \to \infty} \hat{\xi}_n \text{ exists}] = 0$. 
On the other hand, a radial drift of order greater than $\|x\|^{-1}$ can lead to completely different behaviour, in which the random walk is transient with a limiting direction, which can (unlike in Theorem 1.6.1) be random.

In fact, we have the following strong law of large numbers:

**Theorem 1.7.3.** Let $\xi_n$ be a spatially non-homogeneous random walk on $\mathbb{Z}^d$ for which (1.7) and (1.11) hold, with drift of the form

$$\mu(x) = \rho \frac{\hat{x}}{\|x\|^\beta}, \text{ for all } x \text{ with } \|x\| \geq r_0,$$

for some $\beta \in (0, 1)$, $\rho > 0$, and $r_0 \in \mathbb{R}_+$. Suppose that $M(x) = M$ for some positive-definite $M$. Then as $n \to \infty$,

$$\frac{\xi_n}{n^{1/(1+\beta)}} \to v, \text{ a.s.,}$$

for a random vector $v \in \mathbb{R}^d$ of fixed (deterministic, non-zero) magnitude. In particular, $\lim_{n \to \infty} \xi_n = \hat{v}$, a.s., a random limiting direction.

See Figure 1.2 for some simulations illustrating these results.

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**Figure 1.2:** Two 2-dimensional centrally biased random walk simulations, each run for $4 \times 10^4$ steps, under the conditions of Theorem 1.7.2 (left) and Theorem 1.7.3 (right). Both walks are transient; the walk on the left has no limiting direction, while the walk on the right has a limiting direction and satisfies a super-diffusive law of large numbers.
Bibliographical notes

Section 1.1

We give a few references to some of the many applications enjoyed by random walks. Surveys of numerous applications can be found in [15, 202, 315, 287, 23, 139, 314].

Animal behaviour. The use of random walks to model the movement of animals goes back about 100 years to K. Pearson [254]; subsequently random walk models have been used both for the foraging, territorial, and migratory behaviour of macroscopic animals (see e.g. [131]) and for the locomotion of microbes, where experiment suggests that the motion of several kinds of cells consists of roughly straight line segments linked by discrete changes in direction: see e.g. Chapter 6 of [23] and [244, 245]. We refer to [49, 290] for recent surveys and to [32] for further references to the literature.

Molecular conformation and dynamics. Random walks have been used in chemical physics as idealized models for the static and dynamic behaviour of flexible long chain molecules in solution. In the most basic ‘freely-jointed’ chain model, the steps of the walk are independent; more realistic models take ‘restricted’ or ‘correlated’ random walks where the step distribution depends on the previous step(s). We refer to Chapter 15 of [178], Section 2.6 of [139], and to [15, 202, 248] for surveys.

Theory of sound. The passage of a sound wave through an inhomogeneous medium can be modelled using a random walk in phase space formed by summing random wave-vectors of fixed amplitude and random phases. This approach goes back to Lord Rayleigh [264]; subsequent work includes [143].

Other applications. A selection of other applications of random walks includes electronic converters [29], cosmic microwave background [294, 266], and financial mathematics [94].

Section 1.2

Simple random walk was studied by Lord Rayleigh (see [35]) but the seminal early contribution was Pólya’s 1921 paper [259] which gave Theorem 1.2.1. Versions of the usual proof, which is essentially Pólya’s, can be found for example in Chapter 10 of [241]. Simple random walk is undoubtedly the most
studied random walk model; some of the key results from the vast literature are covered in the books of Feller [98] and Révész [269], for example.

This deceptively simple model shows deep and interesting phenomena. For example, there is a remarkable closed-from expression for $p_3$, given by

$$p_3 = 1 - \frac{1}{u_3} \approx 0.340537,$$

where

$$u_3 = \frac{3}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dx dy dz}{3 - \cos x - \cos y - \cos z}$$

$$= \frac{\sqrt{6}}{32\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right);$$

a version of the first equality for $u_3$ was given by McCrea and Whipple [219], the second is a result of Glasser and Zucker [118].

Section 1.3

The approach described in this section is due to Lamperti [190, 192], and is discussed at length in Chapter 3 of this book. The statement of the recurrence classification Theorem 1.3.1 is made precise in Theorem 3.2.3 (in the Markovian case) and Theorems 3.5.1 and 3.5.2 (in the more general case where the Markov property does not hold).

The observation that $\|S_n\|$ is not Markov is an important instance of a general phenomenon; if $Y_n$ is a Markov chain, then $f(Y_n)$ need not be Markov. In the particular case where the state-space is $\mathbb{Z}^d$ and $f(x) = \|x\|$, it is clear what goes ‘wrong’ for $\|S_n\|$, as illustrated in Figure 1.3: the function $x \mapsto \|x\|$ is many-to-one in some places, with pre-images that contain ‘distinguishable’ points. In this case, one can rectify the issue by considering $f(x) = \|\zeta + x\|$ for some $\zeta \in \mathbb{R}^d$ with irrational coordinates; then $f$ is one-to-one over $\mathbb{Z}^d$, and $\|\zeta + S_n\|$ is Markov. So it is misleading to overplay the loss of the Markov property in the case of $\|S_n\|$. However, $\|S_n\|$ is certainly a more natural object than $\|\zeta + S_n\|$, and we will encounter other situations (more general state-spaces, more complicated Lyapunov functions) where it is less straightforward to rescue the Markov property in such a way.

The general question of when a function $f$ applied to a Markov chain $Y_n$ yields a Markov chain $f(Y_n)$ is a topic that has received much interest over the years, under the heading of Markov functions. Here we mention, for example, [274, 196, 272, 156].
Section 1.4

The random walk with spherical increments described in Example 1.4.2 is known as the Pearson–Rayleigh random walk after the exchange in the letters pages of *Nature* between Karl Pearson and Lord Rayleigh in 1905 [253]:

A man starts from a point $O$ and walks $\ell$ yards in a straight line; he then turns through any angle whatever and walks another $\ell$ yards in a second straight line. He then repeats this process $n$ times.

Carazza speculates [35, p. 419] that Pearson’s enthusiasm for this problem may have originated in the game of golf, although Pearson’s academic interest in such problems was directed towards migration of animal species, such as mosquitoes [254]; Rayleigh had earlier considered the acoustic ‘random walks’ in phase space produced by combinations of sound waves of the same amplitude and random phases.

Section 1.5

Theorem 1.5.2 is a special case of the famous Chung–Fuchs theorem, which gives a criterion for recurrence/transience for general spatially homogeneous random walks on $\mathbb{R}^d$; see [47] or Chapter 9 of [150]. In dimension $d = 1$, the second moments condition can be replaced by the weaker condition $\mathbb{E}|\theta_0| < \infty$, while in dimension $d \geq 3$, the moments condition can be dispensed with altogether; in these cases one must reformulate the covariance condition as a condition on the support of the increment distribution being of the full dimension.

Theorem 1.5.3 exhibits the possible anomalous recurrence properties of non-homogeneous random walks; this issue is explored in detail in Chapter 4. The statement of Theorem 1.5.3 is exhibited by both the elliptic random walks of Theorem 4.2.1 (see also [114]) and the controlled random walks of Section 4.3 (see also [257]).

Although certainly appreciated by experts, Theorem 1.5.3 is perhaps not as widely known as it might be. Zeitouni (pp. 91–92 of [320]) describes an example of a transient zero-drift random walk on $\mathbb{Z}^2$, and states that the idea “goes back to Krylov (in the context of diffusions)”.

Theorem 1.5.4 shows that under suitable regularity conditions the Chung–Fuchs theorem extends to spatially non-homogeneous random walks with zero drift and a fixed increment covariance. The proof of Theorem 1.5.4
is given as Example 3.5.4. A stronger version of Theorem 1.5.4 is Theorem 4.1.3 proved in Chapter 4, which relaxes the uniform bound on the increments of the walk to a moments condition.

Section 1.6

We do not claim that Theorem 1.6.1 is new, but we could not find a reference. Thus we present the proof here.

Proof of Theorem 1.6.1. Suppose first that \( \mu \neq 0 \). Then the strong law of large numbers shows that \( n^{-1}\xi_n \to \mu \), a.s., and \( n^{-1}\|\xi_n\| \to \|\mu\| \), a.s. The statement in part (i) now follows, since

\[
\lim_{n \to \infty} \hat{\xi}_n = \lim_{n \to \infty} \frac{n^{-1}\xi_n}{n^{-1}\|\xi_n\|} = \hat{\mu}, \text{ a.s.}
\]

On the other hand, suppose \( \mu = 0 \). Observe that, by exchangeability, the Hewitt–Savage zero-one law (see e.g. \([150, p. 53]\)) shows that \( P[\lim_{n \to \infty} \hat{\xi}_n \text{ exists}] \in \{0, 1\} \), and if the limit exists with probability 1, it is a.s. degenerate.

Suppose, for the purpose of deriving a contradiction, that

\[
P[\lim_{n \to \infty} \hat{\xi}_n \text{ exists}] > 0,
\]

in which case the probability must be equal to 1. Since \( \hat{\xi}_n \in S^{d-1} \cup \{0\} \), and the limit can be 0 only if \( \xi_n = 0 \) for all but finitely many \( n \), which is impossible since \( P[\theta_0 = 0] < 1 \), it follows that \( P[\lim_{n \to \infty} \xi_n = e] = 1 \) for some deterministic \( e \in S^{d-1} \). Then

\[
\lim_{n \to \infty} \frac{e \cdot \xi_n}{\|\xi_n\|} = \|e\|^2 = 1, \text{ a.s.,}
\]

which implies that \( \liminf_{n \to \infty} e \cdot \xi_n \geq 0 \), a.s. But \( e \cdot \xi_n = \sum_{k=0}^{n-1} e \cdot \theta_k \) is a one-dimensional random walk with mean increment \( E[e \cdot \theta_0] = e \cdot \mu = 0 \), which means that \( \liminf_{n \to \infty} e \cdot \xi_n = -\infty \), a.s. (see e.g. \([150, p. 167]\)). This gives the desired contradiction, and yields the statement in part (ii).

Section 1.7

Centrally biased random walks are discussed in detail in Chapter 4; refer to the notes to that chapter for full bibliographic details. The recurrence classification, Theorem 1.7.1, is contained in Theorem 4.4.8. The result on
angular asymptotics, Theorem 1.7.2, is contained in Theorem 4.4.5. The result on the limiting direction in the supercritical case, Theorem 1.7.3, is contained in Theorem 4.4.2.
Chapter 1. Introduction
Chapter 2

Semimartingale approach
and Markov chains

2.1 Definitions

We assume that the reader is familiar with the basic concepts of probability
theory, including convergence of random variables and uniform integrability.

In this section we recall some basic definitions related to Markov pro-
cesses with discrete time and (sub-, super-)martingales, and introduce some
of our notational conventions.

Definition 2.1.1 (Basic concepts for discrete-time stochastic processes).
In the following, all random variables are defined on a common probability
space $(\Omega, \mathcal{F}, P)$. Random variables take values usually in $\mathbb{R}$ or $\mathbb{R}^d$, possibly
extended to allow components $\pm \infty$ to handle limiting variables. We write $E$
for expectation corresponding to $P$, which will be applied to real-valued
random variables, or componentwise in the case of random vectors in $\mathbb{R}^d$.

- We take as a state space a measurable space $(\Sigma, \mathcal{E})$. For technical
  reasons (existence of Markov transition functions), one requires that
  $(\Sigma, \mathcal{E})$ is a Borel space; we will always assume that any state space is
  Borel. In this book, typically $\Sigma \subseteq \mathbb{R}^d$; often, $\Sigma \subseteq \mathbb{R}^d$ is countable. In
  these cases the $\sigma$-field $\mathcal{E}$ is the obvious one (i.e., the appropriate Borel
  sets $\mathcal{B}(\Sigma) = \{ A \cap \Sigma : A \in \mathcal{B}(\mathbb{R}^d) \}$, which reduces to the power set $2^\Sigma$
in the countable case), and will usually not be mentioned explicitly; then we use the term ‘state space’ to refer simply to the set $\Sigma$.

- A discrete-time stochastic process with state space $(\Sigma, \mathcal{E})$ is sequence
  of random variables $X_n : (\Omega, \mathcal{F}) \to (\Sigma, \mathcal{E})$ indexed by $n \in \mathbb{Z}_+$. We
write such sequences as \((X_n, n \geq 0)\), with the understanding that the
time index \(n\) is always an integer.

- A filtration (of \(\mathcal{F}\)) is a sequence of \(\sigma\)-fields \((\mathcal{F}_n, n \geq 0)\) such that
  \(\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}\) for all \(n \geq 0\). We set \(\mathcal{F}_\infty := \sigma(\bigcup_{n \geq 0} \mathcal{F}_n) \subseteq \mathcal{F}\).

- A stochastic process \((X_n, n \geq 0)\) is adapted to a filtration \((\mathcal{F}_n, n \geq 0)\)
  if \(X_n\) is \(\mathcal{F}_n\)-measurable for all \(n \in \mathbb{Z}_+\).

- It is convenient, for definiteness, to introduce the notation \(X_\infty := \limsup_{n \to \infty} X_n\), computed componentwise if \(X_n \in \mathbb{R}^d\), so that \(X_\infty\) has a meaning (as an extended random variable in \(\mathbb{R}^d\)) even when the limit does not exist. Note that if \((X_n, n \geq 0)\) is adapted to \((\mathcal{F}_n, n \geq 0)\),
  then \(X_\infty\) is \(\mathcal{F}_\infty\)-measurable.

- For a (possibly infinite) random variable \(\tau \in \mathbb{Z}_+\), the random variable
  \(X_\tau\) is (as the notation suggests) equal to \(X_n\) on \(\{\tau = n\}\) for finite
  \(n \in \mathbb{Z}_+\) and equal to \(X_\infty := \limsup_{n \to \infty} X_n\) on \(\{\tau = \infty\}\).

- A (possibly infinite) random variable \(\tau \in \mathbb{Z}_+\) is a stopping time with respect to a filtration
  \((\mathcal{F}_n, n \geq 0)\) if \(\{\tau = n\} \in \mathcal{F}_n\) for all \(n \geq 0\) (and \(n = \infty\)). If \(\tau\) is a stopping time, then so are \(\tau + 1, \tau + 2\), etc.

- If \(\tau\) is a stopping time, the corresponding \(\sigma\)-field \(\mathcal{F}_\tau\) consists of all
  events \(A \in \mathcal{F}_\infty\) such that \(A \cap \{\tau \leq n\} \in \mathcal{F}_n\) for all \(n \geq 0\) (and \(n = \infty\)). Note that \(\mathcal{F}_\tau \subseteq \mathcal{F}_\infty\); events in \(\mathcal{F}_\tau\) include \(\{\tau = \infty\}\), as well
  as \(\{X_\tau \in B\}\) for \(B \in \mathcal{E}\), and so on.

For any process \((X_n, n \geq 0)\), one can define a (minimal) filtration to
which this process is adapted via \(\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)\), to give the so-called natural filtration.

If \((X_n, n \geq 0)\) is a process adapted to a filtration \((\mathcal{F}_n, n \geq 0)\) of \(\mathcal{F}\) and \(\tau\)
is a stopping time, then, from the definitions above, \((\mathcal{F}_n, n \geq 0)\) is also a
filtration of \(\mathcal{F}\), to which the sequence \((X_n, n \geq \tau)\) is adapted, where on the
event \(\{\tau = \infty\}\) the latter notation stands for \((X_\infty, X_\infty, \ldots)\).

To keep the notation concise, we will frequently write \(X_n\) and \(\mathcal{F}_n\) instead of
\((X_n, n \geq 0)\) and \((\mathcal{F}_n, n \geq 0)\) (and so on) when no confusion will arise.

**Definition 2.1.2** (Martingales, submartingales, supermartingales). A real-valued stochastic process \(X_n\) adapted to a filtration \(\mathcal{F}_n\) is a martingale (with respect to the given filtration) if, for all \(n \geq 0\),

(i) \(\mathbb{E}|X_n| < \infty\), and
2.1. Definitions

(ii) $\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] = 0.$

If in (ii) “=” is replaced by “$\geq$” (respectively, “$\leq$”), then $X_n$ is called a submartingale (respectively, supermartingale).

We use the term semimartingale in a less precise way, to encompass not only martingales, submartingales and supermartingales, but also adapted processes satisfying ‘drift’ conditions of a similar kind to (ii) above; perhaps, for instance, only locally, i.e., on $\{X_n \notin A\}$ for some measurable set $A$.

Another measure to avoid notational clutter is that we will frequently drop ‘almost surely’ qualifiers in statements asserting equality or inequality of random variables, as in part (ii) above. Moreover, we often will not mention explicitly with respect to which filtration the process is a (sub-, super-)martingale, since normally there will be no possibility of ambiguity.

Consider an $\mathcal{F}_n$-adapted stochastic process $X_n$ taking values in a state space $(\Sigma, \mathcal{E})$. For $A \in \mathcal{E}$ we define

$$\tau_A := \min\{n \geq 0 : X_n \in A\},$$

and

$$\tau_A^+ := \min\{n \geq 1 : X_n \in A\};$$

we may refer to either $\tau_A$ or $\tau_A^+$ as the hitting time of $A$ (also called the passage time into $A$). Then $\tau_A^+$ and $\tau_A$ are $\mathcal{F}_n$-stopping times, given that $A \in \mathcal{E}$ (see e.g. [150, p. 123]).

In the case where $\Sigma \subseteq \mathbb{R}$ we will also frequently use notation for the hitting times of half-lines; namely, for $x \in \mathbb{R}$ we set

$$\rho_x := \tau_{\Sigma \cap [x, \infty)} = \min\{n \geq 0 : X_n \geq x\},$$

$$\lambda_x := \tau_{\Sigma \cap (-\infty, x]} = \min\{n \geq 0 : X_n \leq x\},$$

for right and left passage times, and, for $x \geq 0$,

$$\sigma_x := \tau_{\Sigma \setminus (-x, x)} = \min\{n \geq 0 : |X_n| \geq x\} = \lambda_x \wedge \rho_x.$$  

In the case where $\Sigma \subseteq \mathbb{R}_+$, obviously $\rho_x = \sigma_x$ for $x \geq 0$; in this case our preference is to use the notation $\sigma_x$.

Next we recall some fundamental definitions on Markov processes.

**Definition 2.1.3** (Markov chains).

- An $\mathcal{F}_n$-adapted process $X_n$ taking values in a state space $(\Sigma, \mathcal{E})$ is a **Markov chain** if, for any $B \in \mathcal{E}$, any $n \geq 0$, and any $m \geq 1$,

  $$\mathbb{P}[X_{n+m} \in B \mid \mathcal{F}_n] = \mathbb{P}[X_{n+m} \in B \mid X_n], \text{ a.s.} \quad (2.1)$$

  This is the **Markov property**.
• If there is no dependence on \(n\) in (2.1), the Markov chain is **homogeneous in time** (or **time-homogeneous**). Unless explicitly stated otherwise, all Markov chains considered in this book are assumed to be homogeneous in time. In this case, the Markov property (2.1) reads

\[
P[X_{n+m} \in B \mid \mathcal{F}_n] = P_m(X_n, B), \text{ a.s.,} \tag{2.2}
\]

where \(P_m : \Sigma \times \mathcal{E} \to [0, 1]\) is the \(m\)-step Markov transition function, which satisfies the Chapman–Kolmogorov relation \(P_{n+m}(x, B) = \int_{\Sigma} P_n(x, dy) P_m(y, B)\).

• We use the shorthand \(P_x[\cdot] = P[\cdot \mid X_0 = x]\) and \(E_x[\cdot] = E[\cdot \mid X_0 = x]\) for probability and expectation for the time-homogeneous Markov chain starting from initial state \(x \in \Sigma\).

• If the state space \(\Sigma\) is countable (i.e., finite, or countably infinite), we call \(X_n\) a **countable Markov chain**. For a countable state space \(\Sigma\), we always suppose that \(\mathcal{E} = 2^\Sigma\), the power set of \(\Sigma\).

• If the Markov chain is time-homogeneous and countable, we write \(p(x, y) := P[X_1 = y \mid X_0 = x] = P_1(x, \{y\})\) for the one-step transition probabilities of the Markov chain.

• A time-homogeneous, countable Markov chain is **irreducible** if for all \(x, y \in \Sigma\) there exists \(m = m(x, y) \in \mathbb{N}\) such that \(P_x[X_m = y] > 0\).

For an uncountable state space \(\Sigma\), notions of irreducibility are more technical, and not discussed in this book. For our purposes, if we refer to an **irreducible Markov chain**, it is implicit that the Markov chain is time-homogeneous and countable.

• For an irreducible Markov chain, we define its **period** as the greatest common divisor of \(\{n \in \mathbb{N} : P_x[X_n = x] > 0\}\) (it is straightforward to show that it does not depend on the choice of \(x \in \Sigma\)). An irreducible Markov chain with period 1 is called **aperiodic**.

Suppose now that \(X_n\) is a countable Markov chain; recall the definitions of the hitting times \(\tau^+_A\) and \(\tau_A\), \(A \subseteq \Sigma\). For \(x \in \Sigma\), we use the notation \(\tau^+_x := \tau^+_{\{x\}}\) and \(\tau_x := \tau_{\{x\}}\) for hitting times of one-point sets. Note that for any \(x \in A\), \(P_x[\tau_A = 0] = 1\), while \(\tau^+_A \geq 1\) is then the return time to \(A\). Also note that \(P_x[\tau_A = \tau^+_A] = 1\) for all \(x \in \Sigma \setminus A\).

**Definition 2.1.4.** For a countable Markov chain \(X_n\), a state \(x \in \Sigma\) is called
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- **recurrent** if \( P_x[\tau_x^+ < \infty] = 1 \);
- **transient** if \( P_x[\tau_x^+ < \infty] < 1 \).

A recurrent state \( x \) is classified further as

- **positive recurrent** if \( E_x \tau_x^+ < \infty \);
- **null recurrent** if \( E_x \tau_x^+ = \infty \).

It is straightforward to show that the four properties in Definition 2.1.4 are **class properties**, which entails the following statement.

**Proposition 2.1.5.** For an irreducible Markov chain, if a state \( x \in \Sigma \) is recurrent (respectively, positive recurrent, null recurrent, transient) then all states in \( \Sigma \) are recurrent (respectively, positive recurrent, null recurrent, transient).

By the above fact, it is legitimate to call an irreducible Markov chain itself recurrent (positive recurrent, null recurrent, transient).

A basic fact that we need about positive recurrent Markov chains is the following. As with several other standard results in this chapter, we state the result without proof; see the bibliographical notes at the end of the chapter for references to the literature.

**Theorem 2.1.6.** Suppose that \( X_n \) is an irreducible, positive recurrent Markov chain. Define the stationary measure \((\pi(x), x \in \Sigma)\) by \( \pi(x) = (E_x \tau_x^+)^{-1} \).

(i) It holds that \( \sum_{x \in \Sigma} \pi(x) = 1 \) (so \( \pi \) defines a probability measure on \( \Sigma \)), \( \pi(x) > 0 \) for all \( x \in \Sigma \), and

\[
\pi(x) = \sum_{y \in \Sigma} \pi(y)p(y,x), \text{ for all } x \in \Sigma.
\]

(This implies that if \( X_0 \) is distributed according to the stationary measure \( \pi \), then so is \( X_n \) for all \( n \).)

(ii) If the Markov chain is also aperiodic, then \( \mathbb{P}_x[X_n = y] \to \pi(y) \) as \( n \to \infty \) for all \( x, y \in \Sigma \). For periodic chains, convergence also takes place, but only in the Cesàro sense: \( n^{-1} \sum_{k=0}^{n-1} \mathbf{1}\{X_k = y\} \to \pi(y) \).

**Example 2.1.7.** Let \( p \in (0,1) \), and assume that \( Z_1, Z_2, Z_3, \ldots \) are i.i.d. random variables with \( \mathbb{P}[Z_1 = 1] = 1 - \mathbb{P}[Z_1 = -1] = p \). Let \( \mathcal{F}_0 = \{\emptyset, \Omega\} \), the trivial \( \sigma \)-field, and \( \mathcal{F}_n = \sigma(Z_1, \ldots, Z_n) \) for \( n \geq 1 \). We set \( S_0 = 0 \).
and \( S_n = Z_1 + \cdots + Z_n \) for \( n \geq 1 \); the \( \mathcal{F}_n \)-adapted process \( S_n \) is a one-dimensional simple random walk (SRW) with parameter \( p \). In the case \( p = \frac{1}{2} \), we say that \( S_n \) is (one-dimensional) symmetric SRW, or sometimes just SRW, without any adjectives.

The following observations are straightforward.

- \( S_n \) is a Markov chain with countable state space \( \mathbb{Z} \), and transition probabilities \( p(x, x+1) = 1 - p(x, x-1) = p \) for \( x \in \mathbb{Z} \);

- this Markov chain is irreducible and periodic with period 2;

- with respect to the filtration \( \mathcal{F}_n \), the process \( S_n \) is a supermartingale for \( p \leq \frac{1}{2} \), a submartingale for \( p \geq \frac{1}{2} \), and a martingale for \( p = \frac{1}{2} \);

- for the symmetric SRW, \( S_n^2 \) is a submartingale, and \( (S_n^2 - n) \) is a martingale (in fact, \( S_n^2 = (S_n^2 - n) + n \) is the Doob decomposition of \( S_n^2 \); see Theorem 2.3.1 below);

- \( (1 - p)(\frac{1-p}{p})^{x-1} + p(\frac{1-p}{p})^{x+1} = (\frac{1-p}{p})^x \) for all \( x \in \mathbb{Z} \), so the process \( (\frac{1-p}{p})S_n \) is a martingale (this fact, of course, has non-trivial consequences only for \( p \neq \frac{1}{2} \)). \( \triangle \)

**Example 2.1.8.** Now, consider a modification \( \hat{S}_n \) of the process of the previous example, which lives on \( \Sigma = \mathbb{Z}_+ := \{0, 1, 2, \ldots\} \). Again, fix a parameter \( p \in (0, 1) \); for \( x > 0 \) we keep the same transition probabilities \( p(x, x+1) = 1 - p(x, x-1) = p \), and we set \( p(0, 1) = 1 \); we will call this process the (reflected) simple random walk on the half-line. Then, for \( p \geq \frac{1}{2} \) this process is still a submartingale, but one loses the supermartingale property for \( p \leq \frac{1}{2} \) (as well as the martingale property for \( p = \frac{1}{2} \)) because of the reflection at 0. However, these properties are readily recovered if we consider the stopped process \( \hat{S}_n \wedge \tau_0 \) instead of \( \hat{S}_n \) itself. \( \triangle \)

When the desired (sub-, super-)martingale property breaks down in some region, often it is possible (and useful) to fix it by considering a modification of the original process by means of a suitable stopping time.

The notions of recurrence and transience given in Definition 2.1.4 depend only on the structure of \( \Sigma \) as a set (i.e., are purely ‘topological’). In the context of this book, geometric consequences of these notions are important; we are usually concerned with a given embedding of \( \Sigma \) into Euclidean space \( \mathbb{R}^d \).
A set $A \subseteq \mathbb{R}^d$ is \textit{bounded} if $\sup_{x \in A} \|x\| < \infty$; otherwise it is \textit{unbounded} (recall the convention $\sup \emptyset = 0$). A set $A \subseteq \mathbb{R}^d$ is \textit{locally finite} if $\#(A \cap B) < \infty$ for all bounded $B \subset \mathbb{R}^d$. Of course, a locally finite set is countable, and an unbounded locally finite set is countably infinite.

We state here some basic results for Markov chains, which follow from more general results whose proofs are presented later, in Section 3.6.

**Theorem 2.1.9.** Let $X_n$ be an irreducible time-homogeneous Markov chain on an unbounded, countable state space $\Sigma \subset \mathbb{R}^d$.

(i) If $X_n$ is recurrent, then, a.s.,
\[ \liminf_{n \to \infty} \|X_n\| = \inf_{x \in \Sigma} \|x\|, \text{ and } \limsup_{n \to \infty} \|X_n\| = \infty. \]

(ii) If $\Sigma$ is locally finite and $X_n$ is transient, then $\lim_{n \to \infty} \|X_n\| = \infty$, a.s.

Without local finiteness of $\Sigma$, part (ii) of Theorem 2.1.9 may fail: see the bibliographical notes at the end of this chapter.

Contained in Theorem 2.1.9 is the following basic fact.

**Corollary 2.1.10.** Let $X_n$ be an irreducible time-homogeneous Markov chain on an unbounded, locally finite state space $\Sigma \subset \mathbb{R}^d$. Then $\limsup_{n \to \infty} \|X_n\| = \infty$, a.s.

We end this section with a technical lemma, which the standard notation suggests is obvious, and will often be used without comment.

**Lemma 2.1.11.** Let $\mathcal{F}_n$ be a filtration and $\tau$ a stopping time.

(i) If $X$ is an integrable random variable, then for any $n \in \mathbb{Z}_+$,
\[ \mathbb{E}[X \mid \mathcal{F}_\tau]1\{\tau = n\} = \mathbb{E}[X \mid \mathcal{F}_n]1\{\tau = n\} = \mathbb{E}[X1\{\tau = n\} \mid \mathcal{F}_n]. \]

(ii) If $X_n$ is an $\mathcal{F}_n$-adapted process such that $\lim_{n \to \infty} X_n = X_\infty$ exists a.s., then $\lim_{n \to \infty} X_n \wedge \tau = X_\tau$, a.s.

\textit{Proof.} (i) Fix $n \in \mathbb{Z}_+$. Write $\xi = \mathbb{E}[X \mid \mathcal{F}_n]$ and $\zeta = \mathbb{E}[X \mid \mathcal{F}_\tau]$. Note that $\xi$ and $\zeta$ are integrable, and, by definition of $\mathcal{F}_\tau$, both $\xi 1\{\tau = n\}$ and $\zeta 1\{\tau = n\}$ are both $\mathcal{F}_\tau$-measurable and $\mathcal{F}_n$-measurable. For any $A \in \mathcal{F}_\tau$, $A \cap \{\tau = n\}$ is in $\mathcal{F}_\tau$ and $\mathcal{F}_n$, so that, by definition of conditional expectation,
\[ \mathbb{E}[\zeta 1(A \cap \{\tau = n\})] = \mathbb{E}[X \mathbf{1}(A \cap \{\tau = n\})] = \mathbb{E}[\xi 1(A \cap \{\tau = n\})]. \]
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Hence \( E[(\zeta - \xi) \mathbf{1}_{\{\tau = n\}}] = 0 \). Choosing \( A = \{(\zeta - \xi) \mathbf{1}_{\{\tau = n\}} > 0\} \in \mathcal{F}_\tau \) we deduce that \( E[(\zeta - \xi)^+ \mathbf{1}_{\{\tau = n\}}] = 0 \). A similar argument gives \( E[(\zeta - \xi)^- \mathbf{1}_{\{\tau = n\}}] = 0 \), and hence \( |\zeta - \xi| \mathbf{1}_{\{\tau = n\}} = 0 \), a.s. Exactly the same proof works when \( n = \infty \), but it is worth reading through the argument again in that case.

(ii) Suppose \( \lim_{n \to \infty} X_n = X_\infty \). Then \( \lim_{n \to \infty} X_n \mathbf{1}_{\{\tau > n\}} = X_\infty \mathbf{1}_{\{\tau = \infty\}} \), while \( \lim_{n \to \infty} X_\tau \mathbf{1}_{\{\tau \leq n\}} = X_\tau \mathbf{1}_{\{\tau < \infty\}} \). In other words,

\[
\lim_{n \to \infty} X_{n \wedge \tau} = X_\infty \mathbf{1}_{\{\tau = \infty\}} + X_\tau \mathbf{1}_{\{\tau < \infty\}} = X_\tau,
\]

as claimed.

The point of Lemma 2.1.11(i) will become clearer later on; it allows us to extend statements like \( E[f(X_{n+1}, \ldots, X_{n+k}) \mid \mathcal{F}_n] \leq g(X_n) \) (for all \( n \)) to the ‘stopped’ version \( E[f(X_{\tau+1}, \ldots, X_{\tau+k}) \mid \mathcal{F}_\tau] \leq g(X_\tau) \) for finite stopping times \( \tau \). Here is one example of an application.

Example 2.1.12. Let \( X_n \) be an \( \mathcal{F}_n \)-adapted time-homogeneous Markov chain on state space \((\Sigma, \mathcal{E})\). Let \( \tau \) be a stopping time. Then, on \( \{\tau < \infty\} \),

\[
\mathbb{P}[X_{\tau+m} \in B \mid \mathcal{F}_\tau] = \sum_{n=0}^{\infty} \mathbb{P}[X_{\tau+m} \in B \mid \mathcal{F}_\tau] \mathbf{1}_{\{\tau = n\}}
\]

\[= \sum_{n=0}^{\infty} \mathbb{P}[X_{n+m} \in B \mid \mathcal{F}_n] \mathbf{1}_{\{\tau = n\}},\]

\[= \sum_{n=0}^{\infty} P_m(X_n, B) \mathbf{1}_{\{\tau = n\}},\]

by Lemma 2.1.11(i) and then (2.2). The fact that the Markov property (2.2) extends to stopping times as

\[
\mathbb{P}[X_{\tau+m} \in B \mid \mathcal{F}_\tau] = P_m(X_\tau, B), \text{ on } \{\tau < \infty\},
\]

is the strong Markov property. \( \triangle \)

2.2 An introductory example

In this section we briefly describe a simple example that is rich enough to display a range of interesting behaviour. We will refer back to this example from time to time to illustrate the subsequent discussions in this chapter. This example, and many of the classical methods of its analysis, are likely
2.2. An introductory example

to be familiar to the reader; we present some of the standard results, but will give semimartingale-style proofs here and throughout this chapter to illustrate our approach.

Our example is a nearest-neighbour (i.e., simple) random walk on $\mathbb{Z}_+$, $(X_n, n \geq 0)$. For $x \geq 1$, we set
\[
\mathbb{P}[X_{n+1} = x - 1 \mid X_n = x] = p_x,
\]
\[
\mathbb{P}[X_{n+1} = x + 1 \mid X_n = x] = 1 - p_x =: q_x,
\]
and $\mathbb{P}[X_{n+1} = 0 \mid X_n = 0] = p_0$, $\mathbb{P}[X_{n+1} = 1 \mid X_n = 0] = 1 - p_0 =: q_0$.

Assuming $p_x \in (0, 1)$ for all $x \in \mathbb{Z}_+$, $X_n$ is an irreducible, aperiodic Markov chain on the locally finite state space $\mathbb{Z}_+$; in particular, Corollary 2.1.10 shows that $\limsup_{n \to \infty} X_n = +\infty$, a.s.

For $x \in \mathbb{Z}_+$ we make the following definitions, recalling that an empty sum in 0 and an empty product is 1.

\[d_x := \prod_{y=0}^{x-1} \frac{p_y}{q_y}; \quad (2.4)\]

\[u_x := \sum_{y=0}^{x} q_{x-y-1} \prod_{z=x-y+1}^{x} \frac{p_z}{q_z} = \frac{1}{q_x} + \frac{p_x}{q_x q_{x-1} q_{x-2}} + \cdots + \frac{p_x p_{x-1} \cdots p_1}{q_x q_{x-1} \cdots q_1 q_0}; \quad (2.5)\]

\[v_x := \sum_{y=x}^{\infty} \prod_{z=x}^{y-1} \frac{q_z}{p_z} = \frac{1}{p_x} + \frac{q_x}{p_x p_{x+1}} + \cdots; \quad (2.6)\]

in particular, $d_0 = 1$ and $u_0 = 1/q_0$. Since $p_x \in (0, 1)$ for all $x$, $d_x$ and $u_x$ are finite for all $x$. However, $v_x < \infty$ if and only if
\[
\sum_{y=x}^{\infty} p_y q_{y-1} < \infty \iff \sum_{y=x}^{\infty} (d_y^{-1} + d_{y+1}^{-1}) < \infty,
\]
using the fact that $p_y^{-1} q_{y-1} = d_y^{-1} + d_{y+1}^{-1}$. Hence $v_x < \infty$ if and only if
\[
\sum_{x=1}^{\infty} d_x^{-1} < \infty.
\]

Now for $x \in \mathbb{Z}_+$ define
\[h(x) := \sum_{y=0}^{x} d_y; \quad (2.7)\]

then $h : \mathbb{Z}_+ \to \mathbb{R}_+$ is increasing, so $\lim_{x \to \infty} h(x) = \sum_{x=0}^{\infty} d_x$ exists in $\mathbb{R}_+$.

The function $h$ has a classical interpretation in terms of hitting probabilities. Recall that $\tau_x := \min\{n \geq 0 : X_n = x\}$; here $x \in \mathbb{Z}_+$. Note that $X_{\tau_x} = x$ a.s. and $\tau_x \neq \tau_y$ for any $x \neq y$.  

Lemma 2.2.1. Let $a, b, x$ be integers with $0 \leq a < b$ and $a \leq x \leq b$. Then
\[
P_x[\tau_a < \tau_b] = \frac{h(x) - h(b)}{h(a) - h(b)}.
\]

The classical proof of Lemma 2.2.1 goes via solving a difference equation. We will present a different argument in Example 2.4.2 below, based on the following fact.

Lemma 2.2.2. For any $n \in \mathbb{Z}_+$, and any $x \in \mathbb{Z}_+$,
\[
\mathbb{E}[h(X_{n+1}) - h(X_n) \mid X_n = x] = q_01\{x = 0\}.
\]

Proof. It suffices to take $n = 0$. For $x \geq 1$,
\[
\mathbb{E}_x[h(X_1) - h(X_0)] = p_xh(x-1) + q_xh(x+1) - h(x) = q_xd_{x+1} - p_xd_x = 0.
\]

On the other hand, $\mathbb{E}_0[h(X_1) - h(X_0)] = q_0(h(1) - h(0)) = q_0d_0 = q_0$. \hfill \Box

For $x \in \mathbb{Z}_+$, set
\[
t(x) := \sum_{y=0}^{x-1} u_y, \quad \text{and} \quad r(x) := \sum_{y=1}^{x} v_y,
\]
where $u_y$ and $v_y$ are given by (2.5) and (2.6). By the remarks above, $t(x) < \infty$ for all $x$, while $r(x) < \infty$ for $x \geq 1$ if and only if $\sum_{y=1}^{\infty} d_y^{-1} < \infty$.

Lemma 2.2.3. (i) For $x \in \mathbb{Z}_+$, $\mathbb{E}_0 \tau_x = t(x)$.

(ii) For $x \in \mathbb{Z}_+$, $\mathbb{E}_x \tau_0 = r(x)$.

We give a proof of Lemma 2.2.3 based on the following semimartingale properties of the functions $r$ and $t$.

Lemma 2.2.4. (i) For any $n \in \mathbb{Z}_+$ and any $x \in \mathbb{Z}_+$,
\[
\mathbb{E}[t(X_{n+1}) - t(X_n) \mid X_n = x] = 1.
\]

(ii) Suppose that $\sum_{x=1}^{\infty} d_x^{-1} < \infty$. Then, for any $n \in \mathbb{Z}_+$ and any $x \in \mathbb{Z}_+$,
\[
\mathbb{E}[r(X_{n+1}) - r(X_n) \mid X_n = x] = \begin{cases} -1, & \text{if } x \geq 1, \\ q_0v_1, & \text{if } x = 0. \end{cases}
\]
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Proof. It suffices to take \( n = 0 \). For \( x \geq 1 \), we have

\[
\mathbb{E}_x [t(X_1) - t(X_0)] = p_x (t(x - 1) - t(x)) + q_x (t(x + 1) - t(x)) = q_x u_x - p_x u_{x-1} = 1,
\]

by (2.5). Also,

\[
\mathbb{E}_0 [t(X_1) - t(X_0)] = q_0 t(1) = 1,
\]

since \( t(1) = u_0 = 1/q_0 \). This gives (i). Similarly, for \( x \geq 1 \),

\[
\mathbb{E}_x [r(X_1) - r(X_0)] = p_x (r(x - 1) - r(x)) + q_x (r(x + 1) - r(x)) = q_x v_{x+1} - p_x v_x = -1,
\]

by (2.6). Also,

\[
\mathbb{E}_0 [r(X_1) - r(X_0)] = q_0 (r(1) - r(0)) = q_0 v_1,
\]

which completes the proof of (ii). \( \square \)

Proof of Lemma 2.2.3. Let \( \mathcal{F}_n = \sigma (X_0, \ldots, X_n) \). For part (i), let \( Y_n = t(X_n \wedge \tau_x) \). By Lemma 2.2.4(i), for \( m \geq 0 \),

\[
\mathbb{E}[Y_{m+1} - Y_m | \mathcal{F}_m] = 1 \{ \tau_x > m \}.
\]

Taking expectations conditioned on \( X_0 = 0 \), summing over \( m \) from 0 to \( n - 1 \), and then letting \( n \to \infty \) gives

\[
\mathbb{E}_0 \tau_x = \lim_{n \to \infty} \sum_{m=0}^{n-1} \mathbb{P}_0 [\tau_x > m] = \lim_{n \to \infty} \mathbb{E}_0 t(X_n \wedge \tau_x),
\]

using the fact that \( t(0) = 0 \). Here \( t(X_n \wedge \tau_x) \) is uniformly bounded and converges a.s. to \( t(X_{\tau_x}) = t(x) \), so by the bounded convergence theorem we get part (i).

For part (ii), first suppose that \( \sum_{x=1}^{\infty} d_x^{-1} < \infty \), so that \( r(x) \leq r(\infty) < \infty \) for all \( x \). Let \( Y_n = r(X_n \wedge \tau_0) \). Similarly to the preceding argument, Lemma 2.2.4(ii) shows that \( \mathbb{E}_x [n \wedge \tau_0] = \mathbb{E}_x Y_0 - \mathbb{E}_x Y_n \). Here \( 0 \leq Y_n \leq r(\infty) < \infty \), so \( Y_n \) is uniformly bounded and converges to \( r(0) = 0 \), so again we may apply the bounded convergence theorem to deduce that \( \mathbb{E}_x \tau_0 = r(x) \). The fact that \( \mathbb{E}_x \tau_0 = \infty \) when \( \sum_{x=1}^{\infty} d_x^{-1} = \infty \) is easiest to see from the classical difference equation argument. \( \square \)

Now we can state the basic classification result, in terms of the quantities \( d_x \) defined at (2.4).
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Theorem 2.2.5.  (i) $X_n$ is recurrent if and only if $\sum_{x=1}^{\infty} d_x = \infty$;

(ii) $X_n$ is positive recurrent if and only if $\sum_{x=1}^{\infty} d_x^{-1} < \infty$.

Proof. There are numerous ways to prove Theorem 2.2.5; we describe an instructive approach. Suppose $X_0 = 0$. Since $\limsup_{n \to \infty} X_n = \infty$ a.s., we have $\tau_x < \infty$ a.s. for all $x \in \mathbb{Z}_+$. Also, since $|X_{n+1} - X_n| \leq 1$, we have $\tau_{x+1} \geq \tau_x + 1$. Hence $\tau_x$ is strictly increasing in $x$, with $\lim_{x \to \infty} \tau_x = \infty$. Hence $\tau_0^+ = \infty$ if and only if $\tau_0^+ > \tau_x$ for all $x \in \mathbb{N}$. Since $\{\tau_0 > \tau_x\} \supseteq \{\tau_0 > \tau_x^+\}$, we have

$$P_0[\tau_0^+ = \infty] = q_0 P_1[\cap_{x \geq 2}\{\tau_0 > \tau_x\}] = q_0 \lim_{x \to \infty} P_1[\tau_x < \tau_0],$$

by continuity of probability along monotone limits. Hence by Lemma 2.2.1,

$$P_0[\tau_0^+ = \infty] = q_0 \lim_{x \to \infty} \frac{h(1) - h(0)}{h(x) - h(0)},$$

which is 0 if and only if $\lim_{x \to \infty} h(x) = \infty$, giving part (i) of Theorem 2.2.5. Part (ii) of Theorem 2.2.5 can be derived by direct computation of the stationary measure $\pi$, using the reversibility of the random walk, or by the hitting time result Lemma 2.2.3, since positive recurrence is equivalent to finiteness of $E_0 \tau_0^+ = E_0 \tau_1 + E_1 \tau_0$. \qed

After developing some of the necessary technology, we will give Lyapunov function proofs of parts of Theorem 2.2.5 later, in Examples 2.5.5 and 2.5.13.

2.3 Fundamental semimartingale facts

In this section we state (often, without proof) some basic facts related to (sub-, super-)martingales that we will need in this book. We also give a number of examples to illustrate in a simple way some of the basic ideas of applications of these results.

Recall that the underlying setting is a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_n, n \geq 0)$. Whenever we refer without further qualification to a martingale, submartingale, or supermartingale $X_n$, it is implicit that $X_n$ takes values in $\mathbb{R}$.

Theorem 2.3.1 (Doob decomposition). A submartingale $X_n$ has a unique representation $X_n = M_n + A_n$, $n \geq 0$, where $M_n$ is a martingale, and $A_n$ is a non-decreasing process such that $A_0 = 0$ and $A_n$ is $\mathcal{F}_{n-1}$-measurable for $n \geq 1$ (that is, $A_n$ is previsible).
Recall that a random variable $X$ is integrable if $\mathbb{E}|X| < \infty$ and square-integrable if $\mathbb{E}[X^2] < \infty$; a process $(X_n, n \geq 0)$ is square-integrable if $\mathbb{E}[X_n^2] < \infty$ for all $n \geq 0$.

**Theorem 2.3.2** (Orthogonality of martingale increments). Let $X_n$ be a square-integrable martingale. If $m \leq n$ and $Y$ is $\mathcal{F}_m$-measurable with finite second moment, then $\mathbb{E}[(X_n - X_m)Y] = 0$.

Observe that Theorem 2.3.2 implies that $\mathbb{E}[(X_m+1 - X_m)(X_{k+1} - X_k)] = 0$ for $k \neq m$ for any square-integrable martingale $X_n$ (thus justifying its name).

**Theorem 2.3.3** (Martingales and convexity). (i) If $X_n$ is a martingale and $g: \mathbb{R} \to \mathbb{R}$ is a convex function such that $\mathbb{E}|g(X_n)| < \infty$ for all $n$, then $g(X_n)$ is a submartingale, with respect to the same filtration.

(ii) If $X_n$ is a submartingale and $g: \mathbb{R} \to \mathbb{R}$ is a non-decreasing convex function such that $\mathbb{E}|g(X_n)| < \infty$ for all $n$, then $g(X_n)$ is a submartingale, with respect to the same filtration.

As an immediate corollary, we obtain that if $X_n$ is a square-integrable martingale, then $X_n^2$ is a submartingale.

A fundamental result is the following martingale convergence theorem.

**Theorem 2.3.4** (Martingale convergence theorem). Assume that $X_n$ is a submartingale such that $\sup_n \mathbb{E}[X_n^+] < \infty$. Then there is an integrable random variable $X$ such that $X_n \to X$ a.s. as $n \to \infty$.

**Remarks 2.3.5.** (a) Under the hypotheses of Theorem 2.3.4, the sequence $\mathbb{E}X_n$ is non-decreasing (by the submartingale property) and bounded above by $\sup_n \mathbb{E}[X_n^+]$, so $\lim_{n \to \infty} \mathbb{E}X_n$ exists and is finite; however, it is not necessarily equal to $\mathbb{E}X$.

(b) Note that $X_n^- = X_n^+ - X_n$ so that $\mathbb{E}[X_n^-] \leq \mathbb{E}[X_n^+] - \mathbb{E}X_0$, by the submartingale property, so that the hypothesis $\sup_n \mathbb{E}[X_n^+] < \infty$ actually implies that $\sup_n \mathbb{E}|X_n| < \infty$.

An important corollary to Theorem 2.3.4 and Fatou’s lemma is as follows.

**Theorem 2.3.6** (Convergence of non-negative supermartingales). Assume that $X_n \geq 0$ is a supermartingale. Then there is an integrable random variable $X$ such that $X_n \to X$ a.s. as $n \to \infty$, and $\mathbb{E}X \leq \mathbb{E}X_0$.

Another fundamental result that we will use extensively is the following.
Theorem 2.3.7 (Optional stopping theorem). Suppose that $\sigma \leq \tau$ are stopping times, and $X_{\tau \land n}$ is a uniformly integrable submartingale. Then $E X_\sigma \leq E X_\tau < \infty$ and $X_\sigma \leq E[X_\tau \mid F_\sigma]$ a.s.

Remarks 2.3.8. (a) From the conclusion $X_\sigma \leq E[X_\tau \mid F_\sigma]$ a.s., it follows on taking conditional expectations that $E[X_\sigma \mid F_0] \leq E[X_\tau \mid F_0]$ a.s., which is another useful form of optional stopping.

(b) If $X_n$ is a uniformly integrable submartingale and $\tau$ is any stopping time, then it can be shown that $X_{\tau \land n}$ is also uniformly integrable: see e.g. Section 5.7 of [83].

(c) Observe that two applications of Theorem 2.3.7, one with $\sigma = 0$ and one with $\tau = \infty$, show that for any uniformly integrable submartingale $X_n$ and any stopping time $\tau$, $E X_0 \leq E X_\tau \leq E X_\infty < \infty$, where $X_\infty := \lim_{n \to \infty} X_n = \lim_{n \to \infty} X_n$ exists and is integrable, by Theorem 2.3.4.

Theorem 2.3.7 has the following corollary, obtained on setting $\sigma = 0$ and using well-known sufficient conditions for uniform integrability (see e.g. Sections 4.5 and 4.7 of [83]).

Corollary 2.3.9. Let $X_n$ be a submartingale and $\tau$ a finite stopping time. For a constant $c > 0$, suppose that at least one of the following holds:

(i) $\tau \leq c$ a.s.;

(ii) $|X_{n \land \tau}| \leq c$ a.s. for all $n \geq 0$;

(iii) $E \tau < \infty$ and $E[|X_{n+1} - X_n| \mid F_n] \leq c$ a.s. for all $n \geq 0$.

Then $E X_\tau \geq E X_0$. If $X_n$ is a martingale and at least one of the above conditions (i)-(iii) holds, then $E X_\tau = E X_0$.

Remarks 2.3.10. (a) Corollary 2.3.9(i) exhibits the important fact that for any stopping time $\tau$, if $X_n$ is a (sub-, super-)martingale, then so is $X_{n \land \tau}$.

(b) Suppose that $\tau$ is a stopping time and $X_n$ is a non-negative supermartingale, so that $X_\infty = \lim_{n \to \infty} X_n$ exists by Theorem 2.3.6. Fatou’s lemma with Lemma 2.1.11(ii) shows that $E[X_\tau] \leq \liminf_{n \to \infty} E[X_{n \land \tau}] \leq E[X_0]$, which is a simple version of optional stopping, without the need for uniform integrability. A more general statement of this kind is the next result.

Theorem 2.3.11. Suppose that $X_n$ is a non-negative supermartingale and $\sigma \leq \tau$ are stopping times. Then $E X_\tau \leq E X_\sigma < \infty$ and $E[X_\tau \mid F_\sigma] \leq X_\sigma$, a.s.
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Proof. Under the conditions of the theorem, Theorem 2.3.6 shows that \( \lim_{n \to \infty} X_n = X_\infty \) exists and is integrable. Hence by Lemma 2.1.11(ii) and the (conditional) Fatou lemma, for any \( \sigma \)-field \( \mathcal{G} \subseteq \mathcal{F} \),

\[
E[X_\tau \mid \mathcal{G}] \leq \liminf_{n \to \infty} E[X_{n\wedge \tau} \mid \mathcal{G}].
\]

In particular, for any fixed \( m \geq 0 \),

\[
E[X_\tau \mid \mathcal{F}_{m\wedge \sigma}] \leq \liminf_{n \to \infty} E[X_{n\wedge \tau} \mid \mathcal{F}_{m\wedge \sigma}] \leq X_{m\wedge \sigma},
\]

by Theorem 2.3.7 applied to the uniformly integrable submartingale \( Y_n = -X_{m\wedge n}, n \geq 0 \). Now, by an application of Lemma 2.1.11(i),

\[
E[X_\tau \mid \mathcal{F}_\sigma] 1\{\sigma < \infty\} = \lim_{m \to \infty} E[X_\tau \mid \mathcal{F}_{m\wedge \sigma}] 1\{\sigma \leq m\}
\]

\[
= \lim_{m \to \infty} E[X_\tau \mid \mathcal{F}_{m\wedge \sigma}] 1\{\sigma \leq m\}
\]

\[
\leq \lim_{m \to \infty} X_{m\wedge \sigma} 1\{\sigma < \infty\} = X_\sigma 1\{\sigma < \infty\}.
\]

This gives the claim \( E[X_\tau \mid \mathcal{F}_\sigma] \leq X_\sigma \) on \( \{\sigma < \infty\} \). On \( \{\sigma = \infty\} \), \( \tau = \infty \) too and the claim is justified by the \( n = \infty \) case of Lemma 2.1.11(i):

\[
E[X_\tau \mid \mathcal{F}_\sigma] 1\{\sigma = \infty\} = E[X_\tau \mid \mathcal{F}_\infty] 1\{\sigma = \infty\} = X_\infty 1\{\sigma = \infty\}. \quad \Box
\]

Example 2.3.12. Let \( S_n \) be simple random walk on \( \mathbb{Z} \) with parameter \( p \in (0, 1) \), and, for a fixed \( x > 0 \), let \( \sigma = \sigma_x = \min\{n \geq 0 : |S_n| \geq x\} \).

First suppose \( p = \frac{1}{2} \). Then the martingale \( S_{n\wedge \sigma} \) is uniformly bounded, so \( S_{n\wedge \sigma} \) converges, by Theorem 2.3.4. Also, \( S_{n\wedge \sigma} 1\{\sigma < \infty\} \) converges (to \( S_\sigma 1\{\sigma < \infty\} \)). Hence \( S_{n\wedge \sigma} - S_{n\wedge \sigma} 1\{\sigma < \infty\} \) converges as well. But this last quantity is \( S_n 1\{\sigma = \infty\} \). Since \( |S_{n+1} - S_n| = 1 \), the only way that this convergence can happen is if \( P[\sigma < \infty] = 1 \).

If \( p \neq \frac{1}{2} \), a similar argument based on the martingale \( (1-p)S_{n\wedge \sigma} \) (see Example 2.1.7) gives the same conclusion.

This is an amusing way of showing that \( \limsup_{n \to \infty} |S_n| = +\infty \), a.s. \( \triangle \)

Example 2.3.13. Let \( X_n \) be an \( \mathcal{F}_n \)-adapted stochastic process on state space \( (\Sigma, \mathcal{E}) \), and let \( f : \Sigma \to \mathbb{R} \) be measurable. Suppose that \( f(X_n) \) is integrable for all \( n \). Then

\[
M_n = f(X_n) - f(X_0) - \sum_{m=0}^{n-1} E[f(X_{m+1}) - f(X_m) \mid \mathcal{F}_m]
\]
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is an $\mathcal{F}_n$-adapted martingale. Hence, for any stopping time $\tau$,

$$
\mathbb{E} f(X_{n \wedge \tau}) = \mathbb{E} f(X_0) + \mathbb{E} \left\{ \sum_{m=0}^{(n \wedge \tau) - 1} \mathbb{E} \left[ f(X_{m+1}) - f(X_m) \mid \mathcal{F}_m \right] \right\}, \quad (2.9)
$$

which is a version of Dynkin's formula. Under additional conditions, optional stopping (Corollary 2.3.9) permits $n \to \infty$ preserving equality in (2.9). △

The following maximal inequality, named after Doob, is important.

**Theorem 2.3.14** (Doob's inequality). Let $X_n$ be a submartingale. For any $\lambda > 0$ and any $n \geq 0$ we have

$$
\mathbb{P} \left[ \max_{0 \leq m \leq n} X_m \geq \lambda \mid \mathcal{F}_0 \right] \leq \lambda^{-1} \mathbb{E} [X_n^+ | \mathcal{F}_0].
$$

*Proof.* We give a (short) proof, since this version, conditional on $\mathcal{F}_0$, is rarely stated explicitly. Fix $\lambda > 0$. Let $\tau = \min\{n \geq 0 : X_n \geq \lambda\}$. Then

$$
X_{n \wedge \tau}^+ \geq 1 \{ \tau \leq n \} \geq \lambda 1 \{ \tau \leq n \}.
$$

Hence

$$
\mathbb{P}[\tau \leq n \mid \mathcal{F}_0] \leq \lambda^{-1} \mathbb{E} [X_{n \wedge \tau}^+ | \mathcal{F}_0].
$$

Since $x \mapsto x^+$ is non-decreasing and convex, and $\mathbb{E} [X_n^+] \leq \mathbb{E} |X_n|$, the fact that $X_n$ is a submartingale implies that $X_n^+$ is also a submartingale, by Theorem 2.3.3(ii). Then optional stopping (Theorem 2.3.7) applied at the bounded stopping times $n \wedge \tau$ and $n$ shows that $\mathbb{E} [X_{n \wedge \tau}^+ | \mathcal{F}_0] \leq \mathbb{E} [X_n^+ | \mathcal{F}_0]$; see Remark 2.3.8(a).

Theorems 2.3.2, 2.3.3, and 2.3.14 have the following corollary.

**Corollary 2.3.15.** For $X_n$ a square-integrable martingale, for any $\lambda > 0$,

$$
\mathbb{P} \left[ \max_{0 \leq m \leq n} |X_m| \geq \lambda \right] \leq \lambda^{-2} \mathbb{E} [X_n^2] = \lambda^{-2} \left( \mathbb{E} [X_0^2] + \sum_{m=0}^{n-1} \mathbb{E}[(X_{m+1} - X_m)^2] \right).
$$

**Example 2.3.16.** Suppose that $X_n$ is a square-integrable martingale with $X_0 = 0$ such that $\mathbb{E}[(X_{n+1} - X_n)^2 \mid \mathcal{F}_n] \leq K$ for all $n \geq 0$, for some constant $K$. Recall $\sigma_x = \min\{n \geq 0 : |X_n| \geq x\}$. Then, Corollary 2.3.15 implies that

$$
\mathbb{P}[\sigma_n \leq an^2] = \mathbb{P} \left[ \max_{0 \leq m \leq an^2} |X_m| \geq n \right] \leq n^{-2} K a n^2 = aK.
$$
This confirms the intuition that martingales with well-behaved increments usually move diffusively: the process is not likely to go out of the interval \((-n,n)\) before time \(an^2\) if \(a\) is small. We return to the question of quantifying the finite-time displacement of a process in Section 2.4. △

**Example 2.3.17.** Continuing the theme of Example 2.3.16, let \(S_n\) be symmetric simple random walk on \(\mathbb{Z}\) with \(S_0 = 0\). Then \(S_n^2\) is a non-negative submartingale with \(\mathbb{E}[S_n^2] = n\), and Doob's inequality, Theorem 2.3.14, which in this case reduces to an instance of Kolmogorov's inequality, gives

\[
\mathbb{P}\left[ \max_{0 \leq m \leq n} |S_m| \geq \lambda \right] \leq \lambda^{-2} n.
\]

For \(\varepsilon > 0\), let \(u(n) = n^{1/2}(\log n)^{(1/2)+\varepsilon}\); then

\[
\mathbb{P}\left[ \max_{0 \leq m \leq n} |S_m| \geq u(n) \right] \leq (\log n)^{-1-2\varepsilon}.
\]

Although this seems a rather weak bound, we can still extract a reasonable result by considering the subsequence \(n = 2^k\), \(k \geq 0\). The Borel–Cantelli lemma then implies that, a.s., \(\max_{0 \leq m \leq 2^k} |S_m| \leq u(2^k)\) for all but finitely many \(k\). Any \(n \in \mathbb{N}\) has \(2^{k_n} \leq n \leq 2^{k_n+1}\) for some \(k_n \in \mathbb{Z}_+\), where \(k_n \to \infty\) as \(n \to \infty\). Hence, a.s., for all but finitely many \(n\),

\[
\max_{0 \leq m \leq n} |S_m| \leq \max_{0 \leq m \leq 2^{k_n+1}} |S_m| \leq u(2 \cdot 2^{k_n}) \leq 2u(n).
\]

So we have shown that for any \(\varepsilon > 0\), a.s., for all but finitely many \(n\),

\[
\max_{0 \leq m \leq n} |S_m| \leq n^{1/2}(\log n)^{(1/2)+\varepsilon}.
\]

This result is certainly not as sharp as the ultimate ‘lim sup’ result for simple random walk, the law of the iterated logarithm, which states that

\[
\limsup_{n \to \infty} \frac{|S_n|}{\sqrt{2n \log \log n}} = 1, \text{ a.s.}
\]

but the proof here is also certainly much simpler, and readily generalizes. We return in some detail to these ideas in Section 2.8. △

We collect some remarks on quadratic variation for square-integrable martingales, which will be useful occasionally later on. Let \(X_n\) be a martingale with \(X_0 = 0\), such that \(\mathbb{E}[X_n^2] < \infty\) for all \(n\). Then the process \(X_n^2\) is
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a submartingale, with Doob decomposition \( X^2_n = M_n + A_n \), where \( M_n \) is a martingale with \( M_0 = 0 \) and

\[
A_n = \sum_{m=1}^{n-1} \mathbb{E}[X_{m+1}^2 - X_m^2 \mid \mathcal{F}_m] = \sum_{m=1}^{n-1} \mathbb{E}[(X_{m+1} - X_m)^2 \mid \mathcal{F}_m],
\]

using the fact that \( X_n \) is a martingale. The non-decreasing process \( A_n \) is the quadratic variation process associated to \( X_n \), and the notation \( \langle X \rangle_n = A_n \) is used. Let \( \tau \) be a finite stopping time. Then \( A_n \land \tau \uparrow A_\tau \) as \( n \to \infty \) and hence, by monotone convergence, \( \mathbb{E}A_n \land \tau \to \mathbb{E}A_\tau \).

Since \( M_n = X_n^2 - A_n \) is a martingale, so is \( M_n \land \tau \), and hence \( \mathbb{E}[X_n^2 \land \tau] = \mathbb{E}A_n \land \tau \) because \( M_0 = 0 \). Then, by Fatou’s lemma,

\[
\mathbb{E}[X_\tau^2] = \lim inf_{n \to \infty} \mathbb{E}A_n \land \tau = \mathbb{E}A_\tau. \tag{2.10}
\]

But \( X_n \land \tau \) is a submartingale, so by optional stopping (Theorem 2.3.7) \( \mathbb{E}[X_\tau^2] \geq \mathbb{E}[X_n^2 \land \tau] = \mathbb{E}A_n \land \tau \) provided \( X_n \land \tau \) is uniformly integrable. Since \( \mathbb{E}A_n \land \tau \to \mathbb{E}A_\tau \), we note the following result.

**Lemma 2.3.18.** Let \( X_n \) be a square-integrable martingale with \( X_0 = 0 \), and let \( \tau \) be a finite stopping time. Then, if \( X_n^2 \land \tau \) is uniformly integrable, \( \mathbb{E}[X_\tau^2] = \mathbb{E}[\langle X \rangle_\tau] \).

We finish this section with a result that we will use several times, which is a refinement due to Lévy of the Borel–Cantelli lemma.

**Theorem 2.3.19.** Let \( (\mathcal{F}_n, n \geq 0) \) be a filtration and \( (E_n, n \geq 0) \) a sequence of events with \( E_n \in \mathcal{F}_n \). Then, up to sets of probability zero,

\[
\{E_n \ i.o.\} = \left\{ \sum_{n=1}^{\infty} \mathbb{P}[E_n \mid \mathcal{F}_{n-1}] = \infty \right\}.
\]

This result is useful for, among other things, formalizing the intuition that an event that has positive probability of occurring soon, having not yet occurred, will eventually happen; see the next example.

**Example 2.3.20.** Let \( X_n \) be an irreducible Markov chain on a countable state space \( \Sigma \). Suppose that there is a finite non-empty set \( A \subset \Sigma \) such that \( \mathbb{P}_y[\tau_A < \infty] = 1 \) for all \( y \in \Sigma \setminus A \). We deduce that \( \mathbb{P}_y[\tau^+_y < \infty] = 1 \) for all \( y \in \Sigma \setminus A \), i.e., \( X_n \) is recurrent (see Definition 2.1.4).

Note that for any \( y \in \Sigma \setminus A \), irreducibility and the finiteness of \( A \) imply that there exist \( m = m(y) < \infty \) and \( \delta = \delta(y) > 0 \) such that \( \mathbb{P}[\tau_y < m |
2.4. Displacement and exit estimates

Let \( \eta_A = \tau_{\Sigma \setminus A} \), the first exit time from \( A \). Then, on \( \{ \eta_A > n \} \), \( \mathbb{P}[\eta_A < n + m \mid \mathcal{F}_n] > \delta \). In particular, setting \( \mathcal{G}_n = \mathcal{F}_{nm} \) and \( E_n = \{ \eta_A < nm \} \), we have \( E_n \in \mathcal{G}_n \) and \( \mathbb{P}[E_n \mid \mathcal{G}_{n-1}] > \delta \) on \( E_{n-1}^c \). Also, \( E_{n-1} \subseteq E_n \) so

\[
\mathbb{P}[E_n \mid \mathcal{G}_{n-1}] \geq \delta \mathbb{1}(E_{n-1}) + 1(E_{n-1}) \geq \delta, \text{ a.s.}
\]

Hence Theorem 2.3.19 implies that \( E_n \) occurs infinitely often, so \( \mathbb{P}[\eta_A < \infty \mid X_0 \in A] = 1 \).

Now take \( X_0 = y \in \Sigma \setminus A \). Let \( \kappa_0 = \tau_A \) and, for \( k \geq 1 \), \( \kappa_k = \min\{ n > \kappa_{k-1} + m : X_n \in A \} \), (a subsequence of) return times to \( A \). It follows from the argument above and the hypothesis that \( \mathbb{P}_z[\tau_A < \infty] = 1 \) for all \( z \in \Sigma \setminus A \) that \( \kappa_k < \infty \), a.s., for all \( k \).

A similar argument now shows that \( X_n \) will eventually return to \( y \) having visited \( A \). To spell this out, for \( k \geq 1 \) let \( E_k = \{ \tau^+_y \leq \kappa_k \} \) and now set \( \mathcal{G}_k = \mathcal{F}_{\kappa_k} \); then \( E_k \in \mathcal{G}_k \) and \( E_{k-1} \subseteq E_k \) so that

\[
\mathbb{P}[E_k \mid \mathcal{G}_{k-1}] \geq \mathbb{P}[\tau^+_y < \kappa_{k-1} + m \mid \mathcal{F}_{\kappa_{k-1}}] \mathbb{1}\{ \tau^+_y > \kappa_{k-1} \} + \mathbb{1}\{ \tau^+_y \leq \kappa_{k-1} \} \geq \delta, \text{ a.s.}
\]

Thus Theorem 2.3.19 implies that \( \mathbb{P}_y[\tau^+_y < \infty] = \mathbb{P}_y[E_n \text{ i.o.}] = 1. \quad \Delta \)

### 2.4 Displacement and exit estimates

A basic ingredient of many of the arguments that we use later in this book is some estimate of how far a process can travel in a certain time, or the probability that a process exits an interval by one end point rather than the other. In this section we collect some examples and results in this vein, to illustrate some important applications of the semimartingale ideas from Section 2.3.

The main tool in this context is the optional stopping theorem. We give several examples showing how optional stopping may be used to deduce information on displacement. We will return to many of the ideas illustrated in these examples later in the book.

**Example 2.4.1.** For SRW \( S_n \) on \( \mathbb{Z} \) with parameter \( p \in (0,1) \), \( p \neq \frac{1}{2} \), abbreviate \( \beta = \frac{1-p}{p} \) and \( X_n = \beta^{S_n} \); as observed in Example 2.1.7, \( X_n \) is a martingale. Also, \( X_{n \wedge \tau_{(a,b)}} \) is bounded and \( \tau_{(a,b)} \) is finite, as observed in Example 2.3.12. So, with \( a < x < b \), optional stopping (Corollary 2.3.9) gives

\[
\beta^x = \mathbb{E}_x X_0 = \mathbb{E}_x X_{\tau_{(a,b)}} = \beta^a \mathbb{P}_x[\tau_a < \tau_b] + \beta^b \mathbb{P}_x[\tau_b < \tau_a],
\]
and so
\[ P_x[\tau_a < \tau_b] = \frac{\beta^x - \beta^b}{\beta^a - \beta^b}, \]
solving the classical gambler’s ruin problem.

In the case \( p = \frac{1}{2} \) the process \( S_n \) is itself a martingale, and the same calculation gives
\[ P_x[\tau_a < \tau_b] = \frac{b-x}{b-a}. \]
It is elementary but instructive to note that one may recast this result in the form, say, for \( x \geq 1, \)
\[ P_0[\sigma_x < \tau_0^+] = P_0[\max_{0 \leq n < \tau_0^+} |S_n| \geq x] = P_1[\tau_x < \tau_0] = \frac{1}{x}, \] (2.11)
a statement about the excursion maximum of the walk. (Here \( \sigma_x = \tau_{\{-x,x\}} \).) Hence, the random variable
\[ M := \max_{0 \leq n < \tau_0^+} |S_n| \]
is almost surely finite and has \( E_0[M] < \infty \) if and only if \( s < 1. \)

With a little extra work, this argument demonstrates the null recurrence of the random walk \( S_n \). Indeed, the \( \sigma_x \) are a.s. finite (by e.g. the argument in Example 2.3.12), and a similar argument to the proof of Theorem 2.2.5 shows that
\[ P_0[\tau_0^+ < \tau_a^+] = P_0[\sigma_x < \tau_0^+] = \lim_{x \to \infty} P_0[\tau_0^+ < \sigma_x] = 1, \] by (2.11), proving recurrence. Moreover, under \( P_0, |S_n| \leq n \) so \( M \leq \tau_0^+ \), and
\[ E_0[\tau_0^+] \geq E_0[M] = \sum_{x \geq 1} P_0[M \geq x] = \infty, \] by (2.11).

The martingale solution to the gambler’s ruin problem from Example 2.4.1 extends to give the proof of Lemma 2.2.1 from Section 2.2, as the next example shows.

**Example 2.4.2.** In the setting of the general nearest-neighbour random walk on \( \mathbb{Z}_+ \) from Section 2.2, let \( a, b, x \) be integers with \( 0 \leq a < b \) and
\( a \leq x \leq b \); let \( X_0 = x \). Then \( h(X_n \wedge \tau_a \wedge \tau_b) \) is a bounded martingale, by Lemma 2.2.2, and \( \tau_a \wedge \tau_b < \infty \) a.s., by Corollary 2.1.10. Then, by optional stopping (Corollary 2.3.9),
\[ h(x) = E_x h(X_0) = E_x h(X_{\tau_a \wedge \tau_b}) = h(a) P_x[\tau_a < \tau_b] + h(b) P_x[\tau_a > \tau_b]. \]
Re-arranging this gives Lemma 2.2.1.

The next example returns to simple symmetric random walk to illustrate another idea.
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Example 2.4.3. Again consider $S_n$ simple symmetric random walk on $\mathbb{Z}$; write $\mathcal{F}_n = \sigma(S_0, \ldots, S_n)$. Example 2.4.1 showed that $\mathbb{E}_0[\tau_0^+] = \infty$. In fact, $\mathbb{E}_0[(\tau_0^+)^{1/2}] = \infty$ (and this is best possible).

To show this, we elaborate the idea of the previous example. Roughly speaking, we know that $S_n$ reaches $\pm x$ before returning to 0 with probability $1/x$; but, if it does so, it should take time about $x^2$ to come back to the origin, by the idea of Example 2.3.16. Formally, let $x \in \mathbb{N}$ and set $E_x = \{\max_{0 \leq n \leq x^2} |2x - S_{\sigma_{2x}^+} - n| < 2x\}$. Then,

$$
\mathbb{P}_0[\tau_0^+ \geq x^2] \geq \mathbb{P}_0[\{\sigma_{2x} < \tau_0^+\} \cap E_x] = \mathbb{E}_0[1_{\{\sigma_{2x} < \tau_0^+\}} \mathbb{P}[E_x | \mathcal{F}_{\sigma_{2x}}]],
$$

since $\{\sigma_{2x} < \tau_0^+\} \in \mathcal{F}_{\sigma_{2x}}$. Let $X_n = (S_{\sigma_{2x}^+} - 2x)^2$. Then, $X_n \leq n^2$ so $\mathbb{E}X_n < \infty$, and $\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_{\sigma_{2x}^+}] = 1$. Hence $(X_n, n \geq 0)$ is a non-negative submartingale adapted to $(\mathcal{F}_{\sigma_{2x}^+}, n \geq 0)$ with $X_0 = 0$, a.s., and we may apply Doob’s inequality (Theorem 2.3.14) to show that

$$
\mathbb{P}\left[\max_{0 \leq n < x^2} X_n \geq 4x^2 \Big| \mathcal{F}_{\sigma_{2x}}\right] \leq \frac{\mathbb{E}[X_{x^2} | \mathcal{F}_{\sigma_{2x}}]}{4x^2} = \frac{1}{4}, \text{ a.s.,}
$$

so that $\mathbb{P}[E_x | \mathcal{F}_{\sigma_{2x}}] \geq 3/4$, a.s. Thus by (2.11),

$$
\mathbb{P}_0[\tau_0^+ \geq x^2] \geq \frac{3}{4} \mathbb{P}_0[\sigma_{2x} < \tau_0^+] = \frac{3}{4x}.
$$

Hence, for any $n \in \mathbb{Z}_+$, $\mathbb{P}_0[\tau_0^+ \geq n] \geq \frac{3}{4}(1 + \sqrt{n})^{-1}$, which shows that $\mathbb{E}_0[(\tau_0^+)^{1/2}] = \infty$. It is clear that the idea of this example can be generalized to martingales satisfying uniformly bounded increments, say; we return to this general theme in Section 2.7. \triangle

It is important to observe that the hitting probability calculation of Example 2.4.1 is “robust”, in the sense that it gives non-trivial and often useful bounds on hitting probabilities with only partial information. The next example takes up this theme.

Example 2.4.4. Suppose that $X_n$ is a submartingale with $X_0 = x$ and $|X_{n+1} - X_n| \leq K$ a.s. for all $n$. Recall also that $\lambda_a = \min\{n : X_n \leq a\}$ and $\rho_b = \min\{n : X_n \geq b\}$. Suppose $a < x < b$. Suppose that $\mathbb{P}[\lambda_a \wedge \rho_b < \infty] = 1$ (this will usually be an easy consequence of some form of non-degeneracy or irreducibility). Then, analogously to Example 2.4.1,

$$
x = \mathbb{E}X_0 \leq \mathbb{E}X_{\lambda_a \wedge \rho_b} \leq a \mathbb{P}[\lambda_a < \rho_b] + (b + K) \mathbb{P}[\rho_b < \lambda_a],
$$
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so

\[ P[\lambda_a < \rho_b] \leq \frac{b + K - x}{b + K - a}. \]

One would need, however, a supermartingale in order to obtain a lower bound for \( P[\lambda_a < \rho_b] \): see Corollary 2.4.6 below for a closely related result.

\[ \Box \]

For estimating hitting probabilities with a suitable Lyapunov function to hand, the optional stopping theorem is one’s friend.

At this point, we present a simple maximal inequality for supermartingales, which is in some sense more elementary than Doob’s inequality (Theorem 2.3.14) and which will be useful at several points later on.

**Theorem 2.4.5.** Suppose that \( X_n \) is a non-negative supermartingale adapted to \( \mathcal{F}_n \). Then, for any \( x > 0 \),

\[ P\left[ \max_{m \geq 0} X_m \geq x \mid \mathcal{F}_0 \right] \leq \frac{X_0}{x}, \text{ a.s.} \]

In particular, \( P[\max_{m \geq 0} X_m \geq x] \leq x^{-1} E X_0 \).

**Proof.** Fix \( x > 0 \). Theorem 2.3.11 applied to the non-negative supermartingale \( X_n \) at stopping times 0 and \( \sigma_x = \min\{n \geq 0 : X_n \geq x\} \) yields

\[ X_0 \geq E[X_{\sigma_x} \mid \mathcal{F}_0] \geq E[X_{\sigma_x} 1\{\sigma_x < \infty\} \mid \mathcal{F}_0] \geq x P[\sigma_x < \infty \mid \mathcal{F}_0]. \]

But \( \sigma_x < \infty \) if and only if \( \max_{m \geq 0} X_m \geq x \). \( \square \)

Theorem 2.4.5 has a useful corollary on the same theme as the preceding examples, but which imposes no assumptions on the size of the increments of the process.

**Corollary 2.4.6.** Suppose that \( X_n \) is an \( \mathcal{F}_n \) adapted process taking values in \( \mathbb{R}_+ \). Suppose that there exists \( y \in \mathbb{R}_+ \) for which, a.s.,

\[ E[X_{n+1} - X_n \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n > y\}. \]

Then, with \( \lambda_y = \min\{n \geq 0 : X_n \leq y\} \), for any \( x > 0 \),

\[ P\left[ \max_{m \geq 0} X_{m \land \lambda_y} \geq x \right] \leq \frac{E X_0}{x}. \]
2.4. Displacement and exit estimates

Proof. Apply Theorem 2.4.5 to the supermartingale \( X_{n \wedge \lambda_y} \in \mathbb{R}_+ \).

The next result includes another important maximal inequality, which shows how an upper bound on the expected increments of a non-negative process gives a probabilistic upper bound on its finite-time maximum. The inequality (2.14) is closely related to the submartingale inequality of Doob (Theorem 2.3.14) and also to the supermartingale inequality Theorem 2.4.5 (take \( B = 0 \)).

**Theorem 2.4.7.** Let \( X_n \) be an integrable \( \mathcal{F}_n \)-adapted process on \( \mathbb{R}_+ \). Suppose that for some \( B \in \mathbb{R}_+ \),

\[
\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] \leq B, \ a.s.
\]

Then, for any stopping time \( \nu \) with \( \mathbb{E}\nu < \infty \), and any \( x > 0 \), a.s.,

\[
P\left[ \max_{0 \leq m \leq \nu} X_m \geq x \mid \mathcal{F}_0 \right] \leq \frac{B \mathbb{E}[\nu | \mathcal{F}_0] + X_0}{x}; \tag{2.12}
\]

and

\[
\mathbb{E}[\sigma_x | \mathcal{F}_0] \geq \frac{x - X_0}{B}. \tag{2.13}
\]

In particular, for any \( n \geq 0 \) and any \( x > 0 \),

\[
P\left[ \max_{0 \leq m \leq n} X_m \geq x \right] \leq \frac{Bn + \mathbb{E}X_0}{x}. \tag{2.14}
\]

We deduce Theorem 2.4.7 from the following important result.

**Theorem 2.4.8.** Let \( X_n \) be an integrable \( \mathcal{F}_n \)-adapted process on \( \mathbb{R} \), and let \( \tau \) be a stopping time. Suppose that for some \( B \in \mathbb{R} \),

\[
\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] \leq B, \text{ on } \{ n < \tau \}. \tag{2.15}
\]

Then, for any \( n \geq 0 \), a.s.,

\[
\mathbb{E}[X_{n \wedge \tau} | \mathcal{F}_0] \leq X_0 + B \mathbb{E}[n \wedge \tau | \mathcal{F}_0]. \tag{2.16}
\]

Proof. We may rewrite the condition (2.15) as

\[
\mathbb{E}[X_{(m+1) \wedge \tau} - X_{m \wedge \tau} | \mathcal{F}_m] \leq B 1_{\{ \tau > m \}}, \text{ for all } m \geq 0. \tag{2.17}
\]

Indeed, (2.17) is clearly equivalent to (2.15) on \( \{ \tau > m \} \), since then \( X_{m \wedge \tau} = X_m \) and \( X_{(m+1) \wedge \tau} = X_{m+1} \), while on \( \{ \tau \leq m \} \) the statement (2.17) is trivial. Conditioning on \( \mathcal{F}_0 \) and taking expectations in (2.17), we obtain

\[
\mathbb{E}[X_{(m+1) \wedge \tau} | \mathcal{F}_0] - \mathbb{E}[X_{m \wedge \tau} | \mathcal{F}_0] \leq B \mathbb{P}[\tau > m | \mathcal{F}_0].
\]
Then summing from $m = 0$ to $m = n - 1$ we obtain

$$
E[X_{n \wedge \tau} \mid F_0] - X_0 \leq B \mathbb{E} \sum_{m=0}^{n-1} 1\{\tau > m\},
$$

which gives (2.16) since $\sum_{m=0}^{n-1} 1\{\tau > m\} = n \wedge \tau$.

**Remark 2.4.9.** A variation of Dynkin’s formula (see Example 2.3.13) gives

$$
E[X_{n \wedge \tau} \mid F_0] = X_0 + B \mathbb{E} \left[ \sum_{m=0}^{(n \wedge \tau)-1} \mathbb{E}[X_{m+1} - X_m \mid F_m] \mid F_0 \right].
$$

Theorem 2.4.8 can be viewed as a consequence of this; the elementary proof presented above is still instructive, and demonstrates how weakly optional stopping is required.

Now we complete the proof of Theorem 2.4.7.

**Proof of Theorem 2.4.7.** Take $\tau = \nu \wedge \sigma_x$; then it follows from (2.16) that

$$
X_0 + B \mathbb{E}[n \wedge \nu \wedge \sigma_x \mid F_0] \geq \mathbb{E}[X_{n \wedge \nu \wedge \sigma_x} \mid F_0] \geq x \mathbb{P}[\sigma_x \leq n \wedge \nu \mid F_0],
$$

(2.18)

since $X_{n \wedge \nu \wedge \sigma_x} \geq x 1\{\sigma_x \leq n \wedge \nu\}$, given that $X_n \geq 0$ for all $n$. In particular, since $\nu \geq n \wedge \nu \wedge \sigma_x$,

$$
X_0 + B \mathbb{E}[\nu \mid F_0] \geq x \mathbb{P}[\sigma_x \leq n \wedge \nu \mid F_0],
$$

which gives the maximal inequality (2.12) on taking $n \to \infty$. The simpler version (2.14) follows from (2.12) applied at the deterministic stopping time $\nu = n$.

To prove (2.13) it suffices to suppose that $\mathbb{P}[\sigma_x < \infty] = 1$. An application of (2.18) with $\nu = n$ gives

$$
X_0 + B \mathbb{E}[\sigma_x \mid F_0] \geq x \mathbb{P}[\sigma_x \leq n \mid F_0],
$$

which on letting $n \to \infty$ yields (2.13). \qed

\textbf{[i]} A Lyapunov function whose drift is uniformly bounded above gives control over the maximum of a process.

Theorems 2.4.7 and 2.4.8 will form basic tools later on; for now we make an aside to give a corollary pertaining to the so-called finite time stochastic stability of Markov chains.
2.4. Displacement and exit estimates

Corollary 2.4.10 (Kushner–Kalashnikov inequality). Let $X_n$ be a time-homogeneous Markov chain on state-space $(\Sigma, \mathcal{E})$. Suppose that for measurable $f : \Sigma \to \mathbb{R}_+, A \in \mathcal{E}$, and $B \in \mathbb{R}_+$,

$$E[f(X_{n+1}) - f(X_n) \mid X_n = x] \leq B, \text{ whenever } x \in A.$$ 

Then, if $\sigma_A = \min\{n \geq 0 : X_n \notin A\}$, for $n \in \mathbb{Z}_+$ and $x \in A$,

$$\mathbb{P}[\sigma_A \leq n \mid X_0 = x] \leq \frac{Bn + f(x)}{\inf_{y \notin A} f(y)}.$$ 

Proof. Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Apply (2.12) to the process $f(X_n)$ at the stopping time $\nu = n \wedge \sigma_A$ to get

$$\mathbb{P}[^{\sigma_A \leq n \mid \mathcal{F}_0}] \leq \mathbb{P}[^{\max_{0 \leq m \leq n \wedge \sigma_A} f(X_m) \geq \inf_{y \notin A} f(y) \mid \mathcal{F}_0}] \leq \frac{B \mathbb{E}[n \wedge \sigma_A \mid \mathcal{F}_0] + f(X_0)}{\inf_{y \notin A} f(y)},$$

which gives the result. \hspace{1cm} $\Box$

Here is a result in the opposite direction, giving an upper bound on the expected time to exit an interval. Again for $x > 0$ we use the notation $\sigma_x = \min\{n \geq 0 : |X_n| \geq x\}$.

Theorem 2.4.11. Let $X_n$ be an $\mathcal{F}_n$-adapted process on $\mathbb{R}_+$ and let $\eta$ be a stopping time. Fix $x \geq 0$. Suppose that there exist a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $\varepsilon > 0$ for which $f(X_n)$ is integrable and

$$E[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \geq \varepsilon, \text{ on } \{n < \eta \wedge \sigma_x\}. \hspace{1cm} (2.19)$$

Then

$$E[\eta \wedge \sigma_x] \leq \varepsilon^{-1} E[f(X_{\sigma_x})]. \hspace{1cm} (2.20)$$

Proof. Let $x \geq 0$, and write $Y_n = f(X_{n \wedge \eta \wedge \sigma_x})$. We may express (2.19) as

$$E[Y_{m+1} - Y_m \mid \mathcal{F}_m] \geq \varepsilon \mathbf{1}_{\{\eta \wedge \sigma_x > m\}},$$

for any $m \geq 0$. We obtain $E Y_{m+1} - E Y_m \geq \varepsilon \mathbb{P}[\eta \wedge \sigma_x > m]$ on taking expectations, and then summing for $m$ from 0 up to $n$ we get

$$E Y_{n+1} - E Y_0 \geq \varepsilon \sum_{m=0}^{n} \mathbb{P}[\eta \wedge \sigma_x > m].$$
Since $Y_0 = f(X_0) \geq 0$, it follows that

$$E[\eta \wedge \sigma_x] = \lim_{n \to \infty} \sum_{m=0}^{n} P[\eta \wedge \sigma_x > m] \leq \varepsilon^{-1} \limsup_{n \to \infty} EY_{n+1}.$$ 

Since $f$ is non-decreasing, $Y_n = f(X_{n \wedge \eta \wedge \sigma_x}) \leq f(X_{\sigma_x})$, a.s., for all $n \in \mathbb{Z}_+$. The result now follows.

A Lyapunov function whose drift is uniformly bounded below quantifies that a process is unlikely to stay in a bounded region for too long.

Some refinements of Theorem 2.4.11 are given in Section 3.10 below. Although the proof of Theorem 2.4.11 is elementary, it is quite powerful. For example, an easy consequence is the following result which shows that a martingale with sufficient variability will exit an interval at least ‘diffusively’. In the particular case where $\eta = \infty$, Theorem 2.4.12 reduces to a semimartingale version of Kolmogorov’s ‘other’ inequality.

**Theorem 2.4.12.** Suppose that $X_n$ is a square-integrable $\mathcal{F}_n$-adapted process on $\mathbb{R}$ with $X_0 = 0$. Suppose that there exist a stopping time $\eta$ and constants $v > 0$, $B \in \mathbb{R}_+$ such that, for all $n \geq 0$,

$$\mathbb{E}[(X_{n+1} - X_n)^2 | \mathcal{F}_n] \geq v1\{n < \eta\}, \quad (2.21)$$

$$P[|X_{n+1} - X_n| \leq B | \mathcal{F}_n] = 1. \quad (2.22)$$

Suppose also that either (i) $X_n$ is a martingale; or (ii) $X_n$ is a non-negative submartingale. Then for any $n \geq 0$ and any $x > 0$,

$$P\left[\max_{0 \leq m \leq n} |X_m| \geq x\right] \geq 1 - \frac{(x + B)^2}{vn} - P[\eta \leq n].$$

The lower bound on second moments in (2.21) is a form of uniform non-degeneracy. The proof of Theorem 2.4.12 uses the following elementary algebraic identity:

$$a^2 - b^2 = (a - b)^2 + 2b(a - b), \quad (a, b \in \mathbb{R}). \quad (2.23)$$

Although trivial, (2.23) will be used repeatedly in this book, so we refer to it as the square-difference identity.
Proof of Theorem 2.4.12. By (2.21), (2.23), and the (sub)martingale assumption, we have that
\[ E[X_{n+1}^2 - X_n^2 \mid F_n] = E[(X_{n+1} - X_n)^2 \mid F_n] + 2X_n E[X_{n+1} - X_n \mid F_n] \geq v, \]
on \{n \leq \eta\}, while \(|X_{\sigma_x}| \leq x + B\), a.s., by (2.22). So we may apply Theorem 2.4.11 to the process \(X_n^2 \in \mathbb{R}_+\) with \(f(x) = x\), to obtain \(E[\eta \wedge \sigma_x] \leq v^{-1}(x + B)^2\). Then, \(P[\sigma_x \leq n] \geq P[\eta \wedge \sigma_x \leq n] - P[\eta \leq n]\), which, with Markov’s inequality, gives the desired result. \(\square\)

Some generalizations of Theorem 2.4.12 are given in Chapter 4.

Example 2.4.13. Let \(X_n\) be an \(F_n\)-adapted process on \(\mathbb{R}_+\). Suppose that there exists an increasing \(f : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(E[f(X_0)] < \infty\) and
\[ E[f(X_{n+1}) - f(X_n) \mid F_n] = B \in (0, \infty), \text{ a.s.} \]

Since \(f\) is increasing, \(\sigma_x = \min\{n \geq 0 : f(X_n) \geq f(x)\}\). Then applications of (2.13) and Theorem 2.4.11 to the process \(f(X_n)\) show that
\[ \frac{f(x) - E[f(X_0)]}{B} \leq E[\sigma_x] \leq \frac{E[f(X_{\sigma_x})]}{B}. \]

Assuming that \(f(x) \to \infty\) as \(x \to \infty\), and
\[ \lim_{x \to \infty} \frac{E[f(X_{\sigma_x})]}{f(x)} = 1, \]
(2.24)
it follows that
\[ \lim_{x \to \infty} \frac{E[\sigma_x]}{f(x)} = \frac{1}{B}, \]
which is sometimes known as a weak renewal theorem. Sufficient for (2.24) is that \(X_{n+1} - X_n\) is uniformly bounded above and \(f\) does not grow too fast; see also Section 3.10. Lemma 2.2.4(i) provides an example. \(\triangle\)

Another very useful result for bounding probabilities of displacement, typically in the situation when the size of displacement is much bigger than one would normally expect, is the Azuma–Hoeffding inequality. Here we state the traditional inequality alongside its one-sided versions; since the latter are not easily found in the literature, we give the proof (which is essentially the standard one).
Theorem 2.4.14 (Azuma–Hoeffding inequalities). Suppose that \((c_k, k \geq 1)\) are finite positive constants. Suppose that \(X_n\) is a supermartingale, such that \(|X_k - X_{k-1}| \leq c_k\) a.s. for all \(k \geq 1\). Then for all \(a > 0\) and all \(n \geq 0\),

\[
P[X_n - X_0 \geq a \mid \mathcal{F}_0] \leq \exp\left(- \frac{a^2}{2 \sum_{k=1}^n c_k^2}\right). \tag{2.25}
\]

If \(X_n\) is a submartingale with \(|X_k - X_{k-1}| \leq c_k\), then

\[
P[X_n - X_0 \leq -a \mid \mathcal{F}_0] \leq \exp\left(- \frac{a^2}{2 \sum_{k=1}^n c_k^2}\right). \tag{2.26}
\]

Finally, if \(X_n\) is a martingale with \(|X_k - X_{k-1}| \leq c_k\), then

\[
P[|X_n - X_0| \geq a \mid \mathcal{F}_0] \leq 2 \exp\left(- \frac{a^2}{2 \sum_{k=1}^n c_k^2}\right). \tag{2.27}
\]

Proof. We prove only (2.25); then, (2.26) follows by considering \((-X_n)\), and (2.27) by the union bound.

For any fixed \(\lambda\), \(g(x) = e^{\lambda x}\) is a convex function. Let \(c > 0\). Since the line segment with endpoints \((-c, e^{-\lambda c})\) and \((c, e^{\lambda c})\) lies above the graph of \(g\), we have that for all \(x \in [-c, c]\),

\[e^{\lambda x} \leq \frac{e^{\lambda c} - e^{-\lambda c}}{2c} x + \frac{e^{\lambda c} + e^{-\lambda c}}{2}.
\]

So, since \(X_n\) is a supermartingale, we obtain

\[
\mathbb{E}\left[e^{\lambda (X_k - X_{k-1})} \mid \mathcal{F}_{k-1}\right] \leq \mathbb{E}\left[\frac{e^{\lambda c_k} - e^{-\lambda c_k}}{2c_k} (X_k - X_{k-1}) + \frac{e^{\lambda c_k} + e^{-\lambda c_k}}{2} \mid \mathcal{F}_{k-1}\right]
\leq \frac{e^{\lambda c_k} + e^{-\lambda c_k}}{2}
\leq e^{\lambda^2 c_k^2/2}, \tag{2.28}
\]

where we used the elementary inequality

\[
\frac{e^y + e^{-y}}{2} = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} \leq \sum_{k=0}^{\infty} \frac{y^{2k}}{2^k k!} = e^{y^2/2}.
\]

With (2.28), we obtain by induction

\[
\mathbb{E}\left[e^{\lambda (X_n - X_0)} \mid \mathcal{F}_0\right] \leq \exp\left(\frac{1}{2} \lambda^2 \sum_{k=1}^n c_k^2\right).
\]
2.4. Displacement and exit estimates

Now the (conditional) Markov inequality gives, for \( a > 0 \) and \( \lambda > 0 \),

\[
P[X_n - X_0 \geq a \mid \mathcal{F}_0] \leq e^{-\lambda a} \mathbb{E}[e^{\lambda(X_n - X_0)} \mid \mathcal{F}_0] \leq \exp(-\lambda a + \frac{1}{2} \lambda^2 \sum_{k=1}^n c_k^2),
\]

and taking \( \lambda = a \left( \sum_{k=1}^n c_k^2 \right)^{-1} \) we conclude the proof. \( \square \)

We finish this section with an example of particularly wide-ranging excursions.

**Example 2.4.15.** Let \( S_n \) be symmetric SRW on \( \mathbb{Z}^2 \), and consider \( f(x) = (\log(1 + \|x\|^2))^\gamma \) for \( \gamma \in \mathbb{R} \). Let \( U_2 := \{ \pm e_1, \pm e_2 \} \) denote the possible jumps of the walk. Then for \( e \in U_2 \), we have

\[
f(x + e) - f(x) = f(x) \left\{ \left( 1 + \frac{\log \left( 1 + \frac{2e \cdot x + 1}{1 + \|x\|^2} \right)}{\log(1 + \|x\|^2)} \right)^\gamma - 1 \right\}.
\]

Here, we have from Taylor’s formula that

\[
\log \left( 1 + \frac{2e \cdot x + 1}{1 + \|x\|^2} \right) = \frac{2e \cdot x + 1}{\|x\|^2} - \frac{2(e \cdot x)^2}{\|x\|^4} + O(\|x\|^{-3}),
\]

so that, with the Taylor formula for \((1 + x)^\gamma\), we obtain

\[
f(x + e) - f(x) = \gamma \|x\|^{-2} (\log(1 + \|x\|^2))^{\gamma - 1} \times \left( 2e \cdot x + 1 - \frac{2(e \cdot x)^2}{\|x\|^2} + \frac{2(\gamma - 1)(e \cdot x)^2}{\|x\|^4 \log(1 + \|x\|^2)} + O(\|x\|^{-1}) \right).
\]

Using the fact that

\[
\sum_{e \in U_2} 1 = 4, \quad \sum_{e \in U_2} e \cdot x = 0, \quad \text{and} \quad \sum_{e \in U_2} (e \cdot x)^2 = 2\|x\|^2,
\]

it follows that, as \( \|x\| \to \infty \),

\[
\mathbb{E}[f(S_{n+1}) - f(S_n) \mid S_n = x] = \frac{1}{4} \sum_{e \in U_2} (f(x + e) - f(x)) = \gamma (\gamma - 1) \|x\|^{-2} (\log(1 + \|x\|^2))^{\gamma - 1} (1 + O(1)), \tag{2.29}
\]

which, for \( \gamma > 1 \), is positive for all \( \|x\| \) sufficiently large.
Suppose \( S_0 \neq 0 \). Then, if \( \tau = \min\{n \geq 0 : S_n = 0\} \) and \( \sigma_x = \min\{n \geq 0 : \|S_n\| \geq x\} \), we have that \( Y_n = f(S_{n \wedge \tau \wedge \sigma_x}) \) is a non-negative submartingale, bounded above by \((\log(2 + x^2))^\gamma\) since \( \|S_{n \wedge \sigma_x}\|^2 \leq 1 + x^2 \). Thus \( Y_n \) is uniformly integrable, and by optional stopping (Theorem 2.3.7),
\[
\log 2 \leq f(S_0) = Y_0 \leq \mathbb{E}Y_{\tau \wedge \sigma_x} \leq \mathbb{P}[\sigma_x < \tau]((\log(2 + x^2))^\gamma),
\]
since \( Y_\tau = f(0) = 0 \). In other words, for any \( \varepsilon > 0 \), there exists \( c > 0 \) such that, for all \( x \geq 2 \),
\[
\mathbb{P}\left[ \max_{0 \leq m \leq \tau} \|S_m\| \geq x \right] \geq c(\log x)^{-1-\varepsilon}. \quad \triangle
\]

### 2.5 Recurrence and transience criteria for Markov chains

We start with a technical result that will allow us to convert results on hitting times of sets (such as we obtain by Lyapunov function arguments) to results on return times to points, and hence statements about recurrence and transience as defined at Definition 2.1.4.

**Lemma 2.5.1.** Let \( X_n \) be an irreducible Markov chain on a countable state space \( \Sigma \).

(i) If for some \( x \in \Sigma \) and some non-empty \( A \subseteq \Sigma \), \( \mathbb{P}_x[\tau_A < \infty] < 1 \), then \( X_n \) is transient.

(ii) If for some finite non-empty \( A \subseteq \Sigma \) and all \( x \in \Sigma \setminus A \), \( \mathbb{P}_x[\tau_A < \infty] = 1 \), then \( X_n \) is recurrent.

**Proof.** We prove part (i). If \( A \) is non-empty and \( \mathbb{P}_x[\tau_A = \infty] > 0 \), then there is some \( y \in A \), \( y \neq x \) (since \( \mathbb{P}_x[\tau_x = \infty] = 0 \)). But \( \tau_y \geq \tau_A \), so \( \mathbb{P}_x[\tau_y = \infty] > 0 \). By irreducibility, for any \( x \neq y \), \( \mathbb{P}_y[\tau_x < \tau_y^+] > 0 \); otherwise the strong Markov property would show that \( \mathbb{P}_y[\tau_x < \infty] = 0 \), contradicting irreducibility. So, by the strong Markov property,
\[
\mathbb{P}_y[\tau_y^+ = \infty] \geq \mathbb{P}_y[\tau_x < \tau_y^+] \mathbb{P}_x[\tau_y = \infty] > 0,
\]
which shows transience by Definition 2.1.4.

Part (ii) is demonstrated in Example 2.3.20. \( \square \)

The two basic criteria for recurrence and transience of Markov chains that we present in this section have at their foundation the supermartingale convergence result Theorem 2.3.6. We first present the recurrence criterion.
Theorem 2.5.2 (Recurrence criterion). An irreducible Markov chain $X_n$ on a countably infinite state space $\Sigma$ is recurrent if and only if there exist a function $f : \Sigma \to \mathbb{R}_+$ and a finite non-empty set $A \subset \Sigma$ such that

$$E[f(X_{n+1}) - f(X_n) \mid X_n = x] \leq 0, \text{ for all } x \in \Sigma \setminus A,$$

and $f(x) \to \infty$ as $x \to \infty$.

We emphasize an elementary but important point about the notation here. The key to correct interpretation of the statement $f(x) \to \infty$ as $x \to \infty$ is the fact that the domain of $f$ is precisely the countable set $\Sigma$, so while the first $\infty$ in the statement is $\sup \mathbb{R}_+$, the second is $\#\Sigma$. Thus the statement $f(x) \to \infty$ as $x \to \infty$ entails that for any enumeration $x_1, x_2, \ldots$ of the elements of $\Sigma$, $\lim_{n \to \infty} f(x_n) = \infty$. Indeed, given such an enumeration, $\lim_{n \to \infty} \inf_{m \geq n} f(x_m) \geq M$ for any $M \in \mathbb{R}_+$, so the set $\{x \in \Sigma : f(x) \leq M\}$ is finite for any $M \in \mathbb{R}_+$. Hence the choice of enumeration of $\Sigma$ did not matter.

If $\Sigma \subset \mathbb{R}^d$ is unbounded and locally finite, then any enumeration $x_1, x_2, \ldots$ of $\Sigma$ has $\lim_{n \to \infty} \|x_n\| = \infty$; thus we may restate an ‘embedded’ version of Theorem 2.5.2 as follows.

Corollary 2.5.3. An irreducible Markov chain $X_n$ on an unbounded and locally finite state space $\Sigma \subset \mathbb{R}^d$ is recurrent if and only if there exist a function $f : \mathbb{R}^d \to \mathbb{R}_+$ and a bounded non-empty set $A \subset \Sigma$ such that (2.30) holds and $f(x) \to \infty$ as $\|x\| \to \infty$.

Example 2.5.4. As in Example 2.4.15, let $S_n$ be symmetric SRW on $\mathbb{Z}^2$, and consider $f(x) = (\log(1 + \|x\|^2))^\gamma$ for $\gamma \in (0, 1)$. Then, by (2.29),

$$E[f(S_{n+1}) - f(S_n) \mid S_n = x] = \gamma(\gamma - 1)\|x\|^{-2}(\log(1 + \|x\|^2))^{\gamma - 1}(1 + o(1)),$$

which is negative for all $\|x\|$ sufficiently large. Thus Corollary 2.5.3 shows that SRW on $\mathbb{Z}^2$ is recurrent; this is a direct Lyapunov-function proof of the $d = 2$ case of Pólya’s theorem (Theorem 1.2.1).

Proof of Theorem 2.5.2. To prove that having a function that satisfies (2.30) is sufficient for the recurrence, take $X_0 = x \in \Sigma$. Set $Y_n = f(X_{n\wedge \tau_A})$ and observe that $Y_n$ is a non-negative supermartingale. Then, by Theorem 2.3.6, there exists a random variable $Y_\infty$ such that $Y_n \to Y_\infty$ a.s. and

$$\mathbb{E}_x Y_\infty \leq \mathbb{E}_x Y_0 = f(x),$$

for any $x \in \Sigma$. On the other hand, since $f \to \infty$, it holds that the set $\{y \in \Sigma : f(y) \leq M\}$ is finite for any $M \in \mathbb{R}_+$; so, the irreducibility implies
that \( \limsup_{n \to \infty} f(X_n) = +\infty \) a.s. on \( \{ \tau_A = \infty \} \). Hence, on \( \{ \tau_A = \infty \} \), we must have \( Y_\infty = \lim_{n \to \infty} Y_n = +\infty \), a.s. This would contradict (2.31) if we assume that \( P_x[\tau_A = \infty] > 0 \), because then \( E_x[Y_\infty] \geq E_x[Y_\infty 1\{\tau_A = \infty\}] = \infty \). Hence \( P_x[\tau_A = \infty] = 0 \) for all \( x \in \Sigma \), which means that the Markov chain is recurrent, by Lemma 2.5.1(ii).

For the ‘only if’ part, see the proof of Theorem 2.2.1 of [96]. \( \square \)

**Example 2.5.5.** Consider the nearest-neighbour random walk on \( \mathbb{Z}_+ \) from Section 2.2, and take \( f(x) = h(x) \) as defined at (2.7). Then Lemma 2.2.2 shows that

\[
E[f(X_{n+1}) - f(X_n) \mid X_n = x] = 0, \text{ for all } x \neq 0,
\]

so Theorem 2.5.2 shows that \( X_n \) is recurrent if \( \lim_{x \to \infty} h(x) = \infty \). This is a Lyapunov function proof of the ‘if’ half of Theorem 2.2.5(i). \( \triangle \)

**Example 2.5.6.** Let \( X_n \) be an irreducible Markov chain on a locally finite subset of \( \mathbb{R}_+ \). Suppose that for some bounded set \( A \),

\[
E[X_{n+1} - X_n \mid X_n = x] = 0, \text{ for all } x \notin A, \tag{2.32}
\]

i.e., the process has zero drift outside \( A \). Corollary 2.5.3 (with \( f(x) = x \)) shows that \( X_n \) is recurrent. However, without additional conditions the same result is not true if the state-space is allowed to be a subset of \( \mathbb{R} \) rather than just \( \mathbb{R}_+ \); see the notes at the end of this chapter. This contrasts with the situation when \( X_n \) is a spatially homogeneous random walk on \( \mathbb{R} \), i.e., a sum of i.i.d. random variables, in which case zero drift does imply recurrence; see e.g. [150, Ch. 9]. \( \triangle \)

To be able to conclude that zero drift implies recurrence for a Markov chain on \( \mathbb{R} \), we need to impose additional moments assumptions on the increments, including a uniform non-degeneracy condition.

**Theorem 2.5.7.** Let \( X_n \) be an irreducible Markov chain on a locally finite subset of \( \mathbb{R} \). Suppose that for some bounded set \( A \), (2.32) holds. Suppose that for some \( p > 2 \) and \( v > 0 \),

\[
\sup_x E[(X_{n+1} - X_n)^p \mid X_n = x] < \infty; \quad \inf_x E[(X_{n+1} - X_n)^2 \mid X_n = x] \geq v.
\]

Then \( X_n \) is recurrent.
Proof. We will apply Corollary 2.5.3 with \( f(x) = \log(1 + |x|) \). Write \( \Delta = X_1 - X_0 \) and \( E_x = \{|\Delta| < |x|\} \). We compute

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid X_n = x] = \mathbb{E}_x[(f(x + \Delta) - f(x))1(E_x)] + \mathbb{E}_x[(f(x + \Delta) - f(x))1(E^c_x)].
\]

On \( \{|\Delta| < |x|\} \) we have that \( x \) and \( x + \Delta \) have the same sign, so

\[
\begin{align*}
\mathbb{E}_x[(f(x + \Delta) - f(x))1(E_x)] &= \mathbb{E}_x \left[ \log \left( \frac{1 + |x + \Delta|}{1 + |x|} \right) 1(E_x) \right] \\
&= \mathbb{E}_x \left[ \log \left( 1 + \frac{\Delta \text{sgn}(x)}{1 + |x|} \right) 1(E_x) \right] \\
&\leq \left( \frac{\text{sgn}(x)}{1 + |x|} \right) \mathbb{E}_x[1(E_x)] - \frac{1}{6}(1 + |x|)^{-2} \mathbb{E}_x[\Delta^2 1(E_x)],
\end{align*}
\]

using the inequality \( \log(1 + y) \leq y - \frac{1}{5}y^2 \) for all \(-1 < y \leq 1\). Here, since \( \mathbb{E}_x \Delta = 0 \) for \( x \notin A \), as \( |x| \to \infty \),

\[ |\mathbb{E}_x[\Delta 1(E_x)]| \leq \mathbb{E}_x[|\Delta| 1(E^c_x)] \leq \mathbb{E}_x[|\Delta|^p |x|^{1-p}] = o(|x|^{-1}). \]

Similarly,

\[ \mathbb{E}_x[\Delta^2 1(E_x)] \geq v - \mathbb{E}_x[\Delta^2 1(E^c_x)] \geq v - o(1), \text{ as } |x| \to \infty. \]

Finally we estimate the term

\[ |\mathbb{E}_x[(f(x + \Delta) - f(x))1(E^c_x)]| \leq \mathbb{E}_x \left[ (\log(1 + |\Delta|) + \log(1 + 2|\Delta|)) 1(E^c_x) \right]. \]

Here,

\[ \log(1 + 2|\Delta|) 1(E^c_x) = \log(1 + 2|\Delta|)|\Delta|^p |\Delta|^{-p} 1(E^c_x) \leq |x|^{-p} \log(1 + 2|x|)|\Delta|^p, \]

for all \( x \) with \( |x| \) greater than some \( x_0 \) sufficiently large, using the fact that \( y \mapsto y^{-p} \log(1 + 2y) \) is eventually decreasing. It follows that

\[ |\mathbb{E}_x[(f(x + \Delta) - f(x))1(E^c_x)]| \leq 2|x|^{-p} \log(1 + 2|x|) \mathbb{E}_x[|\Delta|^p] = o(|x|^{-2}). \]

Combining these calculations we obtain

\[ \mathbb{E}[f(X_{n+1}) - f(X_n) \mid X_n = x] \leq -\frac{v}{6}(1 + |x|)^{-2} + o(|x|^{-2}), \]

which is negative for all \( x \) with \( |x| \) sufficiently large. Thus we may apply Corollary 2.5.3 to deduce recurrence. \( \square \)
Chapter 2. Semimartingale approach and Markov chains

The idea of Theorem 2.5.7, and the interplay between first and second moments of increments for determining recurrence, will be picked up again in detail in Chapter 3. When second moments fail to exist, heavy tailed phenomena come into play; some of these are explored in Chapter 5. In higher dimensions, the analogue of Theorem 2.5.7 fails, and either recurrence or transience is possible: see Chapter 4.

Next we formulate a criterion for transience.

**Theorem 2.5.8 (Transience criterion).** An irreducible Markov chain $X_n$ on a countable state space $\Sigma$ is transient if and only if there exist a function $f : \Sigma \to \mathbb{R}_+$ and a non-empty set $A \subset \Sigma$ such that

$$E[f(X_{n+1}) - f(X_n) \mid X_n = x] \leq 0, \text{ for all } x \in \Sigma \setminus A, \quad (2.33)$$

and

$$f(y) < \inf_{x \in A} f(x), \text{ for at least one site } y \in \Sigma \setminus A. \quad (2.34)$$

Note that, unlike the recurrence criterion, here the set $A$ need not be finite. The following simple example is important.

**Example 2.5.9.** Let $S_n$ be symmetric SRW on $\mathbb{Z}^d$. Write $\mathbf{e}_1, \ldots, \mathbf{e}_d$ for the standard orthonormal basis vectors and $U_d = \{ \pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_d \}$ for the set of possible jumps for the random walk. Let $\alpha > 0$. Consider the Lyapunov function $f : \mathbb{Z}^d \to (0, 1]$ defined by $f(0) = 1$ and $f(x) = \|x\|^{-2\alpha}$ for $x \neq 0$. Then, for any $x \in \mathbb{Z}^d$ with $\|x\| > 1$, for any $\mathbf{e} \in U_d$,

$$f(x + \mathbf{e}) - f(x) = \|x\|^{-2\alpha} \left[ \left( \left( \frac{\|x + \mathbf{e}\|^2}{\|x\|^2} \right)^{-\alpha} - 1 \right) \right]$$

$$= \|x\|^{-2\alpha} \left[ \left( 1 + \frac{2\mathbf{e} \cdot x + 1}{\|x\|^2} \right)^{-\alpha} - 1 \right]$$

$$= \|x\|^{-2\alpha} \left( -\alpha \frac{(2\mathbf{e} \cdot x + 1)}{\|x\|^2} + 2\alpha(\alpha + 1)(\mathbf{e} \cdot x)^2 + O(\|x\|^{-3}) \right),$$

by Taylor’s formula. Using the facts that $\sum_{\mathbf{e} \in U_d} 1 = 2d$, $\sum_{\mathbf{e} \in U_d} \mathbf{e} \cdot x = 0$, and $\sum_{\mathbf{e} \in U_d} (\mathbf{e} \cdot x)^2 = 2\|x\|^2$, we obtain

$$E[f(S_{n+1}) - f(S_n) \mid S_n = x] = \frac{1}{2d} \sum_{\mathbf{e} \in U_d} (f(x + \mathbf{e}) - f(x))$$

$$= \frac{\alpha}{d} \|x\|^{-2\alpha} \left( 2(\alpha + 1) - d + O(\|x\|^{-1}) \right),$$
which is negative for all \( x \) with \( \| x \| \) sufficiently large provided we choose \( \alpha \in (0, d^{-2}) \), which we may do for any \( d \geq 3 \). Thus Theorem 2.5.8 shows that SRW on \( \mathbb{Z}^d \) is transient for \( d \geq 3 \); this is a direct Lyapunov-function proof of the ‘transience’ part of Pólya’s theorem (Theorem 1.2.1).

The central semimartingale idea of the proof of Theorem 2.5.8 is very simple, and will be useful in other contexts; thus we extract it as the following lemma.

**Lemma 2.5.10.** Let \( X_n \) be an \( \mathcal{F}_n \)-adapted process with state space \( (\Sigma, \mathcal{E}) \). Suppose that \( f : \Sigma \to \mathbb{R}_+ \) is measurable, and \( A \in \mathcal{E} \) is such that

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n \in \Sigma \setminus A\},
\]

for all \( n \geq 0 \). Then, a.s.,

\[
\mathbb{P}[\tau_A < \infty \mid \mathcal{F}_0] \leq \frac{f(X_0)}{\inf_{x \in A} f(x)}.
\]

**Proof.** Define the process \( Y_n = f(X_n \wedge \tau_A) \). Then (2.35) implies that \( Y_n \) is a supermartingale adapted to \( \mathcal{F}_n \). Since \( Y_n \) is also non-negative, Theorem 2.3.6 implies that there is a random variable \( Y_\infty \in \mathbb{R}_+ \) such that \( \lim_{n \to \infty} Y_n = Y_\infty \) a.s. Also, by Theorem 2.3.11,

\[
Y_0 \geq \mathbb{E}[Y_\infty \mid \mathcal{F}_0] \geq \mathbb{E}[Y_\infty 1\{\tau_A < \infty\} \mid \mathcal{F}_0].
\]

Here we have that, a.s.,

\[
Y_\infty 1\{\tau_A < \infty\} = \lim_{n \to \infty} Y_n 1\{\tau_A < \infty\} = f(X_{\tau_A}) 1\{\tau_A < \infty\} \geq \inf_{x \in A} f(x) 1\{\tau_A < \infty\}.
\]

So we obtain

\[
f(X_0) = Y_0 \geq \mathbb{P}[\tau_A < \infty \mid \mathcal{F}_0] \inf_{x \in A} f(x),
\]

which yields the result. \( \square \)

A Lyapunov function that is a supermartingale outside a set gives an upper bound on the probability the the processes started outside that set ever reaches its destination.
Proof of Theorem 2.5.8. For the ‘if’ part, we apply Lemma 2.5.10. In this case, with $y \in \Sigma \setminus A$ as in (2.34), we obtain
\[
\mathbb{P}_y[\tau_A < \infty] \leq \frac{f(y)}{\inf_{x \in A} f(x)} < 1,
\]
proving the transience of the Markov chain $X_n$, by Lemma 2.5.1(i).

For the ‘only if’ part, fix an arbitrary $x_0 \in \Sigma$, set $A = \{x_0\}$, and $f(x) = \mathbb{P}_x[\tau_{x_0} < \infty]$ for $x \in \Sigma$ (so, in particular, $f(x_0) = 1$). Then (2.33) holds with equality for all $x \neq x_0$, and, by transience, one can find $y \in \Sigma$ such that $f(y) < 1 = f(x_0)$.

An inspection of the ‘only if’ part of the preceding proof shows that, if a Markov chain is transient, a function $f$ will be an apt Lyapunov function if the value $f(x)$ is given by the probability that the process reaches some fixed set starting from $x$. This suggests the following heuristic rule:

To find a Lyapunov function that proves transience, fix a well-chosen $A \subset \Sigma$, and set $f(x)$ to be your reasonable guess for the probability to ever reach $A$ starting from $x$.

Example 2.5.11. Consider the one-dimensional SRW $S_n$, as in Example 2.1.7.

- To prove recurrence for $p = \frac{1}{2}$, take $A = \{0\}$ and $f(x) = |x|$; then, (2.30) holds with equality for all $x \neq 0$.

- To prove transience of the SRW for $p > \frac{1}{2}$, take $A = (-\infty, 0]$ and $f(x) = \left(\frac{1-p}{p}\right)^x$; then $f(x) \to 0$ as $x \to +\infty$ and (2.33) holds with equality. For $p < \frac{1}{2}$, take $A = [0, +\infty)$ and the same $f$. Or just take $A = \{0\}$ in both cases, and choose any $y > 0$ in the first case and any $y < 0$ in the second one to have (2.34). \[\triangle\]

Theorem 2.5.8 has the following consequence, which is sometimes useful.

**Corollary 2.5.12.** An irreducible Markov chain $X_n$ on a countable state space $\Sigma$ is transient if and only if there exist a non-constant function $h : \Sigma \to \mathbb{R}$ and a state $x_0 \in \Sigma$ such that $\sup_{x \in \Sigma} |h(x)| < \infty$ and
\[
\mathbb{E}[h(X_{n+1}) - h(X_n) \mid X_n = x] = 0, \text{ for all } x \neq x_0. \tag{2.36}
\]

**Proof.** For the ‘if’ part of the proof, by swapping $h$ with $-h$, it suffices to consider the non-constant condition in the form $h(x) < h(x_0)$ for some
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Let $x \not= x_0$. Then taking $f(x) = h(x) - \inf_{x \in \Sigma} h(x)$ we have $f : \Sigma \to \mathbb{R}_+$ satisfies the conditions of Theorem 2.5.8 with $A = \{x_0\}$, and transience follows.

For the ‘only if’ statement, the construction in the proof of Theorem 2.5.8 serves again: for fixed $x_0 \in \Sigma$, set $A = \{x_0\}$, and $h(x) = \mathbb{P}_x[\tau_{x_0} < \infty]$ for $x \in \Sigma$. Then (2.36) holds for all $x \not= x_0$, and, by transience, one can find $y \in \Sigma$ such that $h(y) < 1 = h(x_0)$.

Example 2.5.13. Consider the nearest-neighbour random walk on $\mathbb{Z}_+$ from Section 2.2. With $h$ as defined at (2.7), we see that if $\sup_x h(x) < \infty$, then Lemma 2.2.2 shows that the conditions of Corollary 2.5.12 are satisfied, so that $X_n$ is transient. This is a Lyapunov function proof of the ‘only if’ half of Theorem 2.2.5(i). \(\square\)

Next, we formulate a sufficient condition for transience, first for general processes, and then for Markov chains.

**Theorem 2.5.14.** For a real-valued $\mathcal{F}_n$-adapted process $X_n$, define $\tau_a = \min\{n \geq 1 : X_n - X_0 \leq -a\}$, $a \geq 0$, and suppose that, for some $\varepsilon > 0$,

$$E[X_{n+1} - X_n \mid \mathcal{F}_n] \geq \varepsilon, \text{ on } \{n < \tau_a\},$$

for all $n \geq 0$. Suppose also there exists $K > 0$ such that, for all $n \geq 0$,

$$|X_{n+1} - X_n| \leq K, \text{ on } \{n < \tau_a\}.$$  

Then there exist positive constants $c_1, c_2$ depending only on $\varepsilon, K$, such that

$$\mathbb{P}[\tau_a < \infty \mid \mathcal{F}_0] \leq c_1 e^{-c_2 a}, \text{ for all } a \geq 0.$$  

Also, we have

$$\mathbb{P}[\tau_0 = \infty \mid \mathcal{F}_0] = \mathbb{P}[X_n > X_0 \text{ for all } n \geq 1 \mid \mathcal{F}_0] \geq c_3 > 0,$$

for some constant $c_3$ depending only on $\varepsilon$ and $K$.

**Proof.** Define the process $Y_n = X_n - \varepsilon n$; (2.37) implies that $Y_{n \wedge \tau_a}$ is a submartingale. By (2.37), it holds a.s. that $|Y_{(n+1) \wedge \tau_a} - Y_{n \wedge \tau_a}| \leq K + \varepsilon$, so, using the one-sided Azuma–Hoeffding inequality (2.26) applied to the submartingale $Y_{n \wedge \tau_a}$, together with the union bound, we obtain

$$\mathbb{P}[\tau_a < \infty \mid \mathcal{F}_0] = \mathbb{P}[\text{there exists } n \geq 1 \text{ such that } Y_{n \wedge \tau_a} - Y_0 \leq -a - \varepsilon n \mid \mathcal{F}_0]$$

$$\leq \sum_{n=1}^{\infty} \exp\left(-\frac{(a + \varepsilon n)^2}{2n(\varepsilon + K)^2}\right).$$
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\[ \leq \sum_{n=1}^{\infty} \exp\left(-\frac{2a \varepsilon n + (\varepsilon n)^2}{2(n + K)^2}\right) \]

\[ = \exp\left(-\frac{\varepsilon}{(\varepsilon + K)^2} a\right) \sum_{n=1}^{\infty} \exp\left(-\frac{\varepsilon^2}{2(\varepsilon + K)^2} n\right), \]

proving (2.39) with \( c_1 = \sum_{n=1}^{\infty} \exp\left(-\frac{\varepsilon^2}{2(\varepsilon + K)^2} n\right) \) and \( c_2 = \frac{\varepsilon}{(\varepsilon + K)^2} a \).

To prove (2.40), observe first that, by (2.38), on \( \{ n < \tau_a \} \),

\[ X_{n+1} - X_n \leq \varepsilon \left[ 1 \{ X_{n+1} - X_n \in (0, \varepsilon/2) \} + K \{ X_{n+1} - X_n \geq \varepsilon/2 \} \right], \]

so, taking expectation conditional on \( F_n \) and using (2.37), we obtain

\[ \varepsilon \left[ 1 \{ \tau_a > n \} \right] \leq \frac{\varepsilon}{2} + K \mathbb{P}[X_{n+1} - X_n \geq \varepsilon/2 \mid F_n] \]

and thus

\[ \mathbb{P}[X_{n+1} - X_n \geq \varepsilon/2 \mid F_n] \geq \frac{\varepsilon}{2K} 1\{ \tau_a > n \}. \tag{2.41} \]

Now, choose \( a_0 \) large enough so that \( c_1 e^{-c_2 a} < \frac{1}{2} \) for all \( a \geq a_0 \). Denote \( n_0 = \lceil \frac{a_0}{\varepsilon/2} \rceil \). The idea now is that, by (2.41), the process can initially make at least \( n_0 \) steps to the right of amount at least \( \varepsilon/2 \) with uniformly positive probability (so that its distance from the starting point is at least \( a_0 \)), and then by (2.39) the process will not fall below its initial position with probability at least \( \frac{1}{2} \). To make this argument rigorous, observe that by (2.41),

\[
\mathbb{P}[X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n \mid F_0] = \mathbb{E}\left[ \mathbb{P}[X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n \mid F_{n-1}] \mid F_0 \right] \\
= \mathbb{E}\left[ 1\{ X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n - 1 \} \mathbb{P}[X_n - X_{n-1} \geq \varepsilon/2 \mid F_{n-1}] \mid F_0 \right] \\
\geq \frac{\varepsilon}{2K} \mathbb{P}[X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n - 1 \mid F_0],
\]

and so, by induction,

\[ \mathbb{P}[X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n \mid F_0] \geq \left( \frac{\varepsilon}{2K} \right)^{n}. \tag{2.42} \]

Abbreviating \( \tilde{X}_k = X_{k+n_0} \) and \( \tilde{\tau}_a = \min\{ k \geq 1 : \tilde{X}_k - \tilde{X}_0 \leq -a \} \), we have

\[
\mathbb{P}[\tau_0 = \infty \mid F_0] = \mathbb{E}\left[ \mathbb{E}[1\{ \tau_0 = \infty \} \mid F_{n_0}] \mid F_0 \right]
\]
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\[ \geq \mathbb{E}\left[\mathbf{1}\{\tau_0 = \infty, X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n_0\} | F_{n_0}\right] \mathbb{E}\left[\mathbf{1}| F_{n_0}\right]
\]
\[ \geq \mathbb{E}\left[\mathbf{1}\{X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n_0\} \mathbb{P}[\tau_{a_0} = \infty | F_{n_0}] | F_0\right]
\]
\[ \geq \frac{1}{2} \mathbb{P}[X_k - X_{k-1} \geq \varepsilon/2 \text{ for all } k = 1, \ldots, n_0 | F_0]
\]
\[ \geq \frac{1}{2} \left(\frac{\varepsilon}{2K}\right)^{n_0},\]
proving (2.40).

A consequence of the above result is the following sufficient condition for the transience of Markov chains.

**Theorem 2.5.15.** Let \( X_n \) be an irreducible Markov chain on a countable state space \( \Sigma \). Suppose that there exist a function \( f : \Sigma \to \mathbb{R} \), some \( h \in \mathbb{R} \) for which the set \( A := \{ x \in \Sigma : f(x) \leq h \} \) is neither \( \emptyset \) nor \( \Sigma \), and constants \( \varepsilon > 0 \) and \( K \in \mathbb{R}_+ \) such that

\[ \mathbb{E}[f(X_{n+1}) - f(X_n) | X_n = x] \geq \varepsilon, \text{ for all } x \in \Sigma \setminus A; \text{ and } (2.43) \]
\[ \mathbb{P}(|f(X_{n+1}) - f(X_n)| \leq K | X_n = x) = 1, \text{ for all } x \in \Sigma \setminus A. \] (2.44)

Then, the Markov chain is transient.

**Proof.** Choose any \( x_0 \in \Sigma \setminus A \), so that \( f(x_0) > h \), and take \( X_0 = x_0 \). Let \( a = f(x_0) - h > 0 \), and consider the process \( Y_n = f(X_n) \). Then

\[ \tau_a = \min\{n \geq 0 : Y_n - Y_0 \leq -a\} = \min\{n \geq 0 : X_n \in A\}, \]

so that an application of Theorem 2.5.14 to the process \( Y_n \) shows that with positive probability \( X_n \) started from \( X_0 = x_0 \) never reaches \( A \), i.e., the process is transient. \( \square \)

**Example 2.5.16.** Again, consider the one-dimensional SRW \( S_n \), as in Example 2.1.7. Assume that \( p > \frac{1}{2} \); to prove the transience using Theorem 2.5.15, take \( f(x) = x \) and \( a = 0 \). Then, (2.43) holds with \( \varepsilon = 2p - 1 > 0 \) and (2.44) holds with \( K = 1 \). \( \triangle \)

Heuristically, when something has drift in a fixed direction, it should go to that direction in the limit, and Theorem 2.5.15 is a convenient tool to make this intuition rigorous; however, the restriction on the size of the jumps cannot be completely removed, as the following example shows.
**Example 2.5.17.** Now, consider the random walk on the half-line $\hat{S}$ of Example 2.1.8, with, say, $p = \frac{1}{3}$ (so that it is positive recurrent). It is straightforward to check, however, that (2.43) still holds with uniformly positive $\epsilon$ for $f(x) = 3^x$ and $a = 1$:

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid X_n = x] = \left(\frac{2}{3} \cdot 3^{x-1} + \frac{1}{3} \cdot 3^{x+1} - 3^x\right)
\]

\[
= 2 \cdot 3^{x-2} \geq \frac{2}{3}, \text{ for all } x \geq 1. \quad \triangle
\]

Example 2.5.17 shows that one cannot omit the condition (2.44) from Theorem 2.5.15. However, if one is interested only in transience, rather than exponential estimates such as those in Theorem 2.5.14, then the uniform bounds on the increments in Theorems 2.5.14 and 2.5.15 can be relaxed and replaced by a moments assumption, as in the following result. The idea of the proof is elementary, combining truncation and a Taylor’s formula computation, but important, and previews the methods of Chapter 3.

**Theorem 2.5.18.** Let $X_n$ be a stochastic process on $\mathbb{R}_+$ adapted to a filtration $\mathcal{F}_n$. Suppose that there exist constants $\epsilon > 0$, $\delta > 0$, $B < \infty$, and $x_0 < \infty$ such that, a.s.,

\[
\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \geq \epsilon, \text{ on } \{X_n \geq x_0\}; \tag{2.45}
\]

\[
\mathbb{E}[|X_{n+1} - X_n|^{1+\delta} \mid \mathcal{F}_n] \leq B, \text{ on } \{X_n \geq x_0\}. \tag{2.46}
\]

There exist $\alpha > 0$ and $x_1 \in \mathbb{R}_+$ such that, with $\lambda_{x_1} = \min\{n \geq 0 : X_n \leq x_1\}$,

\[
\mathbb{P}[\lambda_{x_1} < \infty \mid \mathcal{F}_0] \leq \left(\frac{1 + X_0}{1 + x_1}\right)^{-\alpha}, \text{ a.s.}
\]

**Proof.** It suffices to take $\delta \in (0, 1)$. Let $\alpha > 0$, to be specified later. The idea is to show that, for suitable choice of $\alpha$ we can find $x_1 \in \mathbb{R}_+$ for which

\[
\mathbb{E}[(1 + X_{n+1})^{-\alpha} - (1 + X_n)^{-\alpha} \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n \geq x_1\}, \tag{2.47}
\]

and then use Lemma 2.5.10. To ease notation, write $\Delta_n = X_{n+1} - X_n$ for the duration of this proof. Let $\gamma \in (0, 1)$, to be specified later, and let $A_n = \{|\Delta_n| \leq (1 + X_n)^{\gamma}\}$. Then, by Taylor’s formula,

\[
(1 + X_{n+1})^{-\alpha} - (1 + X_n)^{-\alpha}) 1(A_n)
\]

\[
= (1 + X_n)^{-\alpha} \left[1 + \frac{\Delta_n}{1 + X_n}\right]^{-\alpha} - 1 \right] 1(A_n)
\]
= -\alpha (1 + X_n)^{-1-\alpha} \Delta_n \mathbf{1}(A_n) + \zeta_n,$

where there is a constant $C < \infty$ (depending on $\gamma$) for which the error term $\zeta_n$ satisfies

$$|\zeta_n| \leq C(1 + X_n)^{-2-\alpha} (\Delta_n)^2 \mathbf{1}(A_n), \text{ a.s.}$$

Note that, a.s.,

$$[\mathbb{E}[\Delta_n \mathbf{1}(A_n) \mid \mathcal{F}_n] - \mathbb{E}[\Delta_n \mid \mathcal{F}_n]] \leq \mathbb{E}[|\Delta_n^2 \mathbf{1}(A_n) \mid \mathcal{F}_n]
\leq (1 + X_n)^{-\gamma \delta} \mathbb{E}[|\Delta_n|^{1+\delta} \mid \mathcal{F}_n]
\leq B(1 + X_n)^{-\gamma \delta},$$

on $\{X_n \geq x_0\}$, by (2.46). Moreover, a.s.,

$$\mathbb{E}[|\zeta_n| \mid \mathcal{F}_n] \leq C(1 + X_n)^{-2-\alpha+\gamma(1-\delta)} \mathbb{E}[|\Delta_n|^{1+\delta} \mid \mathcal{F}_n]
\leq BC(1 + X_n)^{-2-\alpha+\gamma(1-\delta)},$$

on $\{X_n \geq x_0\}$. So we obtain, for some constant $C' < \infty$, on $\{X_n \geq x_0\}$,

$$\mathbb{E}[((1 + X_{n+1})^{-\alpha} - (1 + X_n)^{-\alpha}) \mathbf{1}(A_n) \mid \mathcal{F}_n]
\leq -\alpha \varepsilon (1 + X_n)^{-1-\alpha} + C'(1 + X_n)^{-1-\alpha-\gamma \delta} + C''(1 + X_n)^{-2-\alpha+\gamma(1-\delta)}.$$

The first term on the right-hand side of the last display dominates as $X_n \to \infty$ provided $\gamma(1 - \delta) < 1$, which is indeed the case. Hence we can find $x_2$ with $x_0 \leq x_2 < \infty$ for which

$$\mathbb{E}[((1 + X_{n+1})^{-\alpha} - (1 + X_n)^{-\alpha}) \mathbf{1}(A_n) \mid \mathcal{F}_n] \leq -(\alpha \varepsilon / 2)(1 + X_n)^{-1-\alpha},$$

on $\{X_n \geq x_2\}$. On the other hand,

$$\mathbb{E}[|(1 + X_{n+1})^{-\alpha} - (1 + X_n)^{-\alpha}| \mathbf{1}(A_n) \mid \mathcal{F}_n] \leq \mathbb{P}[|\Delta_n|^{1+\delta} \geq (1 + X_n)^{\gamma(1+\delta)} \mid \mathcal{F}_n]
\leq B(1 + X_n)^{-\gamma(1+\delta)},$$

on $\{X_n \geq x_0\}$, by Markov’s inequality and (2.46). The result (2.47) now follows provided $\gamma(1 + \delta) > 1 + \alpha$, i.e., $\gamma \geq \frac{1+\alpha}{1+\delta}$, a choice of $\gamma \in (0,1)$ that we can achieve provided we take $\alpha \in (0, \delta)$. \hfill \Box

Theorem 2.5.18 has the following consequence for Markov chains.
Theorem 2.5.19. Assume that, for an irreducible Markov chain $X_n$ on a countable state space $\Sigma$, one can find a function $f : \Sigma \to \mathbb{R}_+$ and $a \in (0, \infty)$, such that the set $A := \{ x \in \Sigma : f(x) < a \} \neq \emptyset$ is a proper subset of $\Sigma$, and for some $\varepsilon > 0$, $\delta > 0$, and $B \in \mathbb{R}_+$, (2.43) holds, and

$$E[|f(X_{n+1}) - f(X_n)|^{1+\delta} | X_n = x] \leq B, \text{ for all } x \in \Sigma \setminus A. \quad (2.48)$$

Then, the Markov chain is transient.

Proof. The idea is to set $Y_n = f(X_n)$, $F_n = \sigma(X_0, \ldots, X_n)$, and apply Theorem 2.5.18 to the process $Y_n$. Then (2.45) and (2.46) hold for $Y_n$ with $x_0 = a$. With $x_1$ the constant arising from the conclusion of Theorem 2.5.18, set $A' = \{ x \in \Sigma : f(x) \leq x_1 \}$; then Theorem 2.5.18 yields

$$P_x[\tau_{A'} < \infty] \leq \left( \frac{1 + f(x)}{1 + f(x_1)} \right)^{-\alpha} < 1,$$

for all $x \in \Sigma \setminus A'$. Then Lemma 2.5.1(i) gives transience. \qed

A Lyapunov function that has strictly positive drift outside some finite set yields transience provided it satisfies a bounded moments condition.

2.6 Expectations of hitting times and positive recurrence

We start this section with a technical lemma for Markov chains that relates the moments of hitting times of finite sets to those of individual states.

Lemma 2.6.1. Let $X_n$ be an irreducible Markov chain on a countable state space $\Sigma$. Let $\alpha > 0$ and $A$ be a finite non-empty subset of $\Sigma$ such that $E_x(\tau_x^A)^\alpha < \infty$ for all $x \in A$. Then $E_x(\tau_x^\Sigma)^\alpha < \infty$ for all $x \in \Sigma$.

Proof. Define $\sigma_0 := 0$ and, for $k \geq 0$,

$$\sigma_{k+1} = \min\{ n > \sigma_k : X_n \in A \},$$

so that $\sigma_k$ is the $k$-th passage time of $A$. Fix an arbitrary $x \in A$ and assume that $X_0 = x$; then, the process $Y_n = X_{\sigma_n}$ is a (finite) Markov chain on $A$ starting from $Y_0 = x$. It is also not hard to see that irreducibility of $Y_n$ follows from irreducibility of $X_n$. Define

$$w = \min\{ n \geq 1 : Y_n = x \}$$
2.6. Positive recurrence

to be the return time to \(x\) for the process \(Y_n\).

Now, since the state space of \(Y_n\) is finite, there exist constants \(K_1 > 0\) and \(\delta > 0\) such that
\[
\mathbb{P}_x[w = k] \leq K_1 e^{-\delta k}.
\] (2.49)

Let \(q(y, z)\) be the one-step transition probabilities from \(y \in A\) to \(z \in A\) for the Markov chain \(Y_n\), and set
\[
r_1 := \min\{q(y, z) : y, z \in A, q(y, z) > 0\},
\]
which is positive since \(A\) is finite. Set \(u_k := \sigma_{k+1} - \sigma_k\). By assumption,
\[
\mathbb{E}[u_k^\alpha | Y_k = y] < \infty, \text{ for all } y \in A.
\] (2.50)

Also, we have for all \(y, z \in A\) with \(q(y, z) > 0\),
\[
\mathbb{E}[u_k^\alpha | Y_k = y] \geq \mathbb{E}[u_k^\alpha \mathbf{1}\{Y_{k+1} = z\} | Y_k = y] = q(y, z) \mathbb{E}[u_k^\alpha | Y_k = y, Y_{k+1} = z],
\]
and hence, since \(r_1 > 0\),
\[
\mathbb{E}[u_k^\alpha | Y_k = y, Y_{k+1} = z] \leq \frac{1}{r_1} \mathbb{E}[u_k^\alpha | Y_k = y] < \infty.
\]

So, since \(A\) is finite, there exists \(r_2 \in \mathbb{R}_+\) such that
\[
\mathbb{E}[u_k^\alpha | Y_k, Y_{k+1}] \leq r_2, \text{ a.s.}
\]

Then, using the simple inequality
\[
(a_1 + \cdots + a_n)^\alpha \leq n^\alpha (a_1^\alpha + \cdots + a_n^\alpha) \quad \text{for all } \alpha > 0 \text{ and } a_i \geq 0,
\] (2.51)
we have for all \(k\),
\[
\mathbb{E}[(\sigma_k)^\alpha | w = k] = \mathbb{E}[(u_0 + \cdots + u_{k-1})^\alpha | w = k] \\
\leq k^\alpha \mathbb{E}[u_0^\alpha + \cdots + u_{k-1}^\alpha | Y_1 \neq x, \ldots, Y_{k-1} \neq x, Y_k = x] \\
\leq k^{\alpha+1} r_2.
\]

Since \(\mathbb{E}_x(\tau_x^+)^\alpha = \mathbb{E}_x \sigma_w^\alpha\) and we have by (2.49)
\[
\mathbb{E}_x \sigma_w^\alpha = \sum_{k=1}^{\infty} \mathbb{P}_x[w = k] \mathbb{E}[\sigma_k^\alpha | w = k]
\]
we obtain that $\mathbb{E}_x(\tau_x^+)^\alpha < \infty$ for all $x \in A$.

Now, fix any $x_0 \in A$, consider an arbitrary $y \in \Sigma$, and define $A' := \{x_0, y\}$. Since the Markov chain is irreducible, there exist $\varepsilon > 0$ and $k \geq 1$ such that

$$\mathbb{P}_{x_0}[X_k = y, X_m \notin A' \text{ for all } m = 1, \ldots, k - 1] \geq \varepsilon.$$ 

Then, we have

$$E_{x_0}(\tau_x^+)^\alpha \geq E_{x_0}[(\tau_x^+)^\alpha; X_k = y, X_m \notin A' \text{ for all } m = 1, \ldots, k - 1] \geq \varepsilon E_y[(\tau_y^+)^\alpha],$$

so we must have $E_y(\tau_y^+)^\alpha < \infty$. But this means that $\mathbb{E}_z(\tau_z^+)^\alpha \leq \mathbb{E}_z(\tau_z^+)^\alpha < \infty$ for all $z \in A'$. Repeating the above argument with $A'$ on the place of $A$, we obtain that $\mathbb{E}_y(\tau_y^+)^\alpha < \infty$, and this concludes the proof. □

Next, we state a couple of simple results, which are, however, very useful for proving positive recurrence of (Markov) processes.

**Theorem 2.6.2.** Let $X_n$ be a real-valued integrable $\mathcal{F}_n$-adapted process, and let $\tau$ be a stopping time with respect to $\mathcal{F}_n$; suppose also that $\mathbb{E}X_{n\wedge \tau} \geq 0$ a.s. for all $n$. Assume that for some $\varepsilon > 0$ and all $n \geq 0$, on $\{\tau > n\}$,

$$E[X_{n+1} - X_n \mid \mathcal{F}_n] \leq -\varepsilon, \text{ a.s.} \tag{2.52}$$

Then it holds that

$$E\tau \leq \frac{E X_0}{\varepsilon} < \infty. \tag{2.53}$$

**Proof.** The result is a version of Theorem 2.4.8, but we recapitulate the (simple) proof. We may rewrite the condition (2.52) as

$$E[X_{(m+1)\wedge \tau} - X_{m\wedge \tau} \mid \mathcal{F}_m] \leq -\varepsilon \mathbb{1}\{\tau > m\}, \text{ a.s., for any } m \geq 0.$$ 

We take expectations

$$E[X_{(m+1)\wedge \tau}] - E[X_{m\wedge \tau}] \leq -\varepsilon \mathbb{P}[\tau > m],$$

and then sum the above over $m$ from 0 to $n$ to obtain that

$$0 \leq E X_{(n+1)\wedge \tau} \leq -\varepsilon \sum_{m=0}^{n} \mathbb{P}[\tau > m] + E X_0.$$
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So, passing to the limit, we have

$$E \tau = \lim_{n \to \infty} \sum_{m=0}^{n} P[\tau > m] \leq \frac{E X_0}{\varepsilon} < \infty,$$

which yields (2.53).

Remark 2.6.3. Under the conditions of Theorem 2.6.2, Dynkin’s formula (2.9) (with $f$ the identity) gives $E[n \wedge \tau] \leq \varepsilon^{-1} E X_0$, which first shows that $P[\tau = \infty] = 0$ and then, with Fatou’s lemma, provides an alternative proof of the theorem.

A consequence of Theorem 2.6.2 is the following positive recurrence criterion for Markov chains.

**Theorem 2.6.4** (Foster’s criterion). An irreducible Markov chain $X_n$ on a countable state space $\Sigma$ is positive recurrent if and only if there exist a positive function $f : \Sigma \to \mathbb{R}_+$, a finite non-empty set $A \subset \Sigma$, and $\varepsilon > 0$ such that

$$E[f(X_{n+1}) - f(X_n) \mid X_n = x] \leq -\varepsilon, \text{ for all } x \in \Sigma \setminus A, \quad (2.54)$$

$$E[f(X_{n+1}) \mid X_n = x] < \infty, \text{ for all } x \in A. \quad (2.55)$$

**Proof.** Let $p(x, y)$ be the transition probabilities of the Markov chain. For $y \in \Sigma$ define the stopping times

$$\tau_y = \min\{n \geq 0 : X_n = y\},$$

and for $B \subset \Sigma$ let

$$\tau_B = \min\{n \geq 0 : X_n \in B\} = \min_{y \in B} \tau_y.$$

To prove the “only if” part, it is enough to set $A = \{x_0\}$ (where $x_0 \in \Sigma$ is any fixed state), and then notice that for $x \neq x_0$,

$$E_x \tau_{x_0} = \sum_{y \in \Sigma} p(x_0, y) E_y [1 + \tau_{x_0}]$$

and that

$$\sum_{y \in \Sigma} p(x_0, y) E_y \tau_{x_0} = E_{x_0} \tau_{x_0}^+ < \infty,$$

so the function $f(x) = E_x \tau_{x_0}$ satisfies (2.54)–(2.55) with $\varepsilon = 1$. 
To prove the “if” part, take \( X_0 = x \in \Sigma \). Then \( f(X_0) \) is integrable, and \( f(X_n) \) is integrable by an induction based on (2.54) and (2.55). Now Theorem 2.6.2 applied to the non-negative, integrable process \( f(X_n) \) with the stopping time \( \tau_A \) implies that

\[
\mathbb{E}_x \tau_A \leq \frac{f(x)}{\varepsilon}, \text{ for all } x \notin A.
\]

(2.56)

So, for any \( y \in A \), by (2.55) and (2.56)

\[
\mathbb{E}_y \tau_A = \sum_{z \in A} p(y, z) + \sum_{z \notin A} p(y, z) \mathbb{E}_z[1 + \tau_A] \\
= 1 + \sum_{z \notin A} p(y, z) \mathbb{E}_z \tau_A \\
\leq 1 + \varepsilon^{-1} \sum_{z \notin A} p(y, z) f(z) < \infty.
\]

Then the \( \alpha = 1 \) case of Lemma 2.6.1 implies that the Markov chain \( X_n \) is positive recurrent, and this concludes the proof of Theorem 2.6.4.

Of course, a successful application of Theorem 2.6.4 for proving the positive recurrence of a Markov chain depends on one’s ability to find a Lyapunov function \( f \) that works. As observed in the above proof, such a function always exists and is given by the expected time to reach some fixed state. While it is not always possible to obtain an exact expression for this expected time, one may try to guess an approximation for it, and check if it works as a Lyapunov function in Theorem 2.6.4:

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**Example 2.6.5.** Consider \( \hat{S}_n \), the nearest-neighbour random walk on the half-line from Example 2.1.8, with \( p < \frac{1}{2} \). Intuitively speaking, the particle just drifts towards the origin with constant speed, so the time of reaching the origin from \( x \) should be proportional to \( x \) (i.e., the distance to the origin). So, consistently with the above heuristic rule, one can set \( A = \{0\} \) and \( f(x) = x \); then, (2.54) holds with \( \varepsilon = 1 - 2p > 0 \), thus proving the positive recurrence of \( \hat{S}_n \) with \( p < \frac{1}{2} \).
Theorem 2.6.2 has many uses beyond the case where $\tau$ is a hitting time of a finite set, as the next two examples illustrate.

**Example 2.6.6.** Let $(\xi_n, n \geq 0)$ be a Markov chain on $\Sigma \subseteq \mathbb{R}^2_+$; denote its increments by $\theta_n := \theta_{n+1} - \theta_n$, as in Section 1.4. Write $\xi_n = (\xi_n^{(1)}, \xi_n^{(2)})$ in Cartesian coordinates. Let $\tau = \min\{n \geq 0 : \xi_n^{(1)} \xi_n^{(2)} = 0\}$, the first hitting time of the boundary of the quarter-plane $\Sigma_0 = \{(x, y) \in \mathbb{R}^2_+ : xy = 0\}$.

Suppose that for some $B \in \mathbb{R}_+$, 

$$
\mathbb{E}[\theta_n \mid \xi_n = x] = 0, \text{ for all } x \in \Sigma \setminus \Sigma_0;
$$

$$
\mathbb{E}[\|\theta_n\|^2 \mid \xi_n = x] \leq B \text{ for all } x \in \Sigma \setminus \Sigma_0.
$$

Suppose also that the components of the increments have a constant covariance, i.e.,

$$
\mathbb{E}[(\xi_{n+1}^{(1)} - \xi_n^{(1)})(\xi_{n+1}^{(2)} - \xi_n^{(2)}) \mid \xi_n = x] = \rho, \text{ for all } x \in \Sigma \setminus \Sigma_0,
$$

for a constant $\rho$; with the notation at (1.9), this means that the off-diagonal element of $M(x)$ is equal to $\rho$. Then, for $x \in \Sigma \setminus \Sigma_0$,

$$
\mathbb{E}[\xi_{n+1}^{(1)} \xi_{n+1}^{(2)} - \xi_n^{(1)} \xi_n^{(2)} \mid \xi_n = x] = \rho. \quad (2.57)
$$

Hence if $\rho < 0$ we may apply Theorem 2.6.2 with $X_n = \xi_n^{(1)} \xi_n^{(2)}$ to deduce that $\mathbb{E}\tau < \infty$.

Remarkably, one may also explicitly compute $\mathbb{E}\tau$ when it exists. Suppose that $\mathbb{E}\tau < \infty$. For each $k \in \{1, 2\}$, the process $\xi_{n \wedge \tau}^{(k)}$ is a non-negative martingale adapted to $\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)$, and converges to $\xi_{\tau}^{(k)}$. The associated quadratic variation process satisfies $\langle \xi_{n \wedge \tau}^{(k)} \rangle_{n \wedge \tau} \leq B(n \wedge \tau)$, so $\mathbb{E}[\langle \xi_{n \wedge \tau}^{(k)} \rangle_{n \wedge \tau}^2] \leq B \mathbb{E}\tau < \infty$. Thus the martingales $\xi_{n \wedge \tau}^{(k)}$ are uniformly bounded in $L^2$, and hence converge in $L^2$ (see e.g. Theorem 5.4.5 of [83]). Hence, $\xi_{n \wedge \tau}^{(1)} \xi_{n \wedge \tau}^{(2)}$ converges in $L^1$, so

$$
\lim_{n \to \infty} \mathbb{E}\xi_{n \wedge \tau}^{(1)} \xi_{n \wedge \tau}^{(2)} = \mathbb{E}\xi_{\tau}^{(1)} \xi_{\tau}^{(2)} = 0. \quad (2.58)
$$

Moreover, it follows from (2.57) that $\xi_{n}^{(1)} \xi_{n}^{(2)} - \rho n$ is a martingale, so

$$
\xi_{0}^{(1)} \xi_{0}^{(2)} = \mathbb{E}\xi_{n \wedge \tau}^{(1)} \xi_{n \wedge \tau}^{(2)} - \rho \mathbb{E}[n \wedge \tau].
$$

Re-arranging, taking $n \to \infty$ and using (2.58) and the monotone convergence theorem, we obtain

$$
\mathbb{E}\tau = \lim_{n \to \infty} \mathbb{E}[n \wedge \tau] = \frac{\xi_0^{(1)} \xi_0^{(2)}}{|\rho|}, \text{ when } \rho < 0.
$$
A similar argument shows that the result $E\tau < \infty$ when $\rho < 0$ is sharp: we show that if $\rho \geq 0$ and $\xi_0 \not\in \Sigma_0$, then $E\tau = \infty$. For the purposes of deriving a contradiction, suppose that $E\tau < \infty$. Since $\rho \geq 0$, (2.57) shows that $\xi_n^{(1)} \xi_n^{(2)}$ is a submartingale, which converges in $L^1$ by (2.58). Hence $0 = E\xi_n^{(1)} \xi_n^{(2)} \geq E\xi_0^{(1)} \xi_0^{(2)} > 0$, since $\xi_0 \not\in \Sigma_0$, which is a contradiction. So $E\tau = \infty$. △

**Example 2.6.7.** Let $X_n$ be a square-integrable martingale with $X_0 = 0$, such that $|X_{n+1} - X_n| \leq K$, a.s., and $E[(X_{n+1} - X_n)^2 | F_n] = v^2$, a.s., for some $K, v \in (0, \infty)$. For $a \in \mathbb{R}_+$, define

$$\kappa_a := \min\{n \geq 0 : |X_n| > an^{1/2}\},$$

the exit time from a *square-root boundary*. In this example we prove that

$$E\kappa_a < \infty$$

if and only if $a < v$. (2.60)

To obtain the ‘only if’ half of (2.60), it suffices to show that $E\kappa_v = \infty$. To this end, let $X_n$ be a square-integrable martingale with $E[(X_{n+1} - X_n)^2 | F_n] = v^2$ (for this part of the argument the uniform jumps bound is unnecessary).

Suppose, for the purpose of deriving a contradiction, that $E\kappa_v < \infty$. We make use of the discussion leading up to Lemma 2.3.18. In this case, $\langle X \rangle_n = nv^2$, and $\langle X \rangle_{\kappa_v} = \kappa_v v^2$. Then, from (2.10), $E[X_{\kappa_v}^2] \leq v^2 E\kappa_v < \infty$.

It is straightforward to show that a sufficient condition for $X_{n \wedge \kappa_v}$ to be uniformly integrable is that

$$E[X_{\kappa_v}^2] < \infty, \quad \text{and} \quad X_{n \wedge \kappa_v}^2 1\{\kappa_v > n\} \text{ is uniformly integrable.}$$

Moreover, by definition of $\kappa_v$,

$$X_n^2 1\{\kappa_v > n\} \leq v^2 n 1\{\kappa_v > n\} \leq v^2 \kappa_v,$$

which is integrable under the assumption $E\kappa_v < \infty$. So $X_{n \wedge \kappa_v}$ is uniformly integrable, and we may apply Lemma 2.3.18 to conclude

$$E[X_{\kappa_v}^2] = E[\langle X \rangle_{\kappa_v}] = v^2 E\kappa_v.$$

But, by definition, $X_{\kappa_v}^2 - v^2 \kappa_v > 0$ then has expectation 0, and so must be 0 a.s., providing the desired contradiction. Thus $E\kappa_v = \infty$.

For the ‘if’ half of (2.60), we will actually prove the stronger result that $E\kappa_a < \infty$ for any $a < v$, but where we replace (2.59) by

$$\kappa_a := \min\{n \geq 0 : |X_n| > a(n + c)^{1/2}\} \quad \text{for some } c \in \mathbb{R}_+.$$
We permit $c > 0$ to downgrade the role of trivial cases where $\mathbb{P}[|X_1| > a] = 1$; but note that $\mathbb{E}[|X_1|^2] = v^2$ so $\mathbb{P}[|X_1| > v] < 1$, and there can be no trivial contradiction to $\mathbb{E}\kappa_v = \infty$.

Suppose $a \in (0,v)$. Take $b \in (a,v)$ and $c' > c$. Define $Y_n = b^2(n + c') - X_n^2$. Then, using the square-difference identity (2.23),

$$Y_{n+1} - Y_n = b^2 - 2X_n(X_{n+1} - X_n) - (X_{n+1} - X_n)^2,$$

and hence, by our hypotheses on $X_n$, $\mathbb{E}[Y_{n+1} - Y_n \mid \mathcal{F}_n] = b^2 - v^2$, which is uniformly (strictly) negative since $b < v$. Moreover, a.s.,

$$Y_{n\wedge\kappa_a} \geq b^2(n \wedge \kappa_a + c') - (a(n \wedge \kappa_a + c)^{1/2} + K)^2 \geq 0,$$

provided $c' > c$ is chosen sufficiently large (depending on $a$, $b$, $c$ and $K$). Then we can apply Theorem 2.6.2 to see that, if $a \in (0,v)$, $\mathbb{E}\kappa_a < \infty$. \triangle

Foster’s original argument in [108] for his version of Theorem 2.6.4 is rather different from the proof given above; instead it is more similar in spirit to the following result, which gives a lower bound on expected occupation times. Note that the set $A$ in this formulation need not be finite.

**Lemma 2.6.8.** Let $X_n$ be an $\mathcal{F}_n$-adapted process on state space $(\Sigma, \mathcal{E})$. Suppose that there exist $f : \Sigma \to \mathbb{R}_+$, a non-empty $A \in \mathcal{E}$, and $\varepsilon > 0$ such that $f(X_0)$ is integrable, and

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq -\varepsilon, \text{ on } \{X_n \in \Sigma \setminus A\}.$$

Suppose also that for some constant $K < \infty$,

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq K, \text{ on } \{X_n \in A\}.$$

Then it holds that

$$\liminf_{n \to \infty} \frac{1}{n} \mathbb{E} \sum_{m=0}^{n-1} 1\{X_m \in A\} \geq \frac{\varepsilon}{\varepsilon + K}.$$

**Proof.** Combining the conditions in the lemma, we have

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq -\varepsilon + (\varepsilon + K) 1\{X_n \in A\}.$$

In particular, it follows that $f(X_n)$ is integrable for all $n$, given that $f(X_0)$ is integrable, and we obtain

$$\mathbb{E} f(X_{n+1}) - \mathbb{E} f(X_n) \leq -\varepsilon + (\varepsilon + K) \mathbb{P}[X_n \in A],$$
and then summing yields

\[ \mathbb{E} f(X_n) - \mathbb{E} f(X_0) \leq -\varepsilon n + (\varepsilon + K) \sum_{m=0}^{n-1} \mathbb{P}[X_m \in A]. \]

Re-arranging and using the non-negativity of \( f \), we obtain

\[ \frac{\varepsilon + K}{n} \sum_{m=0}^{n-1} \mathbb{P}[X_m \in A] \geq \varepsilon - \frac{1}{n} \mathbb{E} f(X_0). \]

Taking \( n \to \infty \) yields the result. \( \square \)

**Example 2.6.9.** Suppose that \( X_n \) is an irreducible Markov chain on a countable state space \( \Sigma \), and \( A \subset \Sigma \) is finite. It is a classical result that, for \( \pi \) a sub-probability measure on \( \Sigma \),

\[ \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \sum_{m=0}^{n-1} 1\{X_m \in A\} = \pi(A); \]

\( X_n \) is positive recurrent if and only if \( \pi(x) > 0 \) for some (hence all) \( x \in \Sigma \), in which case \( \pi \) is a proper probability measure. Thus in this case the conclusion of Lemma 2.6.8 shows that \( \pi(A) > 0 \) and hence \( \pi(x) > 0 \) for some \( x \in A \), which yields positive recurrence. So the “if” half of Theorem 2.6.4 can be deduced from Lemma 2.6.8. \( \triangle \)

We end this section with a partial result in the opposite direction.

**Theorem 2.6.10.** Let \( X_n \) be an \( F_n \)-adapted process taking values in \( \mathbb{R}_+ \), and let \( \tau \) be a stopping time. Suppose that there exists \( B \in \mathbb{R}_+ \) such that

\[ \mathbb{E}[X_n+1 - X_n \mid F_n] \geq 0, \text{ on } \{n < \tau\}; \]
\[ \mathbb{E}(X_n+1 - X_n)^+ \mid F_n \leq B, \text{ on } \{n < \tau\}. \]

Suppose that there is some \( c \geq 0 \) such that \( \mathbb{P}[X_\tau \leq c \mid \tau < \infty] = 1 \) and \( \mathbb{E} X_0 > c \). Then \( \mathbb{E} \tau = \infty. \)

**Proof.** Let \( Y_n = X_{n\wedge \tau} \). Then, by hypothesis,

\[ \mathbb{E}[(Y_{n+1} - Y_n)^+ \mid F_n] \leq B 1\{n < \tau\}. \tag{2.61} \]

Suppose, for the purpose of deriving a contradiction, that \( \mathbb{E} \tau < \infty; \) so \( \mathbb{P}[\tau < \infty] = 1 \). Now, taking expectations in (2.61) and summing from \( m = 0 \) to \( n - 1 \) we obtain

\[ \mathbb{E} \sum_{m=0}^{n-1} (Y_{m+1} - Y_m)^+ \leq B \sum_{m=0}^{n-1} \mathbb{P}[\tau > m] \leq B \mathbb{E} \tau. \]
Moreover, since $Y_0 = X_0$,

\[ 0 \leq Y_n = Y_0 + \sum_{m=0}^{n-1} (Y_{m+1} - Y_m) \leq X_0 + \sum_{m=0}^{\infty} (Y_{m+1} - Y_m)^+ =: Z. \]

Here $Z \geq 0$ is an integrable random variable, since, by Fatou's lemma,

\[ \mathbb{E}Z \leq X_0 + \liminf_{n \to \infty} \mathbb{E} \sum_{m=0}^{n-1} (Y_{m+1} - Y_m)^+ \leq X_0 + B \mathbb{E} \tau < \infty. \]

Hence $|X_{n \wedge \tau}|$ is dominated by an integrable random variable, so the submartingale $X_{n \wedge \tau}$ is uniformly integrable. Then by optional stopping (Theorem 2.3.7),

\[ \mathbb{E}X_\tau \geq \mathbb{E}X_0 > c, \]

by hypothesis. But, since $\mathbb{P}[\tau < \infty] = 1$, $X_\tau \leq c$, a.s., by hypothesis, giving the desired contradiction. So $\mathbb{E} \tau = \infty$.

This leads to a sufficient condition for a Markov chain to be null, i.e., null recurrent or transient.

**Corollary 2.6.11.** Let $X_n$ be an irreducible Markov chain on a countable state space $\Sigma$. Suppose that there exists a non-empty $A \subset \Sigma$ and a function $f : \Sigma \to \mathbb{R}_+$ such that $f(X_n)$ is integrable, and

\[
\begin{align*}
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid X_n = x] &\geq 0, \text{ for all } x \in \Sigma \setminus A; \\
\mathbb{E}[(f(X_{n+1}) - f(X_n))^+ \mid X_n = x] &\leq B, \text{ for all } x \in \Sigma \setminus A; \\
f(y) &> \max_{x \in A} f(x), \text{ for some } y \in \Sigma \setminus A.
\end{align*}
\]

Then $\mathbb{E}_x \tau_x^+ = \infty$ for any $x \in \Sigma$, i.e., $X_n$ is not positive recurrent.

**Proof.** Take $X_0 = y$ and let $\tau = \tau_A$. Then $f(X_n)$ satisfies the hypotheses of Theorem 2.6.10 (with $c = \max_{x \in A} f(x)$), which implies that $\mathbb{E}_y \tau_A = \infty$.

But $\tau_A = \min_{x \in A} \tau_x$, so that $\mathbb{E}_y \tau_x = \infty$ for any $x \in A$. Take one such $x \in A$. Then $\mathbb{P}_x[\tau_y < \tau_x^+] > 0$, by irreducibility. Hence, by the strong Markov property,

\[ \mathbb{E}_x \tau_x^+ \geq \mathbb{P}_x[\tau_y < \tau_x^+] \mathbb{E}_y \tau_x^+ = \infty. \]

But then we must have $\mathbb{E}_x \tau_x^+ = \infty$ for all $x \in \Sigma$, since (by Proposition 2.1.5) the property $\mathbb{E}_y \tau_y^+ < \infty$ is a class property. \qed
Example 2.6.12. Let $S_n$ be symmetric simple random walk on $\mathbb{Z}$. The hypotheses of Corollary 2.6.11 are satisfied with $X_n = S_n$, $f(x) = |x|$, $A = \{0\}$, $B = \frac{1}{2}$, and any $y \geq 1$, so we conclude $\mathbb{E}_1 \tau_0 = \infty$. Hence

$$\mathbb{E}_0 \tau_0^+ \geq 1 + \frac{1}{2} \mathbb{E}_1 \tau_0 = \infty.$$  \hfill $\triangle$

2.7 Moments of hitting times

If a Markov chain is recurrent, it is of interest to quantify recurrence beyond the distinction between null and positive recurrent by estimating tails of the return time or determining which moments of return times exist. The question of existence and non-existence of moments of hitting times is also of interest beyond the Markov setting. This section gives some results in this direction.

Recall that $\lambda_x = \min\{n \geq 0 : X_n \leq x\}$. The first result provides a sufficient condition for existence of moments for $\lambda_x$, which, by Markov’s inequality, yield upper tail bounds on $\lambda_x$.

**Theorem 2.7.1.** Let $X_n$ be an $\mathcal{F}_n$-adapted stochastic process, taking values in an unbounded subset of $\mathbb{R}_+$. Suppose that there exist $\delta > 0$, $x > 0$, and $s_0 > 0$ such that for any $n \geq 0$, $X_n^{2s_0}$ is integrable and

$$\mathbb{E}[X_{n+1}^{2s_0} - X_n^{2s_0} | \mathcal{F}_n] \leq -\delta X_n^{2s_0 - 2}, \text{ on } \{n < \lambda_x\}.$$  \hspace{1cm} (2.62)

For any $s \in [0, s_0)$, there exists a positive constant $c = c(\delta, s, s_0)$ such that

$$\mathbb{E}[\lambda_x^s | \mathcal{F}_0] \leq c X_0^{2s_0}.$$  

Moreover, in the case when $s_0 \geq 1$, for any $s \in [0, s_0]$, there exists a positive constant $c' = c'(\delta, s, s_0)$ such that

$$\mathbb{E}[\lambda_x^s | \mathcal{F}_0] \leq c' X_0^{2s}.$$  

**Remark 2.7.2.** It follows from (2.62) that $X_n \geq \delta^{-1/2}$ on $\{n < \lambda_x\}$. So implicit in the hypotheses of Theorem 2.7.1 is that $x \geq \delta^{-1/2}$ is not too small.

Before proving Theorem 2.7.1, we state a useful corollary, which can be seen as a direct generalization of Theorem 2.6.2 (the case $\gamma = 0$).
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Corollary 2.7.3. Let \( X_n \) be an integrable \( \mathcal{F}_n \)-adapted stochastic process, taking values in an unbounded subset of \( \mathbb{R}_+ \), with \( X_0 = x_0 \) fixed. Suppose that there exist \( \delta > 0, x > 0 \), and \( \gamma < 1 \) such that for any \( n \geq 0 \),

\[
\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \leq -\delta X_n^\gamma, \text{ on } \{ n < \lambda_x \}.
\] (2.63)

Then, for any \( p \in [0, 1/(1 - \gamma)) \), \( \mathbb{E}[\lambda_x^p] < \infty \).

Proof. Let \( s_0 = 1/(1 - \gamma) > 0 \), and \( Y_n = X_n^{(1-\gamma)/2} \). Then \( X_n = Y_n^{2s_0} \) and \( X_n^\gamma = Y_n^{2s_0-2} \), so (2.63) implies that (2.62) holds for \( Y_n \), and Theorem 2.7.1 yields the result. \( \square \)

Proof of Theorem 2.7.1. It suffices to suppose that \( X_0 > x \). We treat separately the two cases: \( s_0 \geq 1 \) and \( s_0 < 1 \).

Case 1. \( s_0 \geq 1 \). For \( s \in (0, s_0] \) define \( U_n^{(s)} \) for \( n \geq 0 \) by

\[
U_n^{(s)} = \left( X_n^{2s} + \frac{\delta}{s_0} n^{\lambda_x} \right)^s.
\]

We show that \( U_n^{(s_0)} \) is a supermartingale. By (2.62), on \( \{ n < \lambda_x \} \),

\[
\mathbb{E}[X_{n+1}^{2s_0} \mid \mathcal{F}_n] \leq X_n^{2s_0} - \delta X_n^{2s_0-2} = X_n^{2s_0} \left( 1 - \delta X_n^{-2} \right).
\]

The last quantity is non-positive if \( X_n \leq \sqrt{\delta/s_0} \), while if \( X_n \geq \sqrt{\delta/s_0} \) we may apply the simple inequality

\[
1 - sy \leq (1 - y)^s \text{ for } s \geq 1 \text{ and } 0 < y < 1,
\]

with \( y = (\delta/s_0) X_n^{-2} \in (0, 1) \) and \( s = s_0 \) to see that

\[
1 - \delta X_n^{-2} \leq \left( 1 - \delta X_n^{-2} \right) 1\{X_n > \sqrt{\delta/s_0}\} \leq \left( 1 - \frac{\delta}{s_0} X_n^{-2} \right)^{s_0},
\]

on \( \{ n < \lambda_x \} \). It follows that

\[
\mathbb{E}[X_{n+1}^{2s_0} \mid \mathcal{F}_n] \leq \left( X_n^2 - \frac{\delta}{s_0} \right)^{s_0}, \text{ on } \{ n < \lambda_x \}.
\] (2.64)

Now, on \( \{ n < \lambda_x \} \), we apply the \( L^{s_0} \) version of Minkowski’s inequality for conditional expectations to obtain

\[
\mathbb{E}[U_{n+1}^{(s_0)} \mid \mathcal{F}_n] = \mathbb{E}\left[ \left( X_{n+1}^2 + \frac{\delta}{s_0} (n + 1) \right)^{s_0} \mid \mathcal{F}_n \right]
\]
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\[ \leq \left( E[X_{n+1}^{2s_0} \mid F_n] + (\delta/s_0)(n+1) \right)^{s_0}, \text{ on } \{n < \lambda_x\}. \]

Then using (2.64), we obtain on \{\lambda_x > n\},

\[ E[U_{n+1}^{(s_0)} \mid F_n] \leq \left( X_n^2 + \frac{\delta}{s_0}n \right)^{s_0} = U_n^{(s_0)}. \]

Hence \( U_n^{(s_0)} \) is a non-negative supermartingale, and it follows from Theorem 2.3.3 that for \( s \in (0, s_0] \), the process \( U_n^{(s)} \) is also a non-negative supermartingale. Hence, by optional stopping (Theorem 2.3.11),

\[ \left( \frac{\delta}{s_0} \right)^s E[(n \land \lambda_x)^s \mid F_0] \leq E[U_n^{(s)} \mid F_0] \leq E[U_0^{(s)} \mid F_0] = X_0^{2s}, \]

and it follows from the (conditional) Fatou lemma that

\[ E[\lambda_x^s \mid F_0] \leq \left( \frac{s_0}{\delta} \right)^s X_0^{2s}, \text{ for any } s \in (0, s_0]. \]

Case 2. \( s_0 < 1 \). We fix some \( s \in (0, s_0) \), and define, for \( n \geq 0 \),

\[ V_n^{(s)} = X_{n\land\lambda_x}^{2s_0} + \frac{\delta}{s_0}(n \land \lambda_x)^s. \]

First we observe, on \{\lambda_x > n\},

\[ E[X_{n+1}^{2s_0} - X_n^{2s_0} \mid F_n] = E[(X_{n+1}^{2s_0} - X_n^{2s_0})1\{X_n \in (x, n^{1/2})\} \mid F_n] + E[(X_{n+1}^{2s_0} - X_n^{2s_0})1\{X_n \geq n^{1/2}\} \mid F_n]; \]

note that for small \( n \) the event \{\( X_n \in (x, n^{1/2}) \)\} may be empty. A simple consequence of (2.62) is

\[ E[X_{n+1}^{2s_0} - X_n^{2s_0} \mid F_n] \leq 0, \text{ on } \{n < \lambda_x\}. \hspace{1cm} (2.65) \]

So applying (2.65) and (2.62) we have, on \{\lambda_x > n\},

\[ E[X_{n+1}^{2s_0} - X_n^{2s_0} \mid F_n] \leq -\delta X_n^{2s_0-2}1\{X_n \in (x, n^{1/2})\}. \]

Hence, on \{\lambda_x > n\},

\[ E[V_{n+1}^{(s)} - V_n^{(s)} \mid F_n] \leq -\delta X_n^{2s_0-2}1\{X_n \in (x, n^{1/2})\} + \frac{\delta}{s_0}((n+1)^s - n^s). \]

So, using the simple inequality

\[ (n+1)^s - n^s \leq sn^{s-1}, \text{ for all } n \geq 1, \]
we obtain, for \( s < s_0 \), for \( n \geq 1 \),

\[
\mathbb{E}[V^{(s)}_{n+1} - V^{(s)}_n | \mathcal{F}_n] \leq \delta \left( n^{s-1} - X_n^{2s_0 - 2} I\{X_n \in (x, n^{1/2})\} \right) 1\{n < \lambda_x\}
\leq \delta \left( n^{s-1} - X_n^{2s_0 - 2} \right) I\{X_n \wedge \lambda_x \in (x, n^{1/2})\}
+ \delta n^{s-1} I\{X_n \wedge \lambda_x \geq n^{1/2}\}.
\]

Now taking expectations conditioned on \( \mathcal{F}_0 \) and using Markov’s inequality we have, for \( n \geq 1 \),

\[
\mathbb{E}[V^{(s)}_{n+1} - V^{(s)}_n | \mathcal{F}_0] \leq \delta n^{s-s_0-1} \mathbb{E}[X_{n \wedge \lambda_x}^{2s_0} | \mathcal{F}_0]
\leq \delta n^{s-s_0-1} X_0^{2s_0},
\]

as follows from (2.65). Thus, for \( n \geq 1 \),

\[
\mathbb{E}[V^{(s)}_n | \mathcal{F}_0] \leq \mathbb{E}[V^{(s)}_1 | \mathcal{F}_0] + \delta X_0^{2s_0} \sum_{k=1}^{\infty} k^{s-s_0-1}
\leq \mathbb{E}[V^{(s)}_1 | \mathcal{F}_0] + c_1 X_0^{2s_0},
\]

for \( c_1 = c_1(\delta, s, s_0) < \infty \). Here, by (2.62),

\[
\mathbb{E}[V^{(s)}_1 | \mathcal{F}_0] = \mathbb{E}[X_1^{2s_0} | \mathcal{F}_0] + \frac{\delta}{s_0} \leq X_0^{2s_0} \left( 1 - \delta X_0^{-2} + \frac{\delta}{s_0} X_0^{-2s_0} \right).
\]

Since \( X_0 > x \geq \delta^{-1/2} \) (see Remark 2.7.2), the term in the last displayed bracket is bounded above by some \( c_2 = c_2(\delta, s_0) < \infty \), so finally we obtain, for some \( c' = c'(\delta, s, s_0) < \infty \),

\[
\mathbb{E}[V^{(s)}_n | \mathcal{F}_0] \leq c' X_0^{2s_0}, \text{ on } \{X_0 > x\}, \text{ for all } n \geq 0.
\]

The statement now follows similarly to the case \( s_0 \geq 1 \).

Now we turn to the opposite problem. The first result states sufficient conditions for non-existence of moments of hitting times \( \lambda_x = \min\{n \geq 0 : X_n \leq x\} \).

**Theorem 2.7.4.** Let \( X_n \) be a \( \mathcal{F}_n \)-adapted stochastic process taking values in an unbounded subset of \( \mathbb{R}_+ \). Suppose that there exist constants \( x \in (0, \infty) \), \( B \in (0, \infty) \) and \( c \in \mathbb{R} \) such that, for any \( n \geq 0 \),

\[
\mathbb{E}[(X_{n+1} - X_n)^2 | \mathcal{F}_n] \leq B, \text{ on } \{X_n \geq x\}; \quad (2.66)
\]
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\[ E[X_{n+1} - X_n \mid \mathcal{F}_n] \geq -\frac{c}{X_n}, \quad \text{on } \{X_n \geq x\}. \tag{2.67} \]

Suppose in addition that for some \( s_0 > 0 \), the process \( X^{2s_0}_{n \wedge \lambda_s} \) is a submartingale. Then, for any \( s > s_0 \), we have \( E[\lambda_s^n \mid X_0 = x_0] = \infty \) for any \( x_0 > x \).

A key role in the proof of this theorem is the following estimate, which gives conditions to ensure that a process in unlikely to return rapidly to a neighbourhood of the origin. The proof is based on an application of the maximal inequality from Theorem 2.4.8.

**Lemma 2.7.5.** Let \( X_n \) be an \( \mathcal{F}_n \)-adapted stochastic process taking values in \( \mathbb{R}_+ \). Let \( \delta \in (0, 1) \). Suppose that there exist finite positive constants \( x, B, c \) such that, for any \( n \geq 0 \), (2.66) and (2.67) hold. There exists \( \varepsilon > 0 \) (depending only on \( B, c, \) and \( \delta \)) so that, for any \( n \),

\[ P\left[ \min_{n \leq m \leq n + \varepsilon X_n^2} X_m \geq \frac{1}{2} X_n \bigg| \mathcal{F}_n \right] \geq 1 - \delta, \quad \text{on } \{X_n > 2x\}. \tag{2.68} \]

In particular, on \( \{X_{n \wedge \lambda_s} > 2x\} \),

\[ P[\lambda_s^n > n + \varepsilon X_n^2 \mid \mathcal{F}_n] \geq 1 - \delta. \tag{2.69} \]

**Proof.** To ease notation, we prove (2.68) and (2.69) for \( n = 0 \); the argument in the general case is the same. Hence suppose \( X_0 > 2x \). Write \( \lambda = \lambda_{X_0/2} \), and note that \( \lambda_x \geq \lambda \) provided \( X_0 > 2x \).

Write \( \Delta_k = X_{k+1} - X_k \) for the duration of this proof. Define the function \( w(x) := (X_0 - x)^21\{x < X_0\} \) for \( x \in \mathbb{R}_+ \), and let \( W_k = w(X_k) \). Since \( w \) is non-increasing, on \( \{X_k \geq X_0\} \), \( w(X_k + \Delta_k) \leq w(X_0 + \Delta_k) \leq \Delta_k^2 \). Hence, on \( \{X_k \geq X_0\} \),

\[ E[W_{k+1} - W_k \mid \mathcal{F}_k] \leq E[\Delta_k^2 \mid \mathcal{F}_k] \leq B, \quad \text{a.s.} \]

by (2.66). On the other hand, on \( \{X_k < X_0\} \),

\[ W_{k+1} - W_k \leq (W_{k+1} - W_k)1\{X_{k+1} < X_0\} \]

\[ = (-2(X_0 - X_k)\Delta_k + \Delta_k^2)1\{\Delta_k < X_0 - X_k\}. \]

Here we have that

\[ -2(X_0 - X_k)\Delta_k 1\{\Delta_k < X_0 - X_k\} \]

\[ = -2(X_0 - X_k)\Delta_k + 2(X_0 - X_k)\Delta_k 1\{\Delta_k \geq X_0 - X_k\} \]

\[ \leq -2(X_0 - X_k)\Delta_k + 2\Delta_k^2 1\{\Delta_k \geq X_0 - X_k\}. \]
Combining these bounds we get, on \( \{ X_k < X_0 \} \),
\[
W_{k+1} - W_k \leq -2(X_0 - X_k)\Delta_k + 2\Delta_k^2.
\]
Taking expectations we see that, on \( \{ X_0/2 \leq X_k < X_0 \} \), a.s.,
\[
E[W_{k+1} - W_k | F_k] \leq -2(X_0 - X_k)E[\Delta_k | F_k] + 2E[\Delta_k^2 | F_k]
\leq \frac{2c(X_0 - X_k)}{X_k} + 2B.
\]
In particular, there exists \( C < \infty \), depending only on \( B \) and \( c \), such that, on \( \{ X_k > X_0/2 \} \), a.s.,
\[
E[W_{k+1} - W_k | F_k] \leq C,\quad \text{a.s.}
\]
Hence we conclude that, for any \( k \geq 0 \),
\[
E[W_{k+1} \wedge \lambda - W_k \wedge \lambda | F_k] \leq C,\quad \text{a.s.}
\]
We apply the maximal inequality (2.12) from Theorem 2.4.8 to the process \( X_k \wedge \lambda \) at deterministic stopping time \( \nu = m \) to obtain
\[
P[\max_{0 \leq k \leq m} W_{k\wedge \lambda} \geq y^2 | F_0] \leq \frac{Cm + W_0}{y^2} = \frac{Cm}{y^2},\quad \text{a.s.}
\]
on \( \{ X_0 > 2x \} \). With \( y^2 = X_0^2/4 \) and \( m = \varepsilon X_0^2 \) we see that
\[
P[\max_{0 \leq k \leq \varepsilon X_0^2} W_{k\wedge \lambda} < \frac{X_0^2}{4} | F_0] \geq 4C\varepsilon,
\]
on \( \{ X_0 > 2x \} \). Now, \( W_k < y^2 \) implies \( X_k > X_0 - y \), while \( W_\lambda = (X_0 - X_\lambda)^2 \geq X_0^2/4 \), so the event in the last display implies \( \min_{0 \leq k \leq \varepsilon X_0^2} W_k > X_0/2 \). Thus we obtain (2.68), on taking \( \varepsilon < \frac{\delta}{4C} \), and this in turn implies (2.69).

**Proof of Theorem 2.7.4.** Let \( X_0 = x_0 > x \). It suffices to suppose that \( \lambda_x < \infty \) a.s., otherwise \( E \lambda_x = \infty \) is trivial. We proceed by contradiction. Suppose that there exists some \( s > s_0 \) such that \( E \lambda_x^s < \infty \). Lemma 2.7.5 with \( \delta = \frac{1}{2} \) yields the existence of positive \( \varepsilon \) such that for all \( n \geq 0 \),
\[
E \lambda_x^s \geq E \lambda_x^s 1\{X_n \wedge \lambda_x > 2x\}
\geq \frac{\varepsilon^s}{2} E [X_{n \wedge \lambda_x}^2] 1\{X_n \wedge \lambda_x > 2x\}
\geq \frac{\varepsilon^s}{2} E X_{n \wedge \lambda_x}^2 - \frac{\varepsilon^s}{2} (2x)^{2s}.
\]
Under the hypothesis \( E \lambda_x^s < \infty \), the last inequality shows that there exists \( K < \infty \) such that \( E X_{n \wedge \lambda_x}^2 \leq K \) for all \( n \). Hence \( X_{n \wedge \lambda_x}^2 \) is uniformly integrable, and
\[
\lim_{n \to \infty} E X_{n \wedge \lambda_x}^{2s_0} = E X_{\lambda_x}^{2s_0} \leq x^{2s_0},
\]
by definition of $\lambda_x$. On the other hand, since $X_{n\wedge \lambda_x}^2$ is a submartingale, we have $\mathbb{E} X_{n\wedge \lambda_x}^2 \geq \mathbb{E} X_0^2 = x_0^2$. Since $x_0 > x$, this gives the desired contradiction. \hfill \Box

**Example 2.7.6.** Let $S_n$ be one-dimensional SRW. Take a positive $\alpha < \frac{1}{2}$, and fix $c \in (0, \alpha(1-2\alpha))$. We have then

$$
\mathbb{E}[X_{n+1}^{2\alpha} - X_n^{2\alpha} | X_n = x] = \frac{1}{2}x^{2\alpha}\left((1 + \frac{1}{x})^{2\alpha} + (1 - \frac{1}{x})^{2\alpha} - 2\right) 
\leq -cx^{2\alpha-2},
$$

for all large enough $x$, so Theorem 2.7.1 implies that $\mathbb{E}[(\tau_0^+)^\alpha] < \infty$. On the other hand, in order to apply Theorem 2.7.4, take $X_n = |S_n|$. The conditions (2.66) and (2.67) are easily verified, and $X_n$ is itself a submartingale, so we may apply Theorem 2.7.4 with $s_0 = 1/2$ to deduce that $\mathbb{E}[(\tau_0^+)^\beta] = \infty$ for all $\beta > 1/2$. \hfill \triangle

As Example 2.7.6 demonstrates when compared to Example 2.4.3, Theorem 2.7.4 is not quite sharp; in compensation, it is robust. In some situations it is possible to do better by using Lemma 2.7.5 in a slightly different way: the next result shows how tail information about the maximum of an excursion can be used to deduce a lower bound on the tail of $\lambda_x$. The idea is basically an extension of that in Example 2.4.3. In fact, the idea readily yields a lower bound on any non-decreasing additive functional of the excursion, so we state the result in that generality.

**Lemma 2.7.7.** Let $X_n$ be an $\mathcal{F}_n$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$. Suppose that there exist finite positive constants $x$, $B$, and $c$ such that, for any $n \geq 0$, (2.66) and (2.67) hold. Let $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function. Then there exists $\varepsilon > 0$ (not depending on $\alpha$ or $x$) such that, for all $y > x$,

$$
\mathbb{P}\left[\sum_{m=1}^{\lambda_x} \alpha(X_m) \geq 4\varepsilon y^{2\alpha}\alpha(y) \bigg| \mathcal{F}_0\right] \geq \frac{1}{2} \mathbb{P}\left[\max_{0 \leq m \leq \lambda_x} X_m \geq 2y \bigg| \mathcal{F}_0\right], \ a.s.
$$

In particular, there exists $C < \infty$ (not depending on $\alpha$ or $x$) such that, for all $n$ sufficiently large,

$$
\mathbb{P}[\lambda_x \geq n | \mathcal{F}_0] \geq \frac{1}{2} \mathbb{P}\left[\max_{0 \leq m \leq \lambda_x} X_m \geq Cn^{1/2} \bigg| \mathcal{F}_0\right], \ a.s.
$$
Example 2.7.8. Let $y > x$. Lemma 2.7.7 applied to the function $\alpha(x) = 1\{x \geq y\}$ gives the estimate

$$
P \left[ \sum_{m=1}^{\lambda_x} 1\{X_m \geq y\} \geq 4\varepsilon y^2 \right] \geq \frac{1}{2} P \left[ \max_{0 \leq m \leq \lambda_x} X_m \geq 2y \right],$$

and hence, for the expected number of visits to $[y, \infty)$ before reaching $[0, x]$,

$$
E \left[ \sum_{m=1}^{\lambda_x} 1\{X_m \geq y\} \right] \geq 2\varepsilon y^2 P \left[ \max_{0 \leq m \leq \lambda_x} X_m \geq 2y \right]. \quad \triangle
$$

Proof of Lemma 2.7.7. Let $y > x$ and define the stopping time $\kappa = \min\{n \geq 0 : X_n > 2y\}$. Define the event

$$
E(y, \varepsilon) = \left\{ \min_{\kappa \leq m \leq \kappa + \varepsilon X^{2}_{\kappa}} X_m \geq \frac{1}{2} X_{\kappa} \right\}.
$$

Then by (2.68) applied at the stopping time $\kappa$ (if finite), which is justified by Lemma 2.1.11(i), there exists $\varepsilon > 0$ such that

$$
P[E(y, \varepsilon) | \mathcal{F}_{\kappa}] \geq \frac{1}{2}, \text{ on } \{\kappa < \infty\}. \quad (2.70)
$$

Now, $\{\kappa < \lambda_x\} \cap E(y, \varepsilon)$ implies that $\lambda_x > \kappa + \varepsilon X^{2}_{\kappa}$, so that

$$
\sum_{m=1}^{\lambda_x} \alpha(X_m) \geq 1\{\kappa < \lambda_x\} 1(E(y, \varepsilon)) \sum_{\kappa \leq m \leq \kappa + \varepsilon X^{2}_{\kappa}} \alpha(X_m)
$$

$$
\geq \varepsilon X^{2}_{\kappa} \alpha(X_{\kappa}/2) 1\{\kappa < \lambda_x\} 1(E(y, \varepsilon))
$$

$$
\geq 4\varepsilon y^2 \alpha(y) 1\{\kappa < \lambda_x\} 1(E(y, \varepsilon)), \text{ a.s.,}
$$

since $\alpha$ is non-decreasing. Hence, since $\{\kappa < \lambda_x\}$ is $\mathcal{F}_{\kappa}$ measurable,

$$
P \left[ \sum_{m=1}^{\lambda_x} \alpha(X_m) \geq 4\varepsilon y^2 \alpha(y) \bigg| \mathcal{F}_0 \right] \geq P[\{\kappa < \lambda_x\} \cap E(y, \varepsilon) \big| \mathcal{F}_0]
$$

$$
= E[1\{\kappa < \lambda_x\} P[E(y, \varepsilon) \big| \mathcal{F}_{\kappa}] \big| \mathcal{F}_0]
$$

$$
\geq \frac{1}{2} P[\kappa < \lambda_x \big| \mathcal{F}_0],
$$

by (2.70). But $\kappa < \lambda_x$ if and only if $\max_{0 \leq m \leq \lambda_x} X_m \geq 2y$. The result now follows. $\Box$
Chapter 2. Semimartingale approach and Markov chains

A general strategy to prove that certain moments of a hitting time do not exist is to bound below the probability that the process ends up a long way away from its destination, and then takes a while to come back. A local submartingale can help with the first part, and a maximal inequality with the second.

Example 2.7.9. As in Example 2.4.15, let $S_n$ be symmetric SRW on $\mathbb{Z}^2$. Let $\tau = \min\{n \geq 1 : S_n = 0\}$. It is not hard to show that $X_n = \|S_n\|$ satisfies the conditions of Lemma 2.7.7, so that with the excursion lower bound in Example 2.4.15, we obtain that for any $\varepsilon > 0$ there is a constant $c > 0$ such that, for all $n \geq 2$,

$$\mathbb{P}[\tau \geq n] \geq c(\log n)^{-1-\varepsilon}.$$ 

In particular, $\mathbb{E}[\tau^s] = \infty$ for all $s > 0$. $\triangle$

2.8 Growth bounds on trajectories

In this section we turn to the question of quantifying the ‘speed’ of an $\mathbb{R}^+$-valued discrete-time stochastic process $(X_n, n \geq 0)$ adapted to a filtration $(\mathcal{F}_n, n \geq 0)$; later we will apply these results to multidimensional models.

Under mild conditions, $\limsup_{n \to \infty} X_n = \infty$, i.e., $\max_{0 \leq m \leq n} X_m$ tends to infinity. To quantify the ‘speed’ of the process, we present general semimartingale criteria for obtaining upper and lower almost-sure bounds on $\max_{0 \leq m \leq n} X_m$. For example, we give conditions for finding an increasing function $h$ such that $\max_{0 \leq m \leq n} X_m \leq h(n)$ for all but finitely many $n \in \mathbb{Z}^+$, a.s. (which implies $\limsup_{n \to \infty} X_n/h(n) \leq 1$). Similarly, we present lower bounds that may be cast as ‘lim inf’ statements.

To start with, we consider almost-sure upper bounds. The statement involves a function $a$ satisfying the following condition.

(A0) Suppose that for some $n_a \in \mathbb{Z}^+$ the function $a : [n_a, \infty) \to (0, \infty)$ is such that (i) $x \mapsto a(x)$ is increasing on $x \geq n_a$; (ii) $\lim_{x \to \infty} a(x) = \infty$; and (iii) $\sum_{n \geq n_a} \frac{1}{a(n)} < \infty$.

Condition (A0) is satisfied, for example, by $a(x) = x^{1+\varepsilon}$ (with $n_a = 1$) or $a(x) = x(\log x)^{1+\varepsilon}$ (with $n_a = 2$), where $\varepsilon > 0$. 
Theorem 2.8.1. Let $X_n$ be an $\mathcal{F}_n$-adapted process on $\mathbb{R}_+$. Suppose that there exist a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $B \in \mathbb{R}_+$ for which $f(X_0)$ is integrable, and
\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq B, \text{ a.s.,} \tag{2.71}
\]
for all $n \geq 0$. Define the non-decreasing function $f^{-1}$ for $x > 0$ by
\[
f^{-1}(x) := \sup\{y \geq 0 : f(y) < x\}. \tag{2.72}
\]
Let $a$ satisfy (A0). Then, a.s., for all but finitely many $n \geq 0$,
\[
\max_{0 \leq m \leq n} X_m \leq f^{-1}(a(2^n)). \tag{2.73}
\]

Proof. Note that
\[
\mathbb{E} f(X_n) \leq \mathbb{E} f(X_0) + \sum_{m=0}^{n-1} \mathbb{E}[f(X_{m+1}) - f(X_m)] \leq \mathbb{E} f(X_0) + Bn,
\]
by (2.71), so that $f(X_n)$ is integrable for all $n$ provided $f(X_0)$ is integrable. The maximal inequality (2.14) from Theorem 2.4.7 applied to the process $f(X_n)$ shows that, for any $n \geq n_a$,
\[
\mathbb{P}\left[\max_{0 \leq m \leq n} f(X_m) \geq a(n)\right] \leq \frac{Bn + \mathbb{E} f(X_0)}{a(n)}. \tag{2.74}
\]
Also, since $f$ is non-negative and non-decreasing, for $n \geq n_a$,
\[
\mathbb{P}\left[\max_{0 \leq m \leq n} f(X_m) \geq a(n)\right] = \mathbb{P}\left[f\left(\max_{0 \leq m \leq n} X_m\right) \geq a(n)\right]. \tag{2.75}
\]
With $f^{-1}$ defined by (2.72), let $E_n$ denote the event
\[
E_n := \left\{\max_{0 \leq m \leq n} X_m > f^{-1}(a(n))\right\}.
\]
Then since $z > f^{-1}(r)$ implies $f(z) \geq r$, we obtain from (2.74) and (2.75) that for all $n \geq n_a$,
\[
\mathbb{P}[E_n] \leq \mathbb{P}\left[f\left(\max_{0 \leq m \leq n} X_m\right) \geq a(n)\right] \leq \frac{Bn + \mathbb{E} f(X_0)}{a(n)}. \tag{2.76}
\]
Now for any $\ell_a \geq \log_2 n_a$,
\[
\sum_{\ell=\ell_a}^{\infty} \frac{2^\ell}{a(2^\ell)} < \infty \iff \sum_{\ell=\ell_a}^{\infty} \frac{1}{a(2^\ell)} < \infty. \tag{2.77}
\]
By (2.76) and (2.77), along the subsequence \( n = 2^{\ell} \) for \( \ell \in \mathbb{Z}_+ \), (A0) and the Borel–Cantelli lemma imply that, a.s., the event \( E_n \) occurs only finitely often, so, for all but finitely many \( \ell \),

\[
\max_{0 \leq m \leq 2^\ell} X_m \leq f^{-1}(a(2^\ell)).
\]

Every \( n \geq 1 \) has \( 2^{\ell_n} \leq n < 2^{\ell_n+1} \) for some \( \ell_n \geq 0 \) with \( \lim_{n \to \infty} \ell_n = \infty \); then,

\[
\max_{0 \leq m \leq n} X_m \leq \max_{0 \leq m \leq 2^{\ell_n+1}} X_m \leq f^{-1}(a(2^{\ell_n+1})), \text{ a.s.,}
\]

for all but finitely many \( n \). Now since \( 2^{\ell_n+1} \leq 2^\ell n \) and \( f^{-1} \) and \( a \) are eventually non-decreasing, (2.73) follows.

We give an application of the preceding result to a many-dimensional random walk with asymptotically vanishing drift.

**Example 2.8.2.** Let \((\xi_n, n \geq 0)\) be a Markov process with state space \( \Sigma \subseteq \mathbb{R}^d \) and increments \( \theta_n := \xi_{n+1} - \xi_n \). Suppose that there exists \( C < \infty \) such that, for all \( x \in \Sigma \),

\[
\|E[\theta_n \mid \xi_n = x]\| \leq C(1 + \|x\|)^{-1}, \text{ and } E[\|\theta_n\|^2 \mid \xi_n = x] \leq C.
\]

Set \( X_n = \|\xi_n\|^2 \). Then, writing \( \cdot \cdot \cdot \) for the scalar product on \( \mathbb{R}^d \),

\[
X_{n+1} - X_n = \xi_{n+1} \cdot \xi_{n+1} - \xi_n \cdot \xi_n = 2\xi_n \cdot (\xi_{n+1} - \xi_n) + \|\xi_{n+1} - \xi_n\|^2.
\]

It follows that

\[
E[X_{n+1} - X_n \mid \xi_n = x] = 2x \cdot E[\theta_n \mid \xi_n = x] + E[\|\theta_n\|^2 \mid \xi_n = x] \leq 3C < \infty,
\]

by hypothesis. In other words, if \( F_n = \sigma(\xi_0, \xi_1, \ldots, \xi_n) \), we have that

\[
E[X_{n+1} - X_n \mid F_n] \leq 3C, \text{ a.s.}
\]

Hence an application of Theorem 2.8.1 with \( f(x) = x \) shows that for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \),

\[
\max_{0 \leq m \leq n} \|\xi_m\| \leq n^{1/2}(\log n)^{(1/2)+\varepsilon}.
\]

The next result is a lower bound for \( \max_{0 \leq m \leq n} X_m \). In addition to the condition (A0), the statement involves the following condition.
2.8. Growth bounds on trajectories

(A1) Suppose that for some \( n_0 \in \mathbb{Z}_+ \) the function \( a : [n_0, \infty) \to (0, \infty) \) is such that (i) \( x \mapsto a(x) \) is increasing on \( x \geq n_0 \); (ii) \( \lim_{x \to \infty} a(x) = \infty \); and (iii) \( \sum_{n \geq n_0} \frac{1}{a(n)} < \infty \).

Note that the summability condition in (A1) is equivalent to the condition that \( \sum_{n \geq n_0} \frac{1}{a(n)} < \infty \) for some (hence all) \( r > 1 \). Condition (A1) is satisfied, for example, by \( a(x) = (\log x)^{1+\varepsilon} \) or \( (\log x)(\log \log x)^{1+\varepsilon} \), \( \varepsilon > 0 \).

Theorem 2.8.3. Let \( X_n \) be an \( \mathcal{F}_n \)-adapted process on \( \mathbb{R}_+ \). Suppose that there exists \( b \in \mathbb{R}_+ \) such that, for all \( n \geq 0 \),

\[
\mathbb{P}[X_{n+1} - X_n \leq b] = 1. \tag{2.78}
\]

Suppose that there exist a non-decreasing function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) and some \( \varepsilon > 0 \) for which, for all \( n \geq 0 \),

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) | \mathcal{F}_n] \geq \varepsilon, \ a.s. \tag{2.79}
\]

For a function \( a : [n_0, \infty) \to (0, \infty) \) with \( n_0 \in \mathbb{Z}_+ \), define \( r_a : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
r_a(x) := \inf\{y \geq n_0 : \varepsilon^{-1} a(y) f(y + b) \geq x\}, \ (x \geq 0). \tag{2.80}
\]

(i) Let \( a \) satisfy (A0). Then, a.s., for all but finitely many \( n \geq 0 \),

\[
\max_{0 \leq m \leq n} X_m \geq r_a(n) - 1. \tag{2.81}
\]

(ii) Let \( a \) satisfy (A1). For any \( \delta > 0 \), a.s., for all but finitely many \( n \geq 0 \),

\[
\max_{0 \leq m \leq n} X_m \geq (1 - \delta) r_a(n). \tag{2.82}
\]

Remark 2.8.4. The two parts of the theorem are complementary. If one expects the growth rate of \( \max_{0 \leq m \leq n} X_m \) with \( n \) to be polynomial, then one should try a polynomial \( f \), and part (ii) of the theorem will usually provide the better bound; for slower growth, one should try a super-polynomial \( f \), and part (i) will usually provide the better bound.

Proof of Theorem 2.8.3. Recall that for \( X_n \in \mathbb{R}_+ \), \( \sigma_x = \min\{n \geq 0 : X_n \geq x\} \). Note that under either (A0) or (A1), the fact that \( f \) is non-decreasing ensures that \( x \mapsto r_a(x) \) is non-decreasing for all \( x \geq 0 \), and \( r_a(x) \to \infty \) as \( x \to \infty \).
We now apply Theorem 2.4.11. Since $f$ satisfies (2.79), we have that (2.19) holds for all $x > 0$ and all $n \geq 0$, while (2.78) and the fact that $f$ is non-decreasing implies that $f(X_{\sigma_x}) \leq f(x + b)$, a.s. Then Theorem 2.4.11 (with stopping time $\eta = \infty$) implies that $\mathbb{E} \sigma_x \leq \varepsilon^{-1} f(x + b)$ for all $x > 0$.

In particular, for all $x > 0$, $\sigma_x < \infty$, and $\sigma_x \leq \sigma_y$ for $x \leq y$. Moreover, the jumps bound (2.78) implies that, for all $x \geq X_0$, $\sigma_{x+b} \geq 1 + \sigma_x$ a.s., so that $\lim_{x \to \infty} \sigma_x = \infty$, a.s.

By Markov’s inequality, for any $x \geq n$, a.s.,

$$\mathbb{P}[\sigma_x > a(x) \mathbb{E} \sigma_x] \leq (a(x))^{-1}. \quad (2.83)$$

To prove part (i), suppose that (A0) holds. Then (2.83) and the Borel–Cantelli lemma imply that, a.s., for all but finitely many $\ell \in \mathbb{Z}_+$,

$$\sigma_{\ell} \leq a(\ell) \mathbb{E} \sigma_{\ell} \leq \varepsilon^{-1} a(\ell) f(\ell + b).$$

Here, since $\sigma_x \to \infty$ as $x \to \infty$, $\sigma_{\ell} \geq n$ for all but finitely many $\ell$, so, with the definition of $r_a$ at (2.80), a.s., for all but finitely many $\ell$,

$$r_a(\sigma_{\ell}) \leq r_a(\varepsilon^{-1} a(\ell) f(\ell + b)) \leq \ell. \quad (2.84)$$

For $n \in \mathbb{Z}_+$, define $\ell_n = \min\{\ell \geq 0 : \sigma_{\ell} > n\}$. Then $\ell_n \to \infty$ as $n \to \infty$, and $\sigma_{\ell_n-1} \leq n < \sigma_{\ell_n}$. Moreover, a.s.,

$$\max_{0 \leq m \leq n} X_m \geq \max_{0 \leq m \leq \sigma_{\ell_n-1}} X_m \geq X_{\sigma_{\ell_n-1}} \geq \ell_n - 1 \geq r_a(\sigma_{\ell_n}) - 1, \quad (2.85)$$

for all but finitely many $n$, by (2.84). Since $\sigma_{\ell_n} > n$, (2.81) follows.

The proof of part (ii) is similar. Fix $\delta > 0$ and $K > 1$ such that $K^{-1} > 1 - \delta$. Supposing that (A1) holds, (2.83) and the Borel–Cantelli lemma imply that, a.s., for all but finitely many $\ell \in \mathbb{Z}_+$,

$$\sigma_{K^{\ell}} \leq a(K^{\ell}) \mathbb{E} \sigma_{K^{\ell}} \leq \varepsilon^{-1} a(K^{\ell}) f(K^{\ell} + b).$$

Similarly to (2.84), we obtain $r_a(\sigma_{K^{\ell}}) \leq K^{\ell}$, a.s., for all but finitely many $\ell$. Now setting $\ell_n = \min\{\ell \geq 0 : \sigma_{K^{\ell}} > n\}$, we have $\sigma_{K^{\ell_n-1}} \leq n < \sigma_{K^{\ell_n}}$. Then, similarly to (2.85), a.s.,

$$\max_{0 \leq m \leq n} X_m \geq X_{\sigma_{K^{\ell_n}-1}} \geq K^{\ell_n-1} \geq \frac{1}{K} r_a(\sigma_{K^{\ell_n}}) \geq \frac{1}{K} r_a(n),$$

for all $n$ sufficiently large, which yields (2.82). \qed
For the rest of this section we return to the nearest-neighbour random walk on \( \mathbb{Z}_+ \) introduced in Section 2.2 in order to illustrate the results of this section. Consider the case \( p_x = p \in (0, 1) \) for all \( x \in \mathbb{Z}_+ \). If \( p > 1/2 \), the random walk \( X_n \) is positive-recurrent; this fact is an easy consequence of Theorem 2.2.5, or can be seen with the argument in Example 2.6.5, which is for a model with a slightly different reflection rule at the origin. Quantitatively, we will show that

\[
\lim_{n \to \infty} \left( (\log n)^{-1} \max_{0 \leq m \leq n} X_m \right) = \varrho, \text{ a.s.,} \tag{2.86}
\]

where \( \varrho \in (0, \infty) \) is given by

\[
\varrho := (\log \left( \frac{p}{1 - p} \right))^{-1}. \tag{2.87}
\]

We will prove the following more refined result, from which (2.86) is immediate. Recall that \( \log^k x \) denotes the \( k \)-fold iterated logarithm.

**Proposition 2.8.5.** Let \( p_x = p \in (1/2, 1) \) for all \( x \in \mathbb{Z}_+ \). Then with \( \varrho \) given by (2.87), for any integer \( k \geq 3 \) and any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \in \mathbb{Z}_+ \), the following inequalities hold:

\[
\max_{0 \leq m \leq n} X_m \leq \varrho \left( \log n + \sum_{j=2}^{k-1} \log_j n + (1 + \varepsilon) \log_k n \right), \tag{2.88}
\]

\[
\max_{0 \leq m \leq n} X_m \geq \varrho \left( \log n - \sum_{j=2}^{k-1} \log_j n - (1 + \varepsilon) \log_k n \right). \tag{2.89}
\]

**Proof.** From (2.5) with \( p_x = p \in (1/2, 1), q_x = q = 1 - p \) for all \( x \in \mathbb{Z}_+ \),

\[
u_x = q^{-1} \sum_{y=0}^{x} (p/q)^y = \frac{1}{p - q} ((p/q)^{x+1} - 1).
\]

Hence from (2.8) we obtain, for \( x \in \mathbb{N} \),

\[
t(x) = \frac{1}{p - q} \sum_{y=0}^{x-1} ((p/q)^{y+1} - 1) = \frac{p}{(p - q)^2} (p/q)^x - \frac{x}{p - q} - \frac{p}{(p - q)^2}.
\]

Thus there is a constant \( C_0 \in (1, \infty) \) such that for all \( x \geq 1 \),

\[
t_0(x) := C_0^{-1} (p/q)^x \leq t(\lfloor x \rfloor) \leq C_0 (p/q)^x =: t_1(x). \tag{2.90}
\]
Chapter 2. Semimartingale approach and Markov chains

The functions $t_0$ and $t_1$ are continuous and monotone, and so can be inverted. Then, with $\varrho$ given by (2.87), we have that for a constant $C_1 = \varrho \log C_0 \in (0, \infty)$ and all $x \geq 1$,

$$t_0^{-1}(x) = \varrho \log x + C_1, \quad t_1^{-1}(x) = \varrho \log x - C_1. \quad (2.91)$$

We will apply Theorems 2.8.1 and 2.8.3 with $X_n$ as given, with $F_n = \sigma(X_0, \ldots, X_n)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ the non-decreasing function given by $f(x) = t(\lfloor x \rfloor)$. In particular, (2.71) holds with $B = 1$, (2.79) holds with $\varepsilon = 1$, and (2.78) holds with $b = 1$.

For the upper bound we apply Theorem 2.8.1. For $k \geq 2$ and $\varepsilon > 0$, set

$$a(x) = x(\log_{k} x)^{1+\varepsilon} \prod_{j=1}^{k-1} \log_j x.$$ 

Then $a$ satisfies (A0). Define $f^{-1}$ as at (2.72). Since $f(x) = t(\lfloor x \rfloor) \geq t_0(x)$, we have by continuity of $t_0$ that $f^{-1}(x) \leq t_0^{-1}(x)$. Then Theorem 2.8.1 and (2.91) imply that a.s.

$$\max_{0 \leq m \leq n} X_m \leq t_0^{-1}(a(2n)) = \varrho(\log(a(2n))) + C_1, \quad (2.92)$$

for all but finitely many $n$, and here

$$\log(a(2n)) = \sum_{j=1}^{k} \log_j n + (1 + \varepsilon) \log_{k+1} n + O(1),$$

and so from (2.92) we obtain (2.88).

On the other hand, for the lower bound we apply Theorem 2.8.3(i). For $k \geq 3$ and $\varepsilon > 0$, set

$$c(x) := \varrho \left( \log x - \sum_{j=2}^{k} \log_j x - (1 + \varepsilon) \log_{k+1} x \right).$$

Then, for some $C_2 \in (0, \infty)$, for all $x > 0$ sufficiently large,

$$\sup_{0 \leq y \leq c(x)} a(y) \leq a(c(x)) \leq a(\varrho \log x) \leq C_2(\log x)(\log_2 x) \cdots (\log_{k} x)(\log_{k+1} x)^{1+\varepsilon}.$$ 

Also, from (2.90), for all $y \leq c(x),

$$t_1(y) \leq t_1(c(x)) = C_0 x(\log x)^{-1}(\log_2 x)^{-1} \cdots (\log_{k-1} x)^{-1}(\log_k x)^{-1+\varepsilon}.$$
Thus there exists $C_3 \in (0, \infty)$ such that for all $y$ with $y + 1 \leq c(x)$,
\[ a(y)f(y + 1) \leq a(y)t_1(y + 1) \leq C_3x(\log_k x)^{-\varepsilon}(\log_{k+1} x)^{1+\varepsilon} \leq x, \]
for all $x$ sufficiently large. Hence
\[ r_a(x) = \inf\{y \geq n_a : a(y)f(y + 1) \geq x\} \geq c(x) - 1, \]
for all $x$ sufficiently large. Now (2.89) follows from Theorem 2.8.3(i). \qed

Bibliographical notes

Sections 2.1 and 2.3

There are many books that provide the necessary background in probability theory, e.g. [25, 83, 267, 286], and there also one can find the systematic treatment of martingales and countable Markov chains. The books of Durrett [83] and Shiryaev [286] are excellent sources for many of the results quoted without proof in Section 2.3; martingale inequalities are treated thoroughly by Chow and Teicher [41].

Theorem 2.1.6 can be obtained by combining Theorems 1.8.3 and 1.10.2 of [243] for example. Theorem 2.1.6 is a simple Markov chain ergodic theorem; similar results in a more general setting are given in Section 3.6, which also contains the proof of Theorem 2.1.9 and related results.

Without local finiteness of $\Sigma$, part (ii) of Theorem 2.1.9 may fail. To see this, note that a countable set $S \subseteq \mathbb{R}^d$ is locally finite if and only if it contains no finite limit points. Indeed, it is easy to see that if $S \subseteq \mathbb{R}^d$ has a finite limit point, it is not locally finite. On the other hand, suppose that $S \subseteq \mathbb{R}^d$ is not locally finite. Then there is some ball $B_0$ of radius $r_0 = 1$ such that $\#(B_0 \cap S) = \infty$. Cover $B_0$ by a finite number of balls of radius $r_1 = 2^{-1}$ whose centres lie in $B_0$. At least one of the balls in this covering contains infinitely many points of $S$; let $B_1$ be one such ball. Iterating this procedure gives a decreasing sequence of balls $B_n$, each containing infinitely many points of $S$, in which $B_n$ has radius $2^{-n}$. Let $x_n \in \mathbb{R}^d$ be the centre of $B_n$. Then $\|x_n - x_{n+1}\| \leq 2^{-n}$ so $\lim_{n \to \infty} x_n = x \in \mathbb{R}^d$ exists. For any $\varepsilon > 0$, we can take $n$ large enough so that $\|x - x_n\| < \varepsilon/2$ and $r_n < \varepsilon/2$, so $B_n$ (and hence an infinite subset of $S$) is contained in a ball of radius $\varepsilon$ centred at $x$. So $x$ is a finite limit point of $S$.

Many of the fundamental results on discrete-time martingales in Section 2.3 are due to Doob, and can be found in his classic book [78]. In
the more modern reference [83], Theorem 2.3.1 is Theorem 5.2.10, Theorem 2.3.2 is Theorem 5.4.6, Theorem 2.3.3 is a combination of Theorems 5.2.3 and 5.2.4, Theorem 2.3.4 is Theorem 5.2.8, Theorem 2.3.6 is Theorem 5.2.9, Theorem 2.3.7 is Theorem 5.7.4, and Theorem 2.3.19 is Theorem 5.3.2. Theorems 2.3.11 and 2.3.14 generalize slightly Theorems 5.7.6 and 5.4.2 of [83], respectively.

Section 2.2

The recurrence classification of nearest-neighbour random walks on \( \mathbb{Z}_+ \) is classical, with early contributions being [127] and [129]. A standard presentation is given by Chung [45, §I.12].

Section 2.4

Of the numerous examples in Section 2.4, several are standard, and can also be found in [83], for instance; others are more specialized and foreshadow later developments in the present chapter and the rest of the book.

Theorem 2.4.7 is from a vein of results that can be traced back to Kushner [187, 188] and Spitzer [292]. The maximal inequality (2.14) is Lemma 3.1 in [232]. The expectation lower bound (2.13) is inspired by Spitzer [292]; a related result is Lemma 2.2 in [230]. Theorem 2.4.8 extends the result from [232]; inequality (2.12) is a slight generalization of Lemma 4.3 from [133].

Theorem 2.4.5 was used by Kushner [187], who showed that a supermartingale Lyapunov function provides an upper bound on the probability of eventual escape from a set, similar to the \( B = 0 \) case of Corollary 2.4.10. A version of Corollary 2.4.10 for Markov chains on \( \mathbb{R}^d \) was given by Kushner [188], assuming continuity of \( f \); Kalashnikov [145, p. 1406] remarked that this continuity assumption was unnecessary. Kalashnikov [146, 147] later provided some generalizations of the idea of Corollary 2.4.10 and gave applications to reliability theory and to queueing systems.

An alternative proof of (2.14) is to apply Theorem 2.4.5 to the non-negative supermartingale \( (X_{m \land n} + B(m - m \land n), n \geq 0) \), where \( m \) is fixed; this is the method used by Kushner [188] in the Markov setting, but it is awkward to extend to admit the stopping time \( \nu \). The argument in [232, 133] is based on Doob’s inequality. The proof of the more general Theorem 2.4.8 given here is entirely elementary.

The elementary approach to Theorem 2.4.8 is essentially the same idea used in the proofs of the close relatives Theorems 2.4.11 and 2.6.2; as described in Remark 2.4.9, all of these results can essentially be seen as
consequences of Dynkin's formula. Theorem 2.4.11, which is a relative of Lemma 3.2 from [232], can be seen as a ‘reverse’ Foster-type result. Theorem 2.4.11 provides an unusually simple approach to Kolmogorov’s ‘other’ inequality (Theorem 2.4.12); the standard approach via optional stopping can be found in [123, p. 502], for instance. Example 2.4.13 generalizes a result of Spitzer [292] for Markov chains on \( \mathbb{Z}_+ \); the terminology ‘weak renewal theorem’ is Spitzer’s.

The Azuma–Hoeffding martingale concentration inequality (2.27) in Theorem 2.4.14 is due to Hoeffding [130] and Azuma [13]. The submartingale and supermartingale versions given here are not often explicitly stated in the literature. A variety of related concentration inequalities can be found in e.g. [43].

Section 2.5

The basic criteria in Theorems 2.5.2 and 2.5.8 have their origin in work of F. G. Foster [108], who was a student of D. G. Kendall. Foster [108] proved a version of the ‘if’ half of the recurrence criterion Theorem 2.5.2, in the case where the exceptional set \( A \) is a single point. Extension to the case of a finite set \( A \) appears in Pakes [250]. The ‘only if’ part of Theorem 2.5.2 is due to Mertens et al. [238].

The transience criterion Theorem 2.5.8 is also due to Foster [108], again in the case where \( A \) is a singleton. The case of \( \text{finite } A \) is treated by Harris and Marlin [126] and Mertens et al. [238]; the condition that \( A \) be finite is not needed, as had already been realised in a different but analogous context by Kalashnikov [148], who proved a version of Theorem 2.5.8 with an inessential restriction on the form of \( f \).

Theorem 2.5.15, which gives transience of a Markov chain under a positive drift condition, assuming a uniform jumps bound, can be found in [213], following earlier work of Malyshev [214]. Both Theorem 2.5.2 and Theorem 2.5.15 were usefully extended to multiple-step drift conditions by Menshikov and Malyshev [226, 213]; see also Chapter 2 of Fayolle et al. [96]. Theorem 2.5.19 replaces the jumps bound in Theorem 2.5.15 by a (much weaker) moments assumption.

As mentioned in the text, Theorems 2.5.7 and 2.5.18 prefigure the developments in Chapter 3, and the basic ideas behind these two results owe much to Lamperti [190]. Theorem 2.5.7 is contained in Theorem 3.1 of [190]. A counterexample showing that zero drift for a Markov chain on \( \mathbb{R} \) does not imply recurrence (unlike on \( \mathbb{R}_+ \), as discussed in Example 2.5.6) can be found in Rogozin and Foss [273], who give a particular example of
Kemperman’s [157] oscillating random walk in which the increment law is one of two distributions (with mean zero but infinite variance) depending on the walk’s present sign. The oscillating random walk and related heavy-tailed phenomena are discussed in Chapter 5. The Markov chain transience condition, Theorem 2.5.19, is close to best possible for a one-step drift condition of this kind: the version here is given by Zachary [318], a (weaker) version requiring second moments is in Robert [271, p. 218], while some improvements are given by Denisov and Foss [106, 73].

The arguments of Foster, and other early authors, are simple enough, but are based on algebraic manipulations with Markov transition probabilities. The more general, and more probabilistic, approach via stopping times and adapted processes came later. The systematic use of martingale and stopping time ideas for investigating recurrence and transience seems to begin with Lamperti [190], who (while acknowledging a debt to ideas of Doob) treated more general processes than Markov chains. It took some time (see Mertens et al. [238] and Filonov [103]) for martingale ideas to become the norm in the proofs of the Markov chain versions of the classification theorems, although Blackwell [27] and Breiman [33], for example, had earlier made use of martingale ideas in proving structural results about Markov chains relating to recurrence.

The presentation in Section 2.5, and also Section 2.6, based on adapted processes, stopping times, and semimartingales, follows the spirit of Chapter 2 of [96], which contains additional results in a similar vein.

Section 2.6

Foster’s positive-recurrence criterion for Markov chains, Theorem 2.6.4, was first proved by Foster in [108] in the case where $A$ is a singleton, and $\varepsilon = 1$ (the latter being no loss of generality here). In fact, Foster appeals to a result of Feller (from the 1950 first edition of [98]) for the ‘only if’ part. Foster himself was aware that his argument could be extended to the case of finite $A$ (see his contribution to the discussion of [158, p. 175]), as was also remarked by Moustafa [242] and Kingman [170]; explicit proofs for finite $A$ were given by Pakes [250] and Kalashnikov [148], although the finite $A$ result is also a consequence of a condition of Mauldon [218] for a (not necessarily irreducible) Markov chain to be non-dissipative, which is equivalent to positive-recurrence in the irreducible case. (The terminology was introduced by Foster [107].)

The formulation of a Foster-type condition for integrability of a stopping time for an adapted process, as in Theorem 2.6.2, can be found for example
in Theorem 2.1.1 of [96]; a version for processes with bounded increments was given by Malyshev [214]. Theorems 2.6.2 and 2.6.4 may both be extended to a condition concerning increments over non-unit time intervals: see Theorems 2.1.2 and 2.2.4 of [96], which are based on earlier results of [226, 213, 103].

The results in Example 2.6.6 on the exit time of a zero drift Markov chain from a quadrant were first announced in the case of nearest-neighbour jumps on $\mathbb{Z}_+^2$ by Klein Haneveld [175], who used generating functions and an analytical method from unpublished work of J. Groeneveld; see [176] for a systematic account. In the generality in Example 2.6.6, the results are due to Klein Haneveld and Pittenger [177], whose proof is the basis for the argument given here, and Cohen [51], who used another elaborate generating function technique; see also [50, 298].

Example 2.6.7 on the exit time of a martingale from a square-root boundary gives a semimartingale approach to a result that dates back to Blackwell and Freedman [28], who showed for one-dimensional symmetric SRW that $E\kappa_a < \infty$ if and only if $a < 1$. The result of Blackwell and Freedman was generalized by Chow et al. [40] and Gundy and Siegmund [122]. The approach here to integrability, via Foster’s criterion, is different from these original papers, while the approach to non-integrability, via quadratic variation, is closely related to the approach in [40].

As mentioned in the main text, Foster’s original argument in [108], and indeed the arguments of Pakes [250] and Kalashnikov [148], are closer to that presented in the proof of Lemma 2.6.8, although all those results are stated only in the Markov chain case. Lemma 2.6.8 is the analogue of Foster’s original argument for adapted processes.

The idea of proving that a stopping time has infinite expectation by exhibiting a contradiction with the optional stopping theorem, as in Theorem 2.6.10, goes back at least to Doob [78, p. 308], who considered a martingale with uniformly bounded increments. The submartingale result stated as Theorem 2.6.10 here is a variation on Theorem 2.4 of Fayolle et al. [95]. The Markov chain formulation, Corollary 2.6.11, is a version of a result of Tweedie [304]; an earlier version, assuming a uniform bound on the increments of $f(X_n)$, was given by Malyshev in his doctoral thesis of 1973. Other results on ‘non-ergodicity’ of Markov chains include those of Sennott et al. [281], which make use of the less elementary Kaplan’s condition [151].
Section 2.7

Results of the form of those in Section 2.7 on existence and non-existence of moments of passage times originate with fundamental work of Lamperti [192], who considered the case of $\mathbb{E}[X_p^p]$ for positive integer $p$. Lamperti’s ideas were extended in [10] to include the case of non-integer $p$, including the important case $p \in (0, 1)$; the presentation here is based closely on [10]. In the Appendix of [10] it is shown that Theorems 2.7.1 and 2.7.4 of Section 2.7 improve the results of Lamperti (cf. Theorems 2.1, 2.2, 3.1, and 3.2 in [192]). General criteria in the vein of Theorem 2.7.1 for existence of generalized moments $\mathbb{E} g(\tau)$ are given in [9].

Lemma 2.7.5, showing that the process takes quadratic time to return to a neighbourhood of zero from a distant point, is closely related to Lemma 2 of [10], which improved Lemma 3.1 of [192]. The proof in [10] is different to the proof of Lemma 2.7.5 given here, which is based on the maximal inequality Theorem 2.4.8.

While existence of moments results are essentially equivalent, via Markov’s inequality, to upper tail bounds on passage times, non-existence of moments results do not directly yield any lower tail bounds. However, both types of result are obtained by combining, with slightly different technical ideas, two elements, namely (i) Lemma 2.7.5 and (ii) a suitable submartingale. The first method, as in the proof of Theorem 2.7.4, uses the submartingale property and the quadratic time estimate to exhibit a contradiction with the optional stopping theorem if one assumes existence of sufficient moments, an argument similar in spirit to Theorem 2.6.10 and an idea that goes back to Doob [78, p. 308]. A second method uses the submartingale to instead provide a lower tail bound on the maximum of an excursion, which with the quadratic time estimate yields a lower tail bound on the return time via Lemma 2.7.7. A similar idea goes back to Lamperti, who obtained a lower tail bound in the course of the proof of his Theorem 3.2 [192]; a variation is used to obtain lower tail bounds in [7]. The explicit connection to excursion maxima, as in Lemma 2.7.7, follows a similar presentation in [60] and [135].

For an irreducible Markov chain $X_n$ on a countable state-space $\Sigma$, the property that $\mathbb{E}_x[\tau^\gamma_x] < \infty$ for $x \in \Sigma$ was studied under the name “$\gamma$-recurrence” (also translated as “$\gamma$-reflexivity”) by Kalashnikov [149] and Popov [260], apparently unaware of Lamperti’s work, who both provided some conditions for $\mathbb{E}_x[\tau^\gamma_x] < \infty$, $\gamma \geq 1$, in terms of space-time Lyapunov functions of the form $f(X_n, n)$; these conditions seem to be much less applicable than the results presented here based on [192, 10]. Popov [260] went on to show that if $X_n$ is $\gamma$-recurrent with $\gamma > 1$, then the rate of convergence
of $\mathbb{P}[X_n = y \mid X_0 = x]$ to the stationary probability $\pi_y$ is $O(n^{-1})$.

For Markov chains on general state-spaces, drift conditions similar to those in Theorem 2.7.1 are used by Jarner and Roberts [142] to establish polynomial rates of convergence to stationarity, in the positive-recurrent case; see also [79].

The idea of Example 2.7.9 is clearly applicable much more widely than the case of SRW on $\mathbb{Z}^2$, and the statement is close to sharp. For SRW, Erdős and Taylor [88], refining an earlier result of Dvoretzky and Erdős [84], showed that $\mathbb{P}[\tau \geq n] = \frac{n}{\log n} + O((\log n)^{-2})$. See also [80].

Analogues of Theorems 2.7.1 and 2.7.4 for diffusion processes on $\mathbb{R}_+$ are given in [224].

Section 2.8

The results of Section 2.8 are based on [232]. The upper bound result, Theorem 2.8.1, is Theorem 3.2 in [232]. The lower bound result, Theorem 2.8.3, is based on Theorem 3.3 of [232]; the proof here is simpler, and the statement corrects the erroneous assertion from [232] that one may set $\delta = 0$ in (2.82).

Example 2.8.2 gives a diffusive upper bound for many-dimensional random walks whose drift decays suitably rapidly with distance from the origin; such walks may be transient, null-recurrent, or positive-recurrent (in which case this upper bound is typically not sharp) depending on the details: see Chapter 4. The application of the results in this section to nearest-neighbour random walk on $\mathbb{Z}$ with negative drift (Proposition 2.8.5) may well be known, but we could find no reference in the literature.

Further remarks

Additional historical and bibliographic details related to this chapter may be found in [96] and [239].

It is worth emphasizing that a large and sophisticated body of work exists on Foster–Lyapunov results for Markov chains in general state spaces. Here the interaction between the process, the space, and the different possible notions of recurrence and transience leads to a rich theory, but one with rather different focus and methodology from this book. Early work in this direction, starting in the mid 1970s, includes that of Tweedie [302, 304], and a recent treatment, including thorough bibliographic details, is the book of Meyn and Tweedie [239].
It is also worth mentioning that there is a well-developed analogue of the Foster–Lyapunov theory for diffusion processes (and Itô processes more generally). Here early contributions include fundamental work of Khas’minskii, such as [167]; see also [316, 317]. Recent accounts of aspects of the theory are presented in the books of Pinsky [258] (for diffusions) and Mao [216] (for Itô processes).
Chapter 3

Lamperti’s problem

3.1 Introduction

The law of a one-dimensional time-homogeneous Markov chain is determined
by the collection, over all points $x$ in the state space, of laws of its increments
at $x$, i.e., the one-step distributions for the chain started from $x$. A very
broad question is to investigate the extent to which the asymptotic behaviour
of the process is determined by coarser information, such as the first few
moments of the increments at $x$, as functions of $x$: can some general form
of ‘invariance principle’ be discerned? To obtain an intuitive grounding, it
is sensible that the variable $x$ should describe a natural scale for the process
in some way. Formalizing such vague notions, and generalizing beyond the
Markovian setting, one is led almost inevitably to what has become known
as Lamperti’s problem, after pioneering work of J. Lamperti published in
the early 1960s (see the bibliographical notes at the end of this chapter for
details).

Let us describe informally Lamperti’s problem. Consider a discrete-time
stochastic process $(X_n, n \geq 0)$ on $\mathbb{R}_+$. For now suppose that $X_n$ is a time-
homogeneous Markov process whose increment moment functions

$$
\mu_k(x) = \mathbb{E}[(X_{n+1} - X_n)^k \mid X_n = x]
$$

(3.1)

are well-defined for $k \geq 0$; one way to ensure this is to impose a uniform
bound on the increments. (We will relax all of these conditions shortly.)
Lamperti’s problem is to determine how the asymptotic behaviour of $X_n$
depends upon $\mu_1$ and $\mu_2$.

It is possible to proceed more generally, but to establish some intuition
we suppose for now that $X_n$ is already on a natural scale, meaning that
\( \mu_2(x) \) is bounded away from 0 and \( \infty \) by finite constants. This is not too much of a restriction, since if this second moments condition does not hold for \( X_n \), one can often choose a suitable \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) for which the condition *does* hold for the process \( Y_n = f(X_n) \). If \( f \) is bijective, then the Markov property of \( X_n \) transfers to \( Y_n \). Even if \( Y_n \) is not itself Markov, we may still proceed; indeed, there are other, more pressing reasons why we wish to relax the Markov assumption, as we describe below.

Assuming that \( \mu_2(x) \) is bounded away from 0 and \( \infty \), the behaviour of \( X_n \) is rather standard when, outside some bounded set, \( \mu_1(x) \equiv 0 \) (the zero drift case) or \( \mu_1(x) \) is uniformly bounded to one side of 0. In the zero-drift case, the Markov chain is null-recurrent; recurrence follows, under mild conditions, from Theorem 2.5.7 and the null property by Corollary 2.6.11. In the case of uniformly negative drift, the chain is positive-recurrent by Theorem 2.6.4. In the case of uniformly positive drift, the chain is transient, by Theorem 2.5.19. Typically, these regimes are also far from critical; positive-recurrence is accompanied by exponential estimates on hitting times and tails of stationary distributions, and transience is accompanied by positive speed of escape.

This motivates the study of the asymptotically zero drift regime, in which \( \mu_1(x) \to 0 \) as \( x \to \infty \), to investigate phase transitions. We shall examine in this chapter part of the rich spectrum of possible behaviours for \( X_n \). It turns out that from the point of view of the recurrence classification of \( X_n \), the case where \( |\mu_1(x)| \) is of order \( 1/x \) is critical.

These Lamperti processes serve as prototypical near-critical stochastic systems, and provide an excellent setting in which to demonstrate the application of the ideas from Chapter 2. There is, however, another reason, of at least equal importance, for studying these processes: they appear regularly in the analysis of near-critical stochastic processes via the Lyapunov function method. This context also provides the main motivation for the desire to work in some generality without imposing, for instance, assumptions of the Markov property, a countable state-space, or uniformly bounded increments. The following simple example demonstrates several important points.

**Example 3.1.1.** As in Example 2.8.2, let \((\xi_n, n \geq 0)\) be a time-homogeneous Markov chain on state space \( \Sigma \subseteq \mathbb{R}^d \) with increments \( \theta_n := \xi_{n+1} - \xi_n \). Suppose that there exists \( B \in \mathbb{R}_+ \) such that, for all \( x \in \Sigma \),

\[
E[||\theta_n||^2 \mid \xi_n = x] \leq B, \quad \text{and} \quad E[\theta_n \mid \xi_n = x] = 0.
\]

For example, \( \xi_n \) might be symmetric simple random walk on \( \mathbb{Z}^d \); in this
3.2. Markovian case

For orientation, we start with the Markov setting. We state in this section the Markov versions of some of the key results in this chapter; we do not...
give the proofs at this stage. The results presented in this section will follow from the more general results presented in the main body of this chapter; we defer the proofs until Section 3.13.

For this section, our basic assumptions on our process $X_n$ and on the structure of its state space $\Sigma$ are as follows.

**(M0)** Let $X_n$ be an irreducible, time-homogeneous Markov chain on $\Sigma$, a locally finite, unbounded subset of $\mathbb{R}_+$, with $0 \in \Sigma$.

We also make some assumptions on the increments of $X_n$. We will typically assume that for some $p > 2$ (at least),

$$
\sup_{x \in \Sigma} \mathbb{E}[|\Delta_n|^p \mid X_n = x] < \infty.
$$

**(3.2)**

**(M1)** Suppose that (3.2) holds for some $p > 2$.

Remark 3.2.1. As stated in Corollary 2.1.10, a consequence of (M0) is that $\limsup_{n \to \infty} X_n = \infty$, a.s.; see Section 3.6 below for a proof.

Given (M1), $\mu_k(x)$ as defined at (3.1) is finite for $k \in \{1, 2\}$.

**Example 3.2.2.** Although the proofs of the results in this section are deferred till later, we present, by way of an example, an application of Foster’s criterion (Theorem 2.6.4) to demonstrate a condition for positive recurrence. Suppose that (M0) and (M1) hold, and there exist positive constants $\varepsilon$ and $x_0$ so that $2x\mu_1(x) + \mu_2(x) < -\varepsilon$ for all $x \geq x_0$. Consider the Lyapunov function $f(x) = x^2$. Then

$$
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid X_n = x] = \mathbb{E}[2X_n\Delta_n + \Delta_n^2 \mid X_n = x] = 2x\mu_1(x) + \mu_2(x) \leq -\varepsilon,
$$

outside the set $[0, x_0] \cap \Sigma$, which is finite since $\Sigma$ is locally finite. Thus we may apply Theorem 2.6.4 to deduce that $X_n$ is positive recurrent. △

The basic recurrence classification is as follows; part (iii) is the statement from Example 3.2.2.

**Theorem 3.2.3.** Suppose that (M0) and (M1) hold. Then the following recurrence conditions are valid.

(i) $X_n$ is transient if there exist $\varepsilon, x_0 \in (0, \infty)$ such that

$$
2x\mu_1(x) - \mu_2(x) > \varepsilon, \text{ for all } x \geq x_0.
$$
3.2. Markovian case

(ii) \( X_n \) is null-recurrent if there exist \( x_0 \in (0, \infty), \theta > 0, \) and \( v > 0 \) such that for all \( x \geq x_0, \mu_2(x) \geq v \) and

\[
2x|\mu_1(x)| \leq \left(1 + \frac{1 - \theta}{\log x}\right)\mu_2(x).
\]

(iii) \( X_n \) is positive-recurrent if there exist \( \epsilon, x_0 \in (0, \infty) \) such that

\[
2x\mu_1(x) + \mu_2(x) < -\epsilon, \text{ for all } x \geq x_0.
\]

Remark 3.2.4. More compactly, (i) says \( \liminf_{x \to \infty} (2x\mu_1(x) - \mu_2(x)) > 0, \) and similarly (iii) says \( \limsup_{x \to \infty} (2x\mu_1(x) + \mu_2(x)) < 0. \)

Example 3.2.5. As in Section 2.2, consider the simple random walk on \( \mathbb{Z}_+ \) with transition probabilities \( p(x, x-1) = p_x = 1 - p(x, x+1) \) for \( x \geq 1 \) and \( p(0,0) = p_0 = 1 - p(0,1), \) where \( p_x \in (0,1) \) for all \( x \geq 0. \) Suppose that, for some \( c \in \mathbb{R}, \)

\[
p_x = \frac{1}{2} \left(1 + \frac{c}{x}\right) + o(x^{-1}).
\]

Then \( \mu_1(x) = 1 - 2p_x = -(c + o(1))x^{-1} \) and \( \mu_2(x) = 1, \) so Theorem 3.2.3 says that this random walk is

- transient if \( c < -1/2; \)
- null recurrent if \( |c| < 1/2; \)
- positive recurrent if \( c > 1/2. \)

The boundary cases \( c = \pm 1/2 \) can go either way depending on the \( o(x^{-1}) \) term in \( p_x; \) if this term is actually \( o((x \log x)^{-1}), \) say, Theorem 3.2.3(ii) shows that the case \( |c| = 1/2 \) is null recurrent. \( \triangle \)

Other results that we present in this chapter include existence of passage-time moments and almost-sure bounds on trajectories. To simplify the statements in the Markovian case, it is convenient to assume the following.

(M2) Suppose that there exist \( a \in \mathbb{R} \) and \( b \in (0, \infty) \) such that

\[
\lim_{x \to \infty} \mu_2(x) = b, \text{ and } \lim_{x \to \infty} x\mu_1(x) = a.
\]

In the case where (M0), (M1) and (M2) all hold, Theorem 3.2.3 implies that \( X_n \) is
• transient if $2a > b$;
• null recurrent if $|2a| < b$;
• positive recurrent if $2a < -b$.

As in Example 3.2.5, to classify the boundary cases needs stronger assumptions. The next result gives existence and non-existence of moments of return times. Recall from Section 2.1 that $\tau^+_x = \min\{n \geq 1 : X_n = x\}$.

**Theorem 3.2.6.** Suppose that (M0), (M1) and (M2) hold for $p > \max\{2, 2s\}$, where $s > 0$.

(i) For any $x \in \Sigma$, $\mathbb{E}_x[(\tau^+_x)^s] < \infty$ if $s < \frac{b-2a}{2b}$.

(ii) For any $x \in \Sigma$, $\mathbb{E}_x[(\tau^+_x)^s] = \infty$ if $s > \frac{b-2a}{2b}$.

The next result gives growth bounds on trajectories. For simplicity, we state the result on the logarithmic scale.

**Theorem 3.2.7.** Suppose that (M0), (M1) and (M2) hold.

(i) If $2a > b$, then, a.s.,
$$\lim_{n \to \infty} \frac{\log X_n}{\log n} = \frac{1}{2}.$$ (ii) If $|2a| \leq b$, then, a.s.,
$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} = \frac{1}{2}.$$ (iii) If $2a < -b$ and (M1) holds with $p > \frac{b-2a}{b}$, then, a.s.,
$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} = \frac{b}{b-2a} \in (0, \frac{1}{2}).$$

**Example 3.2.8.** Continuing from Example 3.2.5, consider the simple random walk on $\mathbb{Z}_+$ with
$$p(x, x-1) = p_x = 1 - p(x, x+1) = \frac{1}{2} \left(1 + \frac{c}{x}\right) + o(x^{-1}).$$

Then $\mu_1(x) = 1 - 2p_x = -(c + o(1))x^{-1}$ and $\mu_2(x) = 1$, so (M2) holds with $a = -c$ and $b = 1$, and Theorem 3.2.7 gives corresponding almost-sure bounds. For example, in the case $c > 1/2$ (positive recurrence), a.s.,
$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} = \frac{1}{1 + 2c} \in (0, \frac{1}{2}),$$
demonstrating the sub-diffusive behaviour of the walk’s upper envelope. △
3.3 General case

We now describe the more general setting that will occupy most of this chapter. Let \((X_n, n \geq 0)\) be a discrete-time stochastic process adapted to a filtration \((\mathcal{F}_n, n \geq 0)\) and taking values in a subset \(S\) of \(\mathbb{R}\) that is bounded in one direction and unbounded in the other. Without loss of generality, we locate the ordinate such that \(S \subseteq \mathbb{R}_+\) with \(\inf S = 0\) and \(\sup S = +\infty\).

Central objects in this chapter will be the conditional increment moments \(E[\Delta^k_n \mid \mathcal{F}_n]\), specifically for \(k = 1\) and \(k = 2\). Many of the conditions in the theorems will suppose that an inequality hold involving the \(\mathcal{F}_n\)-measurable random variables \(E[\Delta^1_n \mid \mathcal{F}_n]\), \(E[\Delta^2_n \mid \mathcal{F}_n]\), and \(X_n\); such inequalities will have to hold a.s. and in an appropriate asymptotic sense (for large values of \(X_n\)). It is convenient therefore to introduce some notation for upper and lower bounds on the increment moments \(E[\Delta^k_n \mid \mathcal{F}_n]\) as ‘functions’ of \(X_n\).

To this end, shortly we will define \(\mu_k : S \to \mathbb{R}\) and \(\bar{\mu}_k : S \to \mathbb{R}\) such that
\[
\mu_k(X_n) \leq E[\Delta^k_n \mid \mathcal{F}_n] \leq \bar{\mu}_k(X_n), \text{ a.s., for all } n \geq 0. \tag{3.3}
\]

If \(X_n\) is Markovian, then \(E[\Delta^k_n \mid \mathcal{F}_n] = E[\Delta^k_n \mid X_n]\), a.s., and we can take
\[
\mu_k(x) = \inf_{n \geq 0} E[\Delta^k_n \mid X_n = x], \text{ and } \bar{\mu}_k(x) = \sup_{n \geq 0} E[\Delta^k_n \mid X_n = x];
\]
if additionally \(X_n\) is time-homogeneous then \(\mu_k(x) = \bar{\mu}_k(x) \equiv \mu_k(x)\) as defined at (3.1).

Loosely speaking, in the general case \(E[\Delta^k_n \mid \mathcal{F}_n]\) involves additional randomness in \(\mathcal{F}_n\), once \(X_n\) has been fixed. Thus \(\bar{\mu}_k(x)\) should be the (essential) supremum over this additional randomness given \(\{X_n = x\}\). For \(\mu_k\) the situation is analogous.

Let us now formally define \(\underline{\mu}_k\) and \(\bar{\mu}_k\). Let \(k\) be a positive integer. Suppose that \(E[\Delta^k_n \mid \mathcal{F}_n]\) exists for all \(n \geq 0\). By standard theory of conditional expectations (see e.g. Section 9.1 of [46]), for each \(n \geq 0\) there exist a Borel-measurable function \(\varphi_{k,n} : S \to \mathbb{R}\) and an \(\mathcal{F}_n\)-measurable random variable \(\psi_{k,n}\) such that, a.s., \(E[\psi_{k,n} \mid X_n] = 0\) and
\[
E[\Delta^k_n \mid \mathcal{F}_n] = E[\Delta^k_n \mid X_n] + \psi_{k,n} = \varphi_{k,n}(X_n) + \psi_{k,n}. \tag{3.4}
\]

Here \(\varphi_{k,n}(x) = E[\Delta^k_n \mid X_n = x]\). Then for \(x \in S\) define
\[
\underline{\mu}_k(x) := \inf_{n \geq 0} \{ \varphi_{k,n}(x) + \text{ess inf}(\psi_{k,n}1\{X_n \geq x}) \}; \tag{3.5}
\]
\[
\bar{\mu}_k(x) := \sup_{n \geq 0} \{ \varphi_{k,n}(x) + \text{ess sup}(\psi_{k,n}1\{X_n \geq x}) \}. \tag{3.6}
\]
Provided that the expectations in question exist, (3.5) and (3.6) define \( \mu_k(x) \) and \( \bar{\mu}_k(x) \) as \( \mathbb{R} \)-valued functions of \( x \in \mathcal{S} \) such that \( \mu_k(x) \leq \bar{\mu}_k(x) \) for all \( x \in \mathcal{S} \), and the property (3.3) is satisfied.

**Example 3.3.1.** With the notation at (3.4), we may write

\[
X_{n+1} = X_n + \varphi_{1,n}(X_n) + \psi_{1,n} + \zeta_{n+1},
\]

where \( \zeta_{n+1} = \Delta_n - \mathbb{E}[\Delta_n \mid \mathcal{F}_n] \) is \( \mathcal{F}_{n+1} \)-measurable with \( \mathbb{E}[\zeta_{n+1} \mid \mathcal{F}_n] = 0 \); i.e., \( \zeta_n \) is a martingale difference sequence.

If \( \varphi_{k,n}(x) = \varphi_k(x) \) does not depend on \( n \), then

\[
|\bar{\mu}_k(x) - \mu_k(x)| \leq 2 \sup_{n \geq 0} \text{ess sup}(|\psi_{k,n}|(\{X_n \geq x\}))
\]

which, in many situations of interest, tends to zero as \( x \to \infty \) (and does so sufficiently fast if \( \bar{\mu}_k(x) \) and \( \mu_k(x) \) also tend to zero) so that \( \bar{\mu}_k(x) \sim \mu_k(x) \). The following simple example is one such case, and also serves to show that even in elementary situations, some technical machinery along the lines of that presented in this section is not easily avoided.

**Example 3.3.2.** Let \( S_n \) be symmetric SRW on \( \mathbb{Z}^d \), \( d \in \mathbb{N} \), and set \( X_n = \|S_n\| \). Let \( \mathcal{F}_n = \sigma(S_0, \ldots, S_n) \). We consider \( \mathbb{E}[\Delta_n \mid \mathcal{F}_n] = \mathbb{E}[\Delta_n \mid S_n] \). Recalling the notation from Section 1.2, for \( e_1, \ldots, e_d \) the standard orthonormal basis vectors, let \( U_d = \{\pm e_1, \ldots, \pm e_d\} \) denote the possible jumps of the walk. Then for \( x \in \mathbb{Z}^d \),

\[
\mathbb{E}[\Delta_n \mid S_n = x] = \frac{1}{2d} \sum_{e \in U_d} (\|x + e\| - \|x\|).
\]

Here, a Taylor’s formula calculation (as in Section 1.3) shows that

\[
\|x + e\| - \|x\| = \|x\| \left( \left( \frac{2e \cdot x}{\|x\|^2} \right)^{1/2} - 1 \right)
\]

\[
= \frac{1}{2} \left( \frac{2e \cdot x + 1}{\|x\|} \right) - \frac{1}{2} \left( \frac{(e \cdot x)^2}{\|x\|^3} \right) + O(\|x\|^{-2}),
\]

(3.8)

where the implicit constant in the \( O(\cdot) \) term depends only on \( d \). So

\[
\mathbb{E}[\Delta_n \mid S_n = x] = \frac{1}{2\|x\|} \left( 1 - \frac{1}{d} \right) + O(\|x\|^{-2});
\]
see equation (1.4). It follows that
\[ \varphi_{1,n}(x) = \frac{1}{2x} \left( 1 - \frac{1}{d} \right) + O(x^{-2}), \]
and \( |\psi_{1,n}|1\{X_n \geq x\} = O(x^{-2}) \),
where, again, implicit constants in the error terms depend only on \( d \). Hence
\[ \lim_{x \to \infty} x \mu_1(x) = \lim_{x \to \infty} x \tilde{\mu}_1(x) = \frac{1}{2} \left( 1 - \frac{1}{d} \right). \]
Note that if \( d \geq 2 \) the difference \( \tilde{\mu}_1(x) - \mu_1(x) \) is genuinely of order \( x^{-2} \): for instance, the third-order Taylor term in (3.8) contains \( \|x\|^{-5} \sum_{e \in U_d} (e \cdot x)^3 = \theta(x) \|x\|^{-2} \) for \( \theta(x) \) whose range includes a positive interval for all \( x \) with \( \|x\| \) sufficiently large.

\[ \square \]

If \( Y_n \) is Markov and \( X_n = f(Y_n) \), then although \( X_n \) is typically not Markov, if \( Y_n \) has reasonably homogeneous behaviour around level curves of \( f \), then one can expect that \( \tilde{\mu}_k(x) \) and \( \mu_k(x) \) have similar asymptotics.

Some further discussion of the definitions in (3.5) and (3.6), and some illustrative examples, can be found at the end of this section.

Our basic assumption is now as follows.

\[ \mathbf{(L0)} \] Let \( X_n \) be a stochastic process adapted to a filtration \( \mathcal{F}_n \) and taking values in \( S \subseteq \mathbb{R}_+ \) with \( \inf S = 0 \) and \( \sup S = +\infty \).

We impose a moments condition analogous to, but somewhat weaker than, condition (M1).

\[ \mathbf{(L1)} \] Suppose that for some \( p > 2, \delta \in (0, p-2] \), and \( C \in \mathbb{R}_+ \),
\[ \mathbb{E}[|\Delta_n|^p | \mathcal{F}_n] \leq C(1 + X_n)^{p-2-\delta}, \text{ a.s., for all } n \geq 0. \quad (3.9) \]
Note that a sufficient condition for (L1) is that \( \mathbb{E}[|\Delta_n|^p | \mathcal{F}_n] \leq C \), a.s., for some \( p > 2 \), some \( C < \infty \), and all \( n \geq 0 \); i.e., the \( \delta = p - 2 \) case of (3.9).

We also will typically assume the following non-confinement condition:

\[ \mathbf{(L2)} \] Suppose that \( \lim \sup_{n \to \infty} X_n = +\infty \), a.s.

Remarks 3.3.3. (a) Since \( X_n < \infty \) for all \( n \), \( \lim \sup_{n \to \infty} X_n = \infty \) is equivalent to \( \sup_{n \geq 0} X_n = \infty \).
(b) Condition (L2) is generally necessary for our questions of interest to be non-trivial, and is usually straightforward to verify in a particular application, as we discuss in Section 3.6 below. For example, if \( X_n \) is an irreducible time-homogeneous Markov chain and \( S \) is locally finite, then (L2) holds automatically: see Corollary 2.1.10.

We state one simple but useful sufficient condition for (L2). This does not rely on irreducibility as such, but the following related local escape property, which says that the process exits any interval \([0, x]\) in a finite number of steps with positive probability, depending only on \( x \).

\[(L3)\] Suppose that for each \( x \in \mathbb{R}_+ \) there exist \( r_x \in \mathbb{N} \) and \( \delta_x > 0 \) such that, for all \( n \geq 0 \),

\[
P\left[ \max_{n \leq m \leq n + r_x} X_m \geq x \, \bigg| \, F_n \right] \geq \delta_x, \quad \text{on } \{X_n \leq x\}. \tag{3.10} \]

Recall that for \( X_n \) a process on \( \mathbb{R}_+ \), \( \sigma_x = \min \{n \geq 0 : X_n \geq x\} \).

**Proposition 3.3.4.** Suppose that (L0) and (L3) hold. Then (L2) holds. Moreover, there exist \( K_x \in \mathbb{R}_+ \) and \( c_x > 0 \) such that, for all \( n \geq 0 \),

\[
P[\sigma_x \geq n \mid F_0] \leq K_x e^{-c_x n}, \quad \text{on } \{X_0 < x\}. \tag{3.11} \]

Finally, a sufficient condition for (L3) is that for each \( x \in \mathbb{R}_+ \) there exist \( m_x \in \mathbb{N} \) and \( \varepsilon_x > 0 \) such that, for all \( n \geq 0 \),

\[
P[X_n + m_x - X_n \geq \varepsilon_x \mid F_n] \geq \varepsilon_x, \quad \text{on } \{X_n \leq x\}. \tag{3.12} \]

**Example 3.3.5.** As in Example 3.1.1, let \((\xi_n, n \geq 0)\) be a time-homogeneous Markov chain on state space \( \Sigma \subseteq \mathbb{R}^d \) with increments \( \theta_n := \xi_{n+1} - \xi_n \). Suppose that there exists \( \varepsilon > 0 \) for which

\[
\inf_{u \in \mathbb{S}^{d-1}} P[\theta_n \cdot u \geq \varepsilon \mid \xi_n = x] \geq \varepsilon, \quad \text{for all } x \in \Sigma; \tag{3.13} \]

the property (3.13) is uniform ellipticity. Then, for \( X_n = \|\xi_n\| \), we have

\[
P[X_{n+1} - X_n \geq \varepsilon \mid \xi_n = x] \geq P[\hat{x} \cdot (x + \theta_n) - \|x\| \geq \varepsilon \mid \xi_n = x] \geq P[\hat{x} \cdot \theta_n \geq \varepsilon \mid \xi_n = x] \geq \varepsilon,
\]

by (3.13). Hence (3.12) holds for all \( x \in \mathbb{R}_+ \) with \( m_x = 1 \) and \( \varepsilon_x = \varepsilon \), and Proposition 3.3.4 shows that \( \limsup_{n \to \infty} \|\xi_n\| = \infty \), a.s. \( \triangle \)
3.3. General case

Proof of Proposition 3.3.4. Fix $x \geq 0$. Write $r = r_x$, $\delta = \delta_x$, and $\sigma = \sigma_x$ for ease of notation. By (3.10), $\mathbb{P}[\max_{n \leq m \leq n+r} X_m \geq x \mid \mathcal{F}_n] \geq \delta$, on \{n < \sigma\}. Hence,

$$\mathbb{P}[\sigma \geq n + r \mid \mathcal{F}_n] \leq 1 - \delta, \text{ on } \{n < \sigma\}. \quad (3.14)$$

Now, the $n = 0$ case of (3.14) says that $\mathbb{P}[\sigma > r \mid \mathcal{F}_0] \leq 1 - \delta$ on $\{X_0 < x\}$. Hence, conditioning on $\mathcal{F}_r$ and using the $n = r$ case of (3.14),

$$\mathbb{P}[\sigma > 2r \mid \mathcal{F}_0] = \mathbb{E}[\mathbf{1}\{\sigma > r\} \mathbb{P}[\sigma > 2r \mid \mathcal{F}_r] \mid \mathcal{F}_0]$$

$$\leq (1 - \delta) \mathbb{P}[\sigma > r \mid \mathcal{F}_0]$$

$$\leq (1 - \delta)^2, \text{ on } \{X_0 < x\}.$$

Iterating this argument shows that, on $\{X_0 < x\}$, $\mathbb{P}[\sigma > kr \mid \mathcal{F}_0] \leq (1 - \delta)^k$ for any integer $k \geq 0$. Any integer $n$ has $k_n r < n \leq (k_n + 1)r$ for some integer $k_n$, and then, on $\{X_0 < x\}$,

$$\mathbb{P}[\sigma > n \mid \mathcal{F}_0] \leq \mathbb{P}[\sigma > k_n r \mid \mathcal{F}_0]$$

$$\leq (1 - \delta)^{k_n} \leq (1 - \delta)^{(n/r) - 1},$$

which yields (3.11) for constants $c_x$ and $K_x$ depending on $\delta = \delta_x$ and $r = r_x$.

Now we deduce (I.2). Let $\lambda_0 = \min\{n \geq 0 : X_n \leq x\}$, and define recursively, for $k \in \mathbb{N}$, $\lambda_k = \min\{n \geq \lambda_{k-1} + r : X_n \leq x\}$, with the usual convention $\min\emptyset = \infty$. Set $L = \inf\{k \geq 0 : \lambda_k = \infty\}$. Thus we have defined stopping times $\lambda_k$ such that $\lambda_{k-1} < \lambda_k$ for all $k \leq L$, and $\lambda_k = \infty$ for all $k \geq L$. Also define for $k \in \mathbb{N}$ the event

$$E_k = \left\{ \max_{\lambda_{k-1} \leq n \leq \lambda_k} X_n \geq x \right\} \cup \{\lambda_k = \infty\}.$$

If $\lambda_k < \infty$, then $X_{\lambda_k} \leq x$ and so, by (3.10), $\mathbb{P}[E_{k+1} \mid \mathcal{F}_{\lambda_k}] \geq \delta$ on $\{\lambda_k < \infty\}$. On the other hand, if $\lambda_k = \infty$ then $\lambda_{k+1} = \infty$ as well, by definition, so that $\mathbb{P}[E_{k+1} \mid \mathcal{F}_{\lambda_k}] = 1$ on $\{\lambda_k = \infty\}$. Hence we obtain

$$\mathbb{P}[E_{k+1} \mid \mathcal{F}_{\lambda_k}] \geq \delta, \text{ a.s.}$$

Since $E_k \in \mathcal{F}_{\lambda_k}$, it follows from Lévy’s version of the Borel–Cantelli lemma (Theorem 2.3.19) that $\{E_k \text{ i.o.}\}$ occurs a.s. Hence either $L < \infty$ or each of $\{X_n \leq x\}$ and $\{X_n \geq x\}$ occurs for infinitely many $n$. In either case, $\limsup_{n \to \infty} X_n \geq x$, a.s. Since $x \in \mathbb{R}_+$ was arbitrary, $\limsup_{n \to \infty} X_n = \infty$, a.s.
It remains to show that (3.12) implies (3.10). Fix \( x \geq 0 \). Write \( m = m_x \), \( \varepsilon = \varepsilon_x \), and \( \sigma = \sigma_x \). For ease of notation, with no loss of generality, we take \( n = 0 \). Define for \( k \in \mathbb{N} \) the events
\[
A_k = \{ X_{km} - X_{(k-1)m} \geq \varepsilon \}, \quad \text{and} \quad B_k = A_k \cup \{ \sigma \leq m(k-1) \}.
\]
Then \( B_k \in \mathcal{F}_{km} \) and
\[
\mathbb{P}[B_{k+1} | \mathcal{F}_{km}] \geq \mathbb{P}[A_{k+1} | \mathcal{F}_{km}] \mathbf{1}\{ km < \sigma \} + \mathbb{P}[\sigma \leq mk | \mathcal{F}_{km}] \mathbf{1}\{ km \geq \sigma \} \geq \varepsilon, \quad \text{a.s.,}
\]
by (3.12), since on \( \{ km < \sigma \} \) we have \( \{ X_{km} \leq x \} \). Hence by a similar iterated conditioning argument to above, we obtain \( \mathbb{P}[\cap_{k=1}^r B_k | \mathcal{F}_0] \geq \varepsilon^r \).

Taking \( r_x = \min\{ r \in \mathbb{N} : r\varepsilon > x \} \) and \( \delta_x = \varepsilon^{r_x} \), we have that with probability at least \( \delta_x \), \( \cap_{k=1}^{r_x} B_k \) occurs, which implies that either \( \cap_{k=1}^{r_x} A_k \) occurs, or \( \{ \sigma < mr_x \} \). In the former case, \( X_{mr_x} \geq X_0 + r_x \varepsilon > x \). Hence
\[
\mathbb{P}[\max_{0 \leq k \leq mr_x} X_k \geq x | \mathcal{F}_0] \geq \delta_x, \quad \text{which implies (3.10).}
\]

To end this section, we briefly discuss further the definitions in (3.5) and (3.6), and give some examples for particular classes of process \( X_n \).

**Moments Markov property**

Somewhat weaker than the full Markov property is the assumption that
\[
\mathbb{E}[\Delta_n | \mathcal{F}_n] = \mu_1(X_n), \quad \text{and} \quad \mathbb{E}[\Delta_n^2 | \mathcal{F}_n] = \mu_2(X_n), \quad \text{a.s.,}
\]
for measurable functions \( \mu_1 \) and \( \mu_2 \). In the notation at (3.4), (3.15) asserts that \( \psi_{n,k} = 0 \) a.s. for \( k \in \{1, 2\} \). This moments Markov property is the standing assumption in papers such as [155] motivated by growth models and phrased in terms of stochastic difference equations. Indeed, under (3.15) one may write, similarly to (3.7),
\[
X_{n+1} = X_n + \mu_1(X_n) + \zeta_{n+1},
\]
where \( \mathbb{E}[\zeta_{n+1} | \mathcal{F}_n] = 0 \) and \( \mathbb{E}[\zeta_{n+1}^2 | \mathcal{F}_n] = \mu_2(X_n) - \mu_1(X_n)^2 \), so \( \zeta_n \) is a martingale difference sequence adapted to \( \mathcal{F}_n \).

**History-dependent processes**

Suppose that \( \mathcal{F}_n = \sigma(X_0, \ldots, X_n) \) and the law of \( X_{n+1} \) depends only upon \( (X_0, \ldots, X_n) \). For convenience, take \( S \) to be countable. Then we can write
\[
\mathbb{E}[\Delta_n | \mathcal{F}_n] = \sum_{x_0, \ldots, x_n \in S} \mathbb{E}[\Delta_n | X_0 = x_0, \ldots, X_n = x_n]
\]
3.4. Lyapunov functions

\[ 1 \{ X_0 = x_0, \ldots, X_n = x_n \} \times 1 \{ X_0 = x_0, \ldots, X_n = x_n \} \]

In this case

\[ \tilde{\mu}_1(x) = \sup_{n \in \mathbb{Z}_+} \sup E[\Delta_n \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x], \]

where the second supremum is taken over \( x_0, \ldots, x_{n-1} \in \mathcal{S} \) for which \( \mathbb{P}[X_0 = x_0, \ldots, X_{n-1} = x_{n-1} \mid X_n = x] > 0 \). There is an analogous expression for \( \underline{\mu}_1 \).

Functions of Markov processes

Suppose that \( Y_n \) is a time-homogeneous Markov process on a general state-space \( \Sigma \), and that for some measurable function \( f : \Sigma \to \mathbb{R}^+ \), \( X_n = f(Y_n) \). Set \( \mathcal{F}_n = \sigma(Y_0, \ldots, Y_n) \). Then \( X_n \) has state-space \( \mathcal{S} = f(\Sigma) \). Now \( E[\Delta_n \mid \mathcal{F}_n] = E[\Delta_n \mid Y_n] \), a.s., and if \( \Sigma \) (hence \( \mathcal{S} \)) is countable, we may write

\[ E[\Delta_n \mid \mathcal{F}_n] = \sum_{x \in \mathcal{S}} \sum_{y \in \Sigma : f(y) = x} E[\Delta_n \mid Y_n = y] 1\{Y_n = y, X_n = x\}. \]

In this case,

\[ \bar{\mu}_1(x) = \sup_{n \in \mathbb{Z}_+} \sup_{y \in \Sigma : f(y) = x} E[\Delta_n \mid Y_n = y], \]

and similarly for \( \underline{\mu}_1 \). This situation often arises in applications, where \( f \) may be, for instance, a Lyapunov-type function applied to a multi-dimensional Markov process.

3.4 Lyapunov functions

The results in this chapter will be obtained by application of the semimartingale results of Chapter 2 to suitably chosen Lyapunov functions of the process \( X_n \). For \( \gamma \in \mathbb{R} \) and \( \nu \in \mathbb{R} \) we define

\[ f_{\gamma, \nu}(x) := \begin{cases} x^\gamma \log x & \text{if } x \geq e; \\ e^{\gamma \nu} & \text{if } x < e. \end{cases} \tag{3.16} \]

The piecewise definition in (3.16) ensures that the function \( f_{\gamma, \nu} : \mathbb{R}^+ \to (0, \infty) \) is well defined for all \( \gamma \) and \( \nu \).

The analysis in this chapter will be built on the fact that \( f_{\gamma, \nu}(X_n) \) satisfies a submartingale or supermartingale condition, for \( X_n \) outside some
bounded set, for appropriate $\gamma$ and $\nu$. The next result gives the necessary increment estimates that we will need throughout the chapter; we state the result in some generality to avoid repeating very similar computations several times.

**Lemma 3.4.1.** Let $\gamma \in \mathbb{R}$ and $\nu \in \mathbb{R}$. Suppose that (L0) and (L1) hold, with $p > 2$ and $\delta \in (0, p - 2]$ such that $-\delta < \gamma < p$. Then there exists $\varepsilon_0 = \varepsilon_0(p, \delta, \gamma) \in (0, 1/2)$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, a.s.,

$$
\mathbb{E}[f_{\gamma, \nu}(X_{n+1}) - f_{\gamma, \nu}(X_n) \mid \mathcal{F}_n] = \gamma \left( X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] + \frac{\gamma - 1}{2} \mathbb{E}[\Delta_n^2 \mid \mathcal{F}_n] \right) f_{\gamma-2, \nu}(X_n)
+ \nu \left( X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] + \frac{2\gamma - 1}{2} \mathbb{E}[\Delta_n^2 \mid \mathcal{F}_n] \right) f_{\gamma-2, \nu-1}(X_n)
+ \left( \nu(\nu - 1) + O_X^2(1) \right) \mathbb{E}[\Delta_n^2 \mid \mathcal{F}_n] f_{\gamma-2, \nu-2}(X_n) + O_{X^n}(X_n^{-\gamma-\varepsilon}). \quad (3.17)
$$

Moreover, if $\gamma \in (0, p)$ then, for any $\varepsilon \in (0, \varepsilon_0)$, a.s.,

$$
\mathbb{E}[X_n^\gamma \mid \mathcal{F}_n] = \gamma \left( X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] + \frac{\gamma - 1}{2} \mathbb{E}[\Delta_n^2 \mid \mathcal{F}_n] \right) X_n^{-2} + O_{X^n}(X_n^{-\gamma-\varepsilon}). \quad (3.18)
$$

To prepare for the proof of Lemma 3.4.1, we first state a technical result. For $\varepsilon \in (0, 1)$, define the event

$$
E_\varepsilon(n) := \{|\Delta_n| \leq (1 + X_n)^{1-\varepsilon}\}. \quad (3.19)
$$

Denote the complementary event by $E^c_\varepsilon(n)$.

**Lemma 3.4.2.** Suppose that (L0) and (L1) hold, with $p > 2$, $\delta \in (0, p - 2]$, and $C \in \mathbb{R}_+$. Then for any $\varepsilon \in (0, 1)$, any $q \in [0, p]$, and all $n \geq 0$,

$$
\mathbb{E}[|\Delta_n|^q \mathbf{1}(E_\varepsilon^c(n)) \mid \mathcal{F}_n] \leq C(1 + X_n)^{-\delta+\varepsilon-\varepsilon} \quad \text{a.s.} \quad (3.20)
$$

Moreover, for any $\varepsilon \in (0, \frac{\delta}{1+p})$ and any $q \in [0, p]$, for all $n \geq 0$,

$$
\mathbb{E}[|\Delta_n|^q \mathbf{1}(E_\varepsilon(n)) \mid \mathcal{F}_n] \leq C(1 + X_n)^{-\delta+\varepsilon} \quad \text{a.s.} \quad (3.21)
$$

**Proof.** For $q \in [0, p]$, by definition of $E_\varepsilon(n)$,

$$
|\Delta_n|^q \mathbf{1}(E_\varepsilon^c(n)) = |\Delta_n|^p |\Delta_n|^{q-p} \mathbf{1}(E_\varepsilon^c(n)) \leq |\Delta_n|^p (1 + X_n)^{(1-\varepsilon)(q-p)}.
$$

Taking expectations conditioned on $\mathcal{F}_n$ and using (3.9) we obtain

$$
\mathbb{E}[|\Delta_n|^q \mathbf{1}(E_\varepsilon^c(n)) \mid \mathcal{F}_n] \leq C(1 + X_n)^{-\delta+\varepsilon+(1-\varepsilon)(q-p)} \quad \text{a.s.},
$$

which implies (3.20), from which (3.21) also follows. \qed
Proof of Lemma 3.4.1. Fix $\gamma, \nu \in \mathbb{R}$. Throughout the proof we write just $f$ for $f_{\gamma, \nu}$. As we are interested in asymptotics when $X_n$ is large, we may suppose throughout that $X_n > 2e$. For any $\varepsilon \in (0, 1)$,

$$
\begin{align*}
&f(X_{n+1}) - f(X_n) = (f(X_n + \Delta_n) - f(X_n)) \mathbf{1}(E_\varepsilon(n)) \\
&\quad + (f(X_n + \Delta_n) - f(X_n)) \mathbf{1}(E_\varepsilon^c(n)).
\end{align*}
$$

We estimate the expectation on $E_\varepsilon(n)$ here using a Taylor expansion, while we use Lemma 3.4.2 to control the expectation on $E_\varepsilon^c(n)$. Write

$$
\varepsilon_0 = \varepsilon_0(p, \delta, \gamma) = \frac{\min\{\delta, \gamma + \delta, 1 + p\}}{2 + 2p},
$$

(3.22)

and suppose that $\varepsilon \in (0, 2\varepsilon_0)$; note that since $-\delta < \gamma < p$, we have $2\varepsilon_0 \in (0, 1)$. Differentiation of $f$ with respect to $x > e$ gives

$$
\begin{align*}
f'(x) &= (\gamma + \nu \log^{-1} x) x^{-1} f(x); \\
f''_{\gamma,\nu}(x) &= (\gamma(\gamma - 1) + \nu(2\gamma - 1) \log^{-1} x + \nu(\nu - 1) \log^{-2} x) x^{-2} f(x);
\end{align*}
$$

and $f'''_{\gamma,\nu}(x) = O(x^{\gamma-3} \log^\nu x)$. Taylor’s formula with Lagrange remainder says that for any $x, h \in \mathbb{R}$ with $x > e$ and $x + h > e$,

$$
f(x + h) - f(x) = hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x + \varphi h),
$$

for some $\varphi = \varphi(x, h) \in [0, 1]$. Taking $x = X_n$ and $h = \Delta_n$, we have on $E_\varepsilon(n)$ that $X_n + \varphi \Delta_n > X_n/2 > e$, say, for all $X_n$ sufficiently large. Hence we can find constants $C$ and $x_1$ so that the third-order term in the Taylor expansion of $f(X_n + \Delta_n) - f(X_n)$ on $E_\varepsilon(n)$ satisfies

$$
|\Delta_n^3 f'''(X_n + \varphi \Delta_n)\mathbf{1}(E_\varepsilon(n))| \leq C|\Delta_n|^3 X_n^{\gamma-3} \log^\nu X_n \mathbf{1}(E_\varepsilon(n))
$$

$$
\leq C\Delta_n^2 X_n^{\gamma-2-\varepsilon} \log^\nu X_n
$$

$$
\leq C\Delta_n^2 X_n^{\gamma-2-(\varepsilon/2)}, \text{ on } \{X_n > x_1\}.
$$

Hence

$$
\begin{align*}
(f(X_n + \Delta_n) - f(X_n)) \mathbf{1}(E_\varepsilon(n)) \\
&= \left(\Delta_n f'(X_n) + \frac{\Delta_n^2}{2} f''(X_n)\right) \mathbf{1}(E_\varepsilon(n)) + \Delta_n^2 O_{X_n}^{F_{n}}(X_n^{\gamma-2-(\varepsilon/2)}).
\end{align*}
$$

Collecting terms, we obtain

$$
(f(X_n + \Delta_n) - f(X_n)) \mathbf{1}(E_\varepsilon(n))
$$
\[
\text{exists} \ C < \exists \text{ of (3.20). By choice of } 112 \text{ that } 0 \text{ this establishes the claim (3.24) in the case which is bounded above by a constant times } X \text{.}
\]

To verify (3.24), first suppose that \( \gamma, \nu > 0 \). Next suppose that \( \gamma, \nu < 0 \). Then \( f_{\gamma, \nu} \) is eventually decreasing, and there exists \( C \in \mathbb{R}_+ \) such that \( 0 \leq f_{\gamma, \nu}(x) \leq C \) for all \( x \in \mathbb{R}_+ \). Hence

\[
\mathbb{E}[|f(X_n + \Delta_n) - f(X_n)| \, 1(E_{\epsilon}(n)) \, | \mathcal{F}_n] = O_{X_n}^{\mathcal{F}}(X_{\frac{2}{1+\nu}} - (\epsilon/2)). \tag{3.24}
\]

To verify (3.24), first suppose that \( \gamma < 0 \). Then \( f_{\gamma, \nu} \) is eventually decreasing, and there exists \( C \in \mathbb{R}_+ \) such that \( 0 \leq f_{\gamma, \nu}(x) \leq C \) for all \( x \in \mathbb{R}_+ \). Hence

\[
\mathbb{E}[|f(X_n + \Delta_n) - f(X_n)| \, 1(E_{\epsilon}(n)) \, | \mathcal{F}_n] = C \mathbb{P}[E_{\epsilon}(n) \, | \mathcal{F}_n],
\]

which is bounded above by a constant times \( X_{\frac{2}{1+\nu}} - (\epsilon/2) \), by the \( q = 0 \) case of (3.20). By choice of \( \epsilon \in (0, 2\epsilon_0) \), we have \( \epsilon < \frac{\gamma + \delta}{1+\nu} \), so that \( -\delta + \nu < \gamma < \epsilon \); this establishes the claim (3.24) in the case \( \gamma \in (-\delta, 0) \).

Next suppose that \( \gamma \geq 0 \). Now, for any \( \epsilon' > 0 \), there exists \( C \in \mathbb{R}_+ \) such that \( 0 \leq f_{\gamma, \nu}(x) \leq C(1 + x)^{\gamma + \epsilon'} \) for all \( x \in \mathbb{R}_+ \). Hence for any \( \epsilon' > 0 \) there exists \( C < \infty \) such that

\[
|f(X_n + \Delta_n) - f(X_n)| \leq C(1 + X_n)^{\gamma + \epsilon'} + C|\Delta_n|^{\gamma + \epsilon'}, \text{ a.s.} \tag{3.25}
\]
Lemma 3.4.3. Let \( \epsilon > 0 \) with a truncated version the function. Specifically, for \( \gamma \geq 2 \), which lies in the interval \((0, 2\epsilon)\), this establishes the claim (3.24) in the case \( \gamma < \epsilon/2 \), say) this establishes the claim (3.24) in the case \( \gamma \in [0, p) \).

Combining (3.24) with (3.23) we obtain (3.17), but with \( \epsilon \) in place of \( \epsilon/2 \), which lies in the interval \((0, \epsilon_0)\). The expression in (3.18) is essentially the \( \nu = 0 \) case of (3.17); the difference near zero between \( f_{\gamma,0} \) as defined at (3.17) and the function \( x^{\gamma} \) does not affect the Taylor estimate in the analogue of (3.23), and the analogue of (3.24) goes through provided \( \gamma > 0 \).

For some of our results we need to study the case where \( f_{\gamma,\nu} \) as defined at (3.16), with positive \( \gamma \), satisfies a local submartingale property; application often requires uniform integrability. A convenient technical device is to work with a truncated version the function. Specifically, for \( x > 0 \) define

\[
 f_{\gamma,\nu}^x(y) := \min\{f_{\gamma,\nu}(y), f_{\gamma,\nu}(2x)\}. 
\]

**Lemma 3.4.3.** Let \( \gamma \in (0, \infty) \) and \( \nu \in \mathbb{R}_+ \). Suppose that (L0) and (L1) hold, with \( p > \max\{2, \gamma\} \). Then there exists \( \epsilon_0 = \epsilon_0(p, \delta, \gamma) > 0 \) such that, for any \( \epsilon \in (0, \epsilon_0) \), there exists \( x_0 = x_0(\epsilon) \in \mathbb{R}_+ \) such that, for any \( x \geq x_0 \), on \( \{X_n \leq x\} \),

\[
 \mathbb{E}[f_{\gamma,\nu}^x(X_{n+1}) - f_{\gamma,\nu}^x(X_n) | \mathcal{F}_n] \\
\geq \gamma \left( X_n \mathbb{E}[\Delta_n | \mathcal{F}_n] + \frac{\gamma - 1}{2} \mathbb{E}[\Delta_n^2 | \mathcal{F}_n] \right) f_{\gamma-2,\nu}(X_n) \\
+ \nu \left( X_n \mathbb{E}[\Delta_n | \mathcal{F}_n] + \frac{2\gamma - 1}{2} \mathbb{E}[\Delta_n^2 | \mathcal{F}_n] \right) f_{\gamma-2,\nu-1}(X_n) \\
+ \left( \nu(\nu - 1) + \sigma_{X_n}^2(1) \right) \mathbb{E}[\Delta_n^2 | \mathcal{F}_n] f_{\gamma-2,\nu-2}(X_n) + O_{X_n}(X_n^{\gamma-2-\epsilon}),
\]

where the implicit constants in the error terms do not depend on \( x \).
Proof. Write $f$ for $f_{\gamma, \nu}$ and $f^x$ for $f_{\gamma, \nu}^x$; note that both $f$ and $f^x$ are non-decreasing, and they coincide on $[0, 2x]$. Let $\varepsilon_0 \in (0, 1/2)$ be the constant in the statement of Lemma 3.4.1, as given at (3.22). Fix $\varepsilon \in (0, 2\varepsilon_0)$. Then $\{X_n \leq x\} \cap E_\varepsilon(n)$ implies that $X_{n+1} \leq 2x$ provided $x \geq x_0$ for some $x_0 = x_0(\varepsilon)$ sufficiently large, so that $f^x(x_{n+1}) = f(x_{n+1})$. Fix $x \geq x_0$. Then, on $\{X_n \leq x\}$,

$$f^x(x_{n+1}) - f^x(x_n) \geq \left( f(x_{n+1}) - f(x_n) \right) 1(E_\varepsilon(n)) - f(x_n) 1\{\Delta_n > (1 + X_n)^{1-\varepsilon}\},$$

so that, on $\{X_n \leq x\}$,

$$\mathbb{E}[f^x(x_{n+1}) - f^x(x_n) \mid \mathcal{F}_n] \geq \mathbb{E}\left[ (f(x_{n+1}) - f(x_n)) 1(E_\varepsilon(n)) \mid \mathcal{F}_n \right] - f(x_n) \mathbb{P}[E_\varepsilon^c(n) \mid \mathcal{F}_n].$$

Since $\varepsilon \in (0, 2\varepsilon_0)$ satisfies $\varepsilon < \frac{\delta}{1+p}$, the $q = 0$ case of (3.21) shows that

$$f(x_n) \mathbb{P}[E_\varepsilon^c(n) \mid \mathcal{F}_n] = O_X^{F_n}(X_n^\gamma - (\varepsilon/2)).$$

Combined with (3.23) this shows that, on $\{X_n \leq x\}$, (3.28) holds, once $\varepsilon/2$ is replaced by $\varepsilon \in (0, \varepsilon_0)$.

\[ \Box \]

### 3.5 Recurrence classification

We need to explain what we mean by ‘recurrence’ or ‘transience’ in the general setting introduced in Section 3.3. In this section we give conditions under which one or other of the following two behaviours (which are \textit{a priori} not exhaustive) occurs (cf. Definition 1.5.1):

- $\lim_{n \to \infty} X_n = +\infty$, a.s., in which case we say that $X_n$ is \textit{transient};
- $\liminf_{n \to \infty} X_n \leq r_0$, a.s., for some constant $r_0 \in \mathbb{R}_+$, when we say $X_n$ is \textit{recurrent}.

First we give a condition for transience.

**Theorem 3.5.1.** Suppose that (L0), (L1) and (L2) hold. Suppose that

$$\limsup_{x \to \infty} \tilde{\mu}_2(x) < \infty, \quad \text{and} \quad \liminf_{x \to \infty} (2x \mu_1(x) - \tilde{\mu}_2(x)) > 0. \quad (3.29)$$

Then $\lim_{n \to \infty} X_n = +\infty$, a.s.
3.5. Recurrence classification

The next result gives a condition for recurrence.

**Theorem 3.5.2.** Suppose that (L0), (L1) and (L2) hold. Suppose that \( \lim \inf_{x \to \infty} \mu_2(x) > 0 \), and that there exist \( x_0 \in \mathbb{R}_+ \) and \( \theta > 0 \) such that
\[
2x \mu_1(x) \leq \left( 1 + \frac{1 - \theta}{\log x} \right) \mu_2(x), \quad \text{for all } x \geq x_0.
\]
(3.30)

Then there exists \( r_0 \in \mathbb{R}_+ \) such that \( \lim \inf_{n \to \infty} X_n \leq r_0 \), a.s.

We give two important examples.

**Example 3.5.3.** Consider \( S_n \) symmetric SRW on \( \mathbb{Z}^d \), and let \( X_n = \|S_n\| \) and \( F_n = \sigma(S_0, \ldots, S_n) \). Either \( S_n \) is transient and \( \lim_{n \to \infty} X_n = \infty \) a.s., or else \( S_n \) is recurrent and \( \lim \inf_{n \to \infty} X_n = 0 \) a.s.: see Section 3.6 for a more general discussion. Example 3.3.2 shows that
\[
2x \mu_1(x) = 1 - \frac{1}{d} + O(x^{-2}), \quad \text{and } 2x \mu_1(x) = 1 - \frac{1}{d} + O(x^{-2}).
\]
Second moments can be computed similarly to in Example 3.3.2 using Taylor’s formula. Alternatively, we can use the square-difference identity (2.23) in the form
\[
\mathbb{E}[\Delta_n^2 | S_n] = \mathbb{E}[X_{n+1}^2 - X_n^2 | S_n] - 2X_n \mathbb{E}[\Delta_n | S_n],
\]
and the fact, obtained by expanding \( \|S_n + (S_{n+1} - S_n)\|^2 \), that
\[
X_{n+1}^2 - X_n^2 = 2S_n \cdot (S_{n+1} - S_n) + \|S_{n+1} - S_n\|^2,
\]
(3.31)
so \( \mathbb{E}[X_{n+1}^2 - X_n^2 | S_n] = 1 \), to obtain
\[
\mu_2(x) = \frac{1}{d} + O(x^{-1}), \quad \text{and } \bar{\mu}_2(x) = \frac{1}{d} + O(x^{-1}).
\]
Thus (3.29) holds provided \( d \geq 3 \), so Theorem 3.5.1 implies transience. On the other hand, (3.30) holds if \( d \in \{1, 2\} \) and Theorem 3.5.2 implies recurrence. This is Lamperti’s proof of Pólya’s theorem on simple random walk, as advertised in Section 1.3.

**Example 3.5.4.** We extend Pólya’s theorem from Example 3.5.3 to a class of zero-drift random walks, as advertised in Section 1.5. Similarly to Example 3.1.1, let \( (\xi_n, n \geq 0) \) be a time-homogeneous Markov chain with state
space $\Sigma \subseteq \mathbb{R}^d$ and increments $\theta_n := \xi_{n+1} - \xi_n$ such that, for some $B \in \mathbb{R}_+$ and all $x \in \Sigma$,
\[ P[\|\theta_n\| \leq B \mid \xi_n = x] = 1, \quad \text{and} \quad E[\theta_n \mid \xi_n = x] = 0. \]

Suppose in addition that the increment covariance matrix is constant, i.e., with $\theta_n$ as a column vector, for all $x \in \Sigma$,
\[ E[\theta_n \theta_n^\top \mid \xi_n = x] = M, \]
for some positive definite, symmetric $d \times d$ matrix $M$. The argument in Remark 1.5.5 shows that it is no loss of generality to take $M = I_d$, the identity.

So suppose $M = I_d$. Let $X_n = \|\xi_n\|$. Similarly to (3.31), we have
\[ E[X_{n+1}^2 - X_n^2 \mid \xi_n] = 2 \xi_n \cdot E[\theta_n \mid \xi_n] + E[\|\theta_n\|^2 \mid \xi_n] = 0 + \text{tr} M = d. \quad (3.32) \]

Moreover, using the bound $\|\theta_n\| \leq B$ and the Taylor expansion $(1 + y)^{1/2} = 1 + (y/2) - (y^2/8) + O(y^3)$, similarly to the calculation in Section 1.5,
\begin{align*}
E[X_{n+1} - X_n \mid \xi_n] &= \|\xi_n\| E\left[\left(1 + \frac{2 \xi_n \cdot \theta_n + \|\theta_n\|^2}{\|\xi_n\|^2}\right)^{1/2} - 1 \mid \xi_n\right] \\
&= \frac{\xi_n \cdot E[\theta_n \mid \xi_n]}{\|\xi_n\|} + \frac{E[\|\theta_n\|^2 \mid \xi_n]}{2\|\xi_n\|^2} - \frac{E[(\xi_n \cdot \theta_n)^2 \mid \xi_n]}{2\|\xi_n\|^3} + O(\|\xi_n\|^{-2}) \\
&= \frac{1}{2X_n} \left(\text{tr} M - \hat{\xi}_n^\top M \hat{\xi}_n\right) + O(X_n^{-2}),
\end{align*}
using the fact that $E[\theta_n \mid \xi_n] = 0$. Hence, since $M = I_d$, we obtain
\[ E[X_{n+1} - X_n \mid \xi_n] = \frac{d - 1}{2X_n} + O(X_n^{-2}). \quad (3.33) \]

Thus with (3.32), (3.33), and the usual trick based on (2.23), we obtain
\[ 2x \mu_1(x) = d - 1 + O(x^{-1}) = 2x \bar{\mu}_1(x), \quad \text{and} \quad \mu_2(x) = 1 + O(x^{-1}) = \bar{\mu}_2(x). \]

Thus (3.29) holds provided $d \geq 3$, so Theorem 3.5.1 implies transience, while (3.30) holds if $d \in \{1, 2\}$ and Theorem 3.5.2 implies recurrence. This example is a preview of Chapter 4; there we relax the uniform increment bound, as well considering walks with non-zero drift. \[ \triangle \]
3.5. Recurrence classification

Remarks 3.5.5. (a) The calculations in Examples 3.5.3 and 3.5.4 suggest informally that the recurrence phase transition for this class of zero-drift many-dimensional random walks occurs at $d = 2$ (rather than elsewhere in the interval $[2, 3]$). This transition is probed more precisely in Chapter 4.

(b) For condition (3.30), it is sufficient (take $\theta = 1$) that
$$2x\bar{\mu}_1(x) - \mu_2(x) \leq 0$$
for all $x$ large enough; however, we needed to take $\theta < 1$ to settle the critical $d = 2$ case in Examples 3.5.3 and 3.5.4.

We now turn to the proofs of Theorems 3.5.1 and 3.5.2. We proceed under the minimal assumption (L0) for much of the argument, in part to highlight exactly where the non-confinement assumption (L2) enters, but also so that the results that follow can be applied to situations where (L2) fails (which is the case, for example, in the context of branching processes or other processes with absorption: see Example 3.5.9).

Consider the random variable $X^* := \sup_{n \geq 0} X_n \in \mathbb{R}_+$. In principle, there are three possibilities:

(a) $\text{ess sup } X^* < \infty$, i.e., $X^*$ is uniformly bounded, a.s.;

(b) $\text{ess inf } X^* < \infty$ but $\text{ess sup } X^* = \infty$;

(c) $\text{ess inf } X^* = \infty$, i.e., $X^* = \infty$ a.s.

Here, (c) is the case when (L2) holds, case (a) is generally uninteresting, and case (b) pertains in, for instance, the case where there is some non-trivial probability that the process gets absorbed or trapped.

The following semimartingale result is a ‘transience’ result related to Theorem 2.5.8, but which applies in case (b) as well as case (c); we prove Theorem 3.5.1 by applying this result with a suitable Lyapunov function.

Recall that $\sigma_x = \min\{n \geq 0 : X_n \geq x\}$.

Theorem 3.5.6. Suppose that (L0) holds. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\sup_x f(x) < \infty$, $\lim_{x \to \infty} f(x) = 0$, and $\inf_{y \leq x} f(y) > 0$ for any $x \in \mathbb{R}_+$. Suppose also that there exists $x_0 \in \mathbb{R}_+$ for which, for all $n \geq 0$,

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n > x_0\}.$$

(i) If $\mathbb{P}[\sigma_x < \infty] > 0$ for all $x \in \mathbb{R}_+$, then $\mathbb{P}[\lim_{n \to \infty} X_n \geq x] > 0$ for all $x \in \mathbb{R}_+$.

(ii) If (L2) holds, then $\mathbb{P}[\lim_{n \to \infty} X_n = \infty] = 1$.

The core of Theorem 3.5.6 is a hitting probability estimate, which is a variation on Lemma 2.5.10.
Lemma 3.5.7. Let $X_n$ be an $\mathcal{F}_n$-adapted process taking values in $\mathbb{R}_+$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\sup_x f(x) < \infty$ and $\lim_{x \to \infty} f(x) = 0$. Suppose that there exists $x_1 \in \mathbb{R}_+$ for which $\inf_{y \leq x_1} f(y) > 0$ and, for all $n \geq 0$,

$$
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n > x_1\}.
$$

Then for any $\varepsilon > 0$ there exists $x \in (x_1, \infty)$ for which, for all $n \geq 0$,

$$
P[\inf_{m \geq n} X_m \geq x_1 \mid \mathcal{F}_n] \geq 1 - \varepsilon, \text{ on } \{X_n > x\}.
$$

Proof. Fix $n \in \mathbb{Z}_+$. For $x_1 \in \mathbb{R}_+$ in the hypothesis of the lemma, write $
\lambda = \min\{m \geq n : X_m \leq x_1\}$ and set $Y_m = f(X_m \wedge \lambda)$. Then $(Y_m, m \geq n)$ is an $(\mathcal{F}_m, m \geq n)$-adapted non-negative supermartingale, and so, by Theorem 2.3.6, converges a.s. as $m \to \infty$ to some $Y_\infty \in \mathbb{R}_+$. Moreover, by Theorem 2.3.11,

$$
Y_n \geq \mathbb{E}[Y_\infty \mid \mathcal{F}_n] \geq \mathbb{E}[Y_\infty 1\{\lambda < \infty\} \mid \mathcal{F}_n], \text{ a.s.}
$$

Here we have that, a.s.,

$$
Y_\infty 1\{\lambda < \infty\} = \lim_{m \to \infty} Y_m 1\{\lambda < \infty\} = f(X_\lambda) 1\{\lambda < \infty\} \geq \inf_{y \leq x_1} f(y) 1\{\lambda < \infty\}.
$$

Combining these inequalities we obtain

$$
\inf_{y \leq x_1} f(y) \mathbb{P}[\lambda < \infty \mid \mathcal{F}_n] \leq Y_n, \text{ a.s.}
$$

In particular, on $\{X_n > x \geq x_1\}$, we have $Y_n = f(X_n)$ and so

$$
\inf_{y \leq x_1} f(y) \mathbb{P}[\lambda < \infty \mid \mathcal{F}_n] \leq f(X_n) \leq \sup_{y \geq x} f(y).
$$

Since $\lim_{y \to \infty} f(y) = 0$ and $\inf_{y \leq x_1} f(y) > 0$, given $\varepsilon > 0$ we can choose $x > x_1$ large enough so that

$$
\frac{\sup_{y \geq x} f(y)}{\inf_{y \leq x_1} f(y)} < \varepsilon;
$$

the choice of $x$ depends only on $f$, $x_1$, and $\varepsilon$, and, in particular, does not depend on $n$. Then, on $\{X_n > x\}$, $\mathbb{P}[\lambda < \infty \mid \mathcal{F}_n] < \varepsilon$, as claimed. 

We now use the local result Lemma 3.5.7 to deduce the sample-path result Theorem 3.5.6.
3.5. Recurrence classification

Proof of Theorem 3.5.6. Under the conditions of Theorem 3.5.6, the hypotheses of Lemma 3.5.7 are satisfied for any \( x_1 > x_0 \). So for any \( x_1 > x_0 \) and any \( \varepsilon > 0 \), there exists \( x \in \mathbb{R}_+, x > x_1 \) such that, on \( \{ X_n \geq x \} \),

\[
P\left[ \inf_{m \geq n} X_m > x_1 \left| \mathcal{F}_n \right. \right] \geq 1 - \varepsilon, \text{ a.s.}
\]

As usual, let \( \sigma_x = \min\{ n \geq 0 : X_n \geq x \} \). Then, on \( \{ \sigma_x < \infty \} \),

\[
P\left[ \inf_{m \geq \sigma_x} X_m > x_1 \left| \mathcal{F}_{\sigma_x} \right. \right] \geq 1 - \varepsilon, \text{ a.s.}
\]

But on \( \{ \sigma_x < \infty \} \cap \{ \inf_{m \geq \sigma_x} X_m > x_1 \} \) we have \( \lim \inf_{m \to \infty} X_m \geq x_1 \), so

\[
P\left[ \lim \inf_{m \to \infty} X_m \geq x_1 \right] \geq \mathbb{E} \left[ P\left[ \inf_{m \geq \sigma_x} X_m > x_1 \left| \mathcal{F}_{\sigma_x} \right. \right] 1\{ \sigma_x < \infty \} \right]
\]

\[
\geq (1 - \varepsilon) P[\sigma_x < \infty].
\]

Under the conditions of part (i) of the theorem, \( P[\sigma_x < \infty] > 0 \), and so the claim in part (i) follows. On the other hand, under the conditions of part (ii) of the theorem, \( P[\sigma_x < \infty] = 1 \), and so, since \( \varepsilon > 0 \) was arbitrary, \( \lim \inf_{m \to \infty} X_m \geq x_1 \), a.s. Then since \( x_1 > x_0 \) was arbitrary, the result follows.

Proof of Theorem 3.5.1. We apply Theorem 3.5.6 with Lyapunov function \( f = f_{0,\nu} \) for \( \nu < 0 \) as defined at (3.16). Taking \( \gamma = 0 \) in (3.17) and using the fact that \( \lim \sup_{x \to \infty} \bar{\mu}_2(x) < \infty \), we see that

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n]
\]

\[
\leq \frac{\nu}{2} \left( 2X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] - \mathbb{E}[\Delta^2_n \mid \mathcal{F}_n] \right) X_n^{-2} \log^{\nu-1} X_n
\]

\[
+ \sigma_{\bar{\mu}_2}^2(X_n^{-2} \log^{\nu-1} X_n). \tag{3.34}
\]

The assumption (3.29) says that there exist \( x_1 \in \mathbb{R}_+ \) and \( \varepsilon_0 > 0 \) such that

\[
2X_n \bar{\mu}_1(X_n) - \bar{\mu}_2(X_n) \geq \varepsilon_0 \text{ on } \{ X_n > x_1 \}. \text{ Hence, by definition of } \bar{\mu} \text{ and } \mu,
\]

\[
2X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] - \mathbb{E}[\Delta^2_n \mid \mathcal{F}_n] \geq 2X_n \bar{\mu}_1(X_n) - \bar{\mu}_2(X_n) \geq \varepsilon_0,
\]

for all \( X_n \) sufficiently large. Since \( \nu < 0 \), it follows that the right-hand side of (3.34) is negative for all \( X_n \) sufficiently large. Thus the conditions of Theorem 3.5.6 are satisfied, and hence \( \lim_{n \to \infty} X_n = \infty \), a.s.

Now we move on to recurrence. First we state a semimartingale relative of Theorem 2.5.2.
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**Theorem 3.5.8.** Suppose that (L0) holds. Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be such that \( f(x) \to \infty \) as \( x \to \infty \) and \( \mathbb{E} f(X_0) < \infty \). Suppose that there exist \( x_0 \in \mathbb{R}_+ \) and \( C < \infty \) for which, for all \( n \geq 0 \),

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n > x_0\}, \text{ a.s.;}
\]

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq C, \text{ on } \{X_n \leq x_0\}, \text{ a.s.}
\]

Then

\[
P\left(\limsup_{n \to \infty} X_n < \infty \cup \liminf_{n \to \infty} X_n \leq x_0\right) = 1.
\]

In particular, if (L2) also holds, we have \( \liminf_{n \to \infty} X_n \leq x_0 \), a.s.

**Proof.** First note that, by hypothesis, \( \mathbb{E} f(X_1) \leq \mathbb{E} f(X_0) + C < \infty \), and, iterating this argument, it follows that \( \mathbb{E} f(X_n) < \infty \) for all \( n \geq 0 \).

Fix \( n \in \mathbb{Z}_+ \). For \( x_0 \in \mathbb{R}_+ \) in the hypothesis of the lemma, write \( \lambda = \min\{m \geq n : X_m \leq x_0\} \). Let \( Y_m = f(X_{m \wedge \lambda}) \). Then \( (Y_m, m \geq n) \) is a non-negative supermartingale. Hence, by Theorem 2.3.6, there exists a finite random variable \( Y_\infty \in \mathbb{R}_+ \) such that \( \lim_{m \to \infty} Y_m = Y_\infty \), a.s. In particular, this means that

\[
\limsup_{m \to \infty} f(X_m) \leq Y_\infty, \text{ on } \{\lambda = \infty\}.
\]

Setting \( \zeta = \sup\{x \geq 0 : f(x) \leq Y_\infty\} \), which satisfies \( \zeta < \infty \) a.s. since \( \lim_{x \to \infty} f(x) = \infty \), it follows that \( \limsup_{m \to \infty} X_m \leq \zeta \) on \( \{\lambda = \infty\} \). Hence

\[
P\left(\limsup_{n \to \infty} X_n < \infty \cup \left\{\inf_{m \geq n} X_m \leq x_0\right\}\right) = 1.
\]

Since \( n \in \mathbb{Z}_+ \) was arbitrary, it follows that

\[
P\left(\limsup_{n \to \infty} X_n < \infty \cup \bigcap_{n \geq 0} \left\{\inf_{m \geq n} X_m \leq x_0\right\}\right) = 1,
\]

which gives the result. \( \Box \)

We can now complete the proof of Theorem 3.5.2; again we use a Lyapunov function of the form \( f_{\gamma, \nu} \).

**Proof of Theorem 3.5.2.** We apply Theorem 3.5.8 with Lyapunov function \( f = f_{0, \nu} \) for \( \nu \in (0, 1) \) as defined at (3.16). Taking \( \gamma = 0 \) in (3.17), we have

\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] = \nu \frac{X_n}{2} (2X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] - \mathbb{E}[\Delta_n^2 \mid \mathcal{F}_n]) X_n^{-2} \log^{\nu-1} X_n
\]
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\[ + \left( \nu (\nu - 1) + \sigma_{X_n}^{F_n}(1) \right) \mathbb{E}[\Delta_n^2 \mid F_n] X_n^{-2} \log^{\nu-2} X_n \]
\[ + \sigma_{X_n}^{F_n}(X_n^{-2} \log^{\nu-2} X_n). \tag{3.35} \]

Here, since \( \nu > 0 \) and \( \nu - 1 < 0 \), we have that, for \( x_1 \in \mathbb{R}_+ \) sufficiently large,
\[ \nu(\nu - 1) + \sigma_{X_n}^{F_n}(1) \leq -\frac{1}{2} \nu(1 - \nu) \]
\text{on} \{X_n \geq x_1\}.

Hence, by definition of \( \bar{\mu}_1 \) and \( \bar{\mu}_2 \),
\[ \mathbb{E}[f(X_{n+1}) - f(X_n) \mid F_n] \leq \frac{\nu}{2} \left( 2X_n \bar{\mu}_1(X_n) - \bar{\mu}_2(X_n) \right) X_n^{-2} \log^{\nu-1} X_n \]
\[ - \frac{1}{2} \nu(1 - \nu) \bar{\mu}_2(X_n) X_n^{-2} \log^{\nu-2} X_n + \sigma_{X_n}^{F_n}(X_n^{-2} \log^{\nu-2} X_n). \]

Hence, by (3.30),
\[ \mathbb{E}[f(X_{n+1}) - f(X_n) \mid F_n] \leq \frac{\nu}{2} X_n^{-2} \log^{\nu-2} X_n \left( (\nu - \theta) \bar{\mu}_2(X_n) + \sigma_{X_n}^{F_n}(1) \right). \]

Choosing \( \nu \in (0, \theta) \), the fact that \( \liminf_{x \to \infty} \bar{\mu}_2(x) > 0 \) shows that the last display is negative for all \( X_n \) sufficiently large, i.e., for some \( r_0 \in \mathbb{R}_+ \),
\[ \mathbb{E}[f(X_{n+1}) - f(X_n) \mid F_n] \leq 0, \text{ on } \{X_n \geq r_0\}. \]

Hence we may apply Theorem 3.5.8 with Lyapunov function \( f \) to conclude that \( \liminf_{n \to \infty} X_n \leq r_0 \), a.s. \( \square \)

We end this section with a branching process example, which is instructive in two respects: how the above ideas may be applied when condition (L2) fails, and how a Lamperti-type problem can often be obtained via an appropriate scaling.

**Example 3.5.9.** Consider the following state-dependent branching process. Let \( \zeta_z, z \in \mathbb{N} \), be a family of random variables, representing the offspring distribution of an individual in a population of size \( z \). The total population size \( (Z_n, n \geq 0) \) is a Markov process on \( \mathbb{Z}_+ \) constructed as follows. Let \( Z_0 = z_0 \in \mathbb{N} \). Given \( Z_n = z \in \mathbb{Z}_+ \), \( Z_{n+1} \) is distributed as the sum of \( Z_n \) independent copies of \( \zeta_z \), as generation \( n \) is succeeded by generation \( n + 1 \).

For simplicity, suppose that \( \mathbb{P}[\zeta_z \leq B] = 1 \) for some \( B \in \mathbb{R}_+ \). Suppose also, to avoid degenerate cases, that \( \mathbb{P}[\zeta_z = 1] < 1 \). The Markov chain \( Z_n \) has an absorbing state at 0 (extinction), and there are two possibilities for
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the long-term behaviour: either \( Z_n \to 0 \), a.s., or else \( \mathbb{P}[Z_n \to \infty] > 0 \) and the process survives with positive probability.

The classical Galton–Watson process has \( \zeta_z = \zeta \) independent of \( z \), and a.s.-extinction occurs if and only if \( m = \mathbb{E} \zeta \leq 1 \). In the state-dependent setting, the case where \( \mathbb{E} \zeta_z \to 1 \) as \( z \to \infty \) is thus of particular interest. So from now on we suppose that the offspring distribution is given by

\[
\zeta_z = 1 + m_z + \theta_z,
\]

where, for a deterministic \( m_z \) and a random \( \theta_z \), \( m_z \sim c/z \), \( \mathbb{E} \theta_z = 0 \) and \( \mathbb{E} \theta_z^2 \to d \), for constants \( c \in \mathbb{R} \) and \( d > 0 \).

In this case, given \( Z_n = z \) we have

\[
D_n := Z_{n+1} - Z_n = \sum_{k=1}^{z} (m_z + \theta_{n,k}^z) = zm_z + \sum_{k=1}^{z} \theta_{n,k}^z,
\]

where \( \theta_{n,k}^z, n \geq 1, z \geq 1 \), are independent copies of \( \theta_z \), representing the offspring of individual \( k \) in generation \( n \). We compute directly,

\[
\mathbb{E}[D_n \mid Z_n = z] = c + o(1);
\]

\[
\mathbb{E}[D_n^2 \mid Z_n = z] = zd + o(z);
\]

\[
\mathbb{E}[D_n^4 \mid Z_n = z] = O(z^2).
\]

A useful change of scale is provided by \( f(z) = z^{1/2} \). Then, \( X_n = f(Z_n) \) is a Markov chain, and, given \( X_n = x \) (so \( Z_n = x^2 \)), by Taylor’s formula,

\[
X_{n+1} - X_n = \frac{D_n}{2x} - \frac{D_n^2}{8x^3} + O(|D_n|^3 x^{-5}).
\]

Here, by Lyapunov’s inequality,

\[
\mathbb{E}[|D_n|^3 \mid Z_n = x^2] \leq (\mathbb{E}[D_n^4 \mid Z_n = x^2])^{3/4} = O(x^3).
\]

It follows that, with the notation at (3.1),

\[
2x\mu_1(x) = c - \frac{d}{4} + o(1);
\]

\[
\mu_2(x) = \frac{d}{4} + o(1).
\]

In this example the condition (L2) does not hold, so we cannot apply Theorems 3.5.1 and 3.5.2 directly, but it is not hard to adapt the proofs to this setting (Theorems 3.5.6 and 3.5.8 are sufficient), and we see that
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- if $c > d/2$, then $P[X_n \to \infty] > 0$;
- if $c < d/2$, then $P[X_n \to 0] = 1$.

The boundary case cannot be classified without sharper assumptions on the rates of convergence of $m_z$ and $E[\theta^2_z]$.

3.6 Irreducibility and regeneration

To translate results from the general semimartingale setting into the Markovian setting, one needs to convert between the notions of recurrence and transience described in Section 3.5 and the usual recurrence and transience for Markov chains involving visits to finite sets. This section explores conditions under which the two descriptions are equivalent. In the Markov chain context, sufficient structure is provided by the (strong) Markov property and irreducibility, although even here there are some subtleties over the necessary nature of the state space. The most convenient general framework is in fact not to assume a full Markov structure but a weaker notion of regeneration.

For most of this section, $X_n$ will be an $\mathcal{F}_n$-adapted process on a countable state-space $S$, not assumed to be a subset of $\mathbb{R}^+$. The form of irreducibility that we will use is as follows.

(I) Suppose that for each $x, y \in S$ there exist constants $m(x, y) \in \mathbb{N}$ and $\varphi(x, y) > 0$, and a collection of $\mathcal{F}_n$-measurable random variables $M_n(y) \in \mathbb{Z}_+$, such that, for all $n \geq 0$, $M_n(y) \leq m(X_n, y)$, a.s., and

$$P[X_n + M_n(y) = y \mid \mathcal{F}_n] \geq \varphi(X_n, y), \text{ a.s.} \quad (3.36)$$

In words, assumption (I) says that for every $y$, given $\mathcal{F}_n$, the process visits $y$ with probability bounded below, in a number of steps bounded above, where the bounds depend on $X_n$ and $y$. A variant of (I) is the following.

(I') Suppose that for each $x, y \in S$ there exist constants $m(x, y) \in \mathbb{N}$ and $\varphi(x, y) > 0$ such that, for all $n \geq 0$,

$$P\left[\bigcup_{k=0}^{\rho(X_n,y)} \{X_{n+k} = y\} \mid \mathcal{F}_n\right] \geq \varphi(X_n, y), \text{ a.s.}$$

Proposition 3.6.1. Conditions (I) and (I') are equivalent.
Proof. Assuming (I), we have that
\[ \mathbb{P}\left[ \bigcup_{k=0}^{m(X_n,y)} \{ X_{n+k} = y \} \mid \mathcal{F}_n \right] \geq \mathbb{P}[X_{n+M_n(y)} = y, M_n(y) \leq m(X_n,y) \mid \mathcal{F}_n] \]
which is bounded below by \( \varphi(X_n,y) \), giving (I'). On the other hand, assuming (I'), take
\[ M_n(y) = \arg \max_{0 \leq k \leq m(X_n,y)} \mathbb{P}[X_{n+k} = y \mid \mathcal{F}_n], \]
which is \( \mathcal{F}_n \)-measurable and has \( M_n(y) \leq m(X_n,y) \), a.s. Then
\[ \mathbb{P}[X_{n+M_n(y)} = y \mid \mathcal{F}_n] = \max_{0 \leq k \leq m(X_n,y)} \mathbb{P}[X_{n+k} = y \mid \mathcal{F}_n] \]
\[ \geq \frac{1}{1 + m(X_n,y)} \mathbb{P}\left[ \bigcup_{k=0}^{m(X_n,y)} \{ X_{n+k} = y \} \mid \mathcal{F}_n \right] \]
\[ \geq \frac{\varphi(X_n,y)}{1 + m(X_n,y)} =: \varphi'(X_n,y), \]
with \( \varphi'(x,y) > 0 \) for all \( x, y \), which gives (I) after a relabelling. \( \square \)

If \( X_n \) is a time-homogeneous Markov process on a countable state-space \( \mathcal{S} \), then (3.36) reduces to the usual sense of irreducibility that, for any \( x, y \in \mathcal{S} \), there exists \( m(x,y) \in \mathbb{N} \) such that \( \mathbb{P}[X_{m(x,y)} = y \mid X_0 = x] > 0 \). The assumption (3.36) allows us to work with more general processes, such as functions of Markov process, as described in the next example.

**Example 3.6.2.** Suppose that \( \xi_n \) is an irreducible Markov chain on a countable state-space \( \Sigma \), and that for some measurable function \( f : \Sigma \to \mathbb{R}_+ \), \( X_n = f(\xi_n) \). Set \( \mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n) \). Then \( X_n \) is \( \mathcal{F}_n \)-adapted and has the countable state-space \( \mathcal{S} = f(\Sigma) \). By irreducibility, given \( u, v \in \Sigma \) there exist \( k(u,v) \in \mathbb{N} \) and \( \omega(u,v) > 0 \) such that \( \mathbb{P}[\xi_{n+k(u,v)} = v \mid \xi_n = u] \geq \omega(x,y) \). Fix \( y \in \mathcal{S} \); choose and fix an arbitrary \( v \in f^{-1}(y) := \{ u \in \Sigma : f(u) = y \} \).

For \( x \in \mathcal{S} \), let
\[ \varphi(x,y) = \inf_{u \in \Sigma : f(u) = x} \omega(u,v), \quad \text{and} \quad M_n(y) = k(\xi_n,v). \]

Then \( M_n(y) \) is \( \mathcal{F}_n \)-measurable, and
\[ \mathbb{P}[X_{n+M_n(y)} = y \mid \xi_n] \geq \mathbb{P}[\xi_{n+k(\xi_n,v)} = v \mid \xi_n] \geq \omega(\xi_n,v) \]
\[ \geq \varphi(X_n,y), \]
which is (3.36). Here \( M_n(y) \leq m(X_n,y) \) where
\[ m(x,y) = \sup_{u \in f^{-1}(x), v \in f^{-1}(y)} k(u,v). \]
Assuming that $f^{-1}(x)$ is finite for each $x$, we have $m(x, y) < \infty$ and $\varphi(x, y) > 0$. Hence (I) is satisfied for $X_n$ and this countable $S \subseteq \mathbb{R}_+$.

In particular, if $\Sigma \subseteq \mathbb{R}^d$ is locally finite and $f(x) = \|x\|$ for $x \in \Sigma$, then for $x \in S = f(\Sigma)$ we have $f^{-1}(x) = \{x \in \Sigma : \|x\| = x\}$ is finite, and so $X_n = \|\xi_n\|$ satisfies (I).

We first note the following basic result. Here ‘i.o.’ and ‘f.o.’ stand for ‘infinitely often’ and ‘finitely often’, respectively.

**Lemma 3.6.3.** Suppose that (I) holds for a countable $S$.

(i) Let $R \subset S$ be finite and non-empty. Then, up to sets of probability 0,

$$\{X_n \in R \text{ i.o.}\} \subseteq \cap_{S \subseteq S, 0 < \#S} \{X_n \in S \text{ i.o.}\}.$$

(ii) Let $R \subset S$ be non-empty. Then, up to sets of probability 0,

$$\{X_n \in R \text{ f.o.}\} \subseteq \cap_{S \subseteq S, 0 < \#S < \infty} \{X_n \in S \text{ f.o.}\}.$$

Hence, a.s., either $X_n \in R$ i.o. for all finite non-empty $R \subset S$, or for none.

**Proof.** First we prove part (i). Let $R \subset S$ be finite. Suppose that $X_n \in R$ i.o., and let $y \in S$. Then, since $R$ is finite and non-empty, we may choose $x \in R$ (any $x \in R$ will do) such that $X_n = x$ i.o.; by (3.36) there exist stopping times $n_1 < n_2 < \cdots$ with $n_{i+1} > n_i + m(x, y)$ such that $X_{n_i} = x$ and, for all $i$,

$$P[X_{n_i + M_{n_i}}(y) = y \mid F_{n_i}] \geq \varphi(x, y) > 0, \text{ and } \{X_{n_i + M_{n_i}}(y) = y\} \in F_{n_i+1},$$

since $M_{n_i}(y) \leq m(x, y)$. Then Lévy’s extension of the Borel–Cantelli lemma (Theorem 2.3.19) shows that $X_n = y$ i.o., a.s. So $X_n = y$ i.o. for all $y$ in the countable set $S$. In particular, $X_n \in S$ i.o. for any non-empty $S \subseteq S$, as claimed.

Next we prove part (ii). Let $R \subset S$ be non-empty. Suppose $X_n \in R$ f.o. If there exists $S$ a finite non-empty subset of $S$ for which $X_n \in S$ i.o., then part (i) would say that $X_n \in R$ i.o., a.s., giving a contradiction. So part (ii) follows.

The next result returns to the case $S \subseteq \mathbb{R}_+$, and relates the present notion of irreducibility to the earlier local escape condition for non-confinement given in Section 3.3, assuming that $S$ is locally finite.

**Lemma 3.6.4.** Suppose that (L0) and (I) hold for a locally finite $S \subset \mathbb{R}_+$. Then (L3) holds.
Proof. Since $S$ is locally finite, $0 = \inf S \in S$. Fix $x \in \mathbb{R}_+$; then, since $S$ is locally finite and contains $0$, $S \cap [0, x]$ is finite and non-empty. Choose $y \in S \cap (x, \infty)$. Take $\delta_x = \min_{z \in S \cap [0, x]} \phi(z, y) > 0$ and $r_x = \max_{z \in S \cap [0, x]} m(z, y) \in \mathbb{N}$, with $m$ and $\varphi$ as given by (3.36). Then, by (3.36), on $\{X_n \in S \cap [0, x]\}$,

$$
P\left[\max_{n \leq k \leq n + r_x} X_k \geq x \mid \mathcal{F}_n\right] \geq \mathbb{P}[X_{n\text{+}m\text{+}(y)} = y \mid \mathcal{F}_n] \geq \varphi(X_n, y) \geq \delta_x,$$

as required for (L3).

In particular, under the conditions of Lemma 3.6.4, Proposition 3.3.4 applies, so that the non-confinement condition (L2) holds. In this section we explore neighbouring results, including a direct proof of this implication, included in the next lemma.

**Lemma 3.6.5.** Suppose that (L0) and (I) hold for a countable $S \subset \mathbb{R}_+$. Then for any (hence every) finite, non-empty $R \subset S$, the following equality holds up to sets of probability 0:

$$\{X_n \in R \text{ i.o.}\} = \{\liminf_{n \to \infty} X_n = 0, \limsup_{n \to \infty} X_n = \infty\}. \tag{3.37}$$

If, in addition, $S$ is locally finite, then for any (hence every) finite, non-empty $R \subset S$, the following equality holds up to sets of probability 0:

$$\{X_n \in R \text{ f.o.}\} = \{\lim_{n \to \infty} X_n = \infty\}. \tag{3.38}$$

In particular, if $S$ is locally finite, then

$$\mathbb{P}\left[\{X_n \to \infty\} \cup \{\liminf_{n \to \infty} X_n = 0, \limsup_{n \to \infty} X_n = \infty\}\right] = 1.$$

Proof. Suppose that $S$ is countable. By Lemma 3.6.3, there are two possibilities, up to events of probability 0: either (i) $X_n \in R$ i.o. for any non-empty $R \subset S$; or (ii) $X_n \in R$ f.o. for any finite non-empty $R \subset S$.

First suppose case (i) occurs. Then, for any $x \in S$, $X_n = x$ i.o., so that $\liminf_{n \to \infty} X_n \leq x \leq \limsup_{n \to \infty} X_n$; since $x \in S$ was arbitrary, together with the obvious bound $X_n \geq 0$, we obtain

$$\liminf_{n \to \infty} X_n = \inf_S = 0, \text{ and } \limsup_{n \to \infty} X_n = \sup_S = \infty.$$ 

This establishes (3.37).
On the other hand, suppose case (ii) occurs, and that $S$ is locally finite; in particular $\inf S = 0 \in S$ in this case. Now, for any $K \in \mathbb{R}_+$, $R_K = S \cap [0, K]$ is finite and non-empty, and so $X_n \in R_K$ f.o. Hence
\[
\liminf_{n \to \infty} X_n \geq K.
\]
Since $K$ was arbitrary, it follows that $\lim_{n \to \infty} X_n = \infty$. This establishes (3.38). Combining (3.37) and (3.38) we deduce the final statement in the lemma.

The next ingredient will be a notion of regeneration, for which we need to introduce notation for the excursions of $X_n$. Consider a general countable state-space $S$, with an identified state $0 \in S$. For convenience, we will assume $X_0 = 0$; we consider excursions away from state 0.

Set $\nu_0 := 0$ and for $k \in \mathbb{N}$ define
\[
\nu_k := \min\{n > \nu_{k-1} : X_n = 0\},
\]
with the usual convention that $\min \emptyset = \infty$. That is, $\nu_0, \nu_1, \nu_2, \ldots$ are the successive times of visit to 0 by $X_n$. Let $N := \min\{k \in \mathbb{N} : \nu_k = \infty\}$; if $N = \infty$ then all the $\nu_k$ are finite and $X_n$ visits 0 infinitely often. Provided $\nu_{k-1} < \infty$, we denote the $k$th excursion ($k \in \mathbb{N}$) by $E_k := (X_n)_{\nu_{k-1} \leq n < \nu_k}$.

For $k \in \mathbb{N}$, let $\kappa_k := \nu_k - \nu_{k-1}$ if $\nu_{k-1} < \infty$ and let $\kappa_k = \infty$ if $\nu_{k-1} = \infty$; when $\nu_{k-1}$ is finite, $\kappa_k$ is the duration of the $k$th excursion.

(Ra) Suppose that, for all $k \in \mathbb{N}$, the distribution of $E_k$ on $\{\nu_{k-1} < \infty\}$ is the same as the distribution of $E_1$.

(Rb) Suppose that, on $\{N = \infty\}$, $(E_k)_{k \in \mathbb{N}}$ is an i.i.d. sequence.

(R) Suppose that $X_0 = 0$ and both (Ra) and (Rb) hold.

Example 3.6.6. In the setting of Example 3.6.2, suppose that $S = f(\Sigma)$ has $0 \in S$. If $f^{-1}(0)$ is a singleton, then the strong Markov property for the underlying Markov chain $\xi_n$ yields the regenerative structure required for (R).

Our irreducibility and regenerative assumptions have the following basic consequence, which includes the recurrence/transience dichotomy in this setting.

Lemma 3.6.7. Suppose that (I) and (Ra) hold for a countable $S$ with $0 \in S$. Then either:
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(i) \( \mathbb{P}[\kappa_1 < \infty] < 1 \) and \( \mathbb{P}[X_n \in R \text{ f.o.}] = 1 \) for every finite, non-empty \( R \subset \mathcal{S} \); or

(ii) \( \mathbb{P}[\kappa_1 < \infty] = 1 \) and \( \mathbb{P}[X_n \in R \text{ i.o.}] = 1 \) for every non-empty \( R \subset \mathcal{S} \).

Proof. A consequence of (Ra) is that
\[
\mathbb{P}[\kappa_k = n | \nu_{k-1} < \infty] = \mathbb{P}[\kappa_1 = n], \quad \text{for all } n \in \mathbb{Z}_+.
\] (3.39)

It follows from (3.39), with a repeated conditioning argument, that
\[
\mathbb{P}[N > k] = \mathbb{P}[\kappa_k < \infty, \nu_{k-1} < \infty, \ldots, \nu_1 < \infty]
\]
\[
= \prod_{j=1}^k \mathbb{P}[\kappa_j < \infty | \nu_{j-1} < \infty] = (\mathbb{P}[\kappa_1 < \infty])^k.
\]

If \( \mathbb{P}[\kappa_1 < \infty] < 1 \), this implies that \( N < \infty \) a.s., so that \( X_n = 0 \) f.o., a.s. The statement in part (i) follows from Lemma 3.6.3(ii). On the other hand, if \( \mathbb{P}[\kappa_1 < \infty] = 1 \) we have that \( \mathbb{P}[N > k] = 1 \) for any \( k \), so \( N = \infty \) a.s. and hence \( X_n = 0 \) i.o., and the statement in part (ii) follows from Lemma 3.6.3(i). \( \square \)

In the case where \( \mathcal{S} \subset \mathbb{R}_+ \) is locally finite, the recurrence/transience dichotomy reads as follows.

**Theorem 3.6.8.** Suppose that (L0) and (I) hold for a locally finite \( \mathcal{S} \subset \mathbb{R}_+ \). Then \( \limsup_{n \to \infty} X_n = +\infty \), a.s. If (Ra) also holds, then either:

(i) (transience) \( \mathbb{P}[\kappa_1 < \infty] < 1 \) and \( \lim_{n \to \infty} X_n = +\infty \), a.s.; or

(ii) (recurrence) \( \mathbb{P}[\kappa_1 < \infty] = 1 \) and \( \liminf_{n \to \infty} X_n = 0 \), a.s.

**Proof.** Lemma 3.6.5 shows that \( \limsup_{n \to \infty} X_n = +\infty \), and also shows that the statements (i) and (ii) in Lemma 3.6.7 translate into statements (i) and (ii) in the theorem, in the case where \( \mathcal{S} \subset \mathbb{R}_+ \) is locally finite. \( \square \)

Now we can also complete the proof of Theorem 2.1.9 (a close relative of Theorem 3.6.8) that was deferred from Chapter 2.

**Proof of Theorem 2.1.9.** Let \( \xi_n \) be an irreducible time-homogeneous Markov chain on \( \Sigma \subset \mathbb{R}^d \), unbounded and countable. Write \( r = \inf_{y \in \Sigma} \|y\| \) and set \( X_n = \|\xi_n\| - r \) and \( \mathcal{F}_n = \sigma(\xi_0, \xi_1, \ldots, \xi_n) \). Then (L0) and (I) hold, with \( \mathcal{S} = \{\|y\| - r : y \in \Sigma\} \) countable. The result now follows from Lemma 3.6.5, since recurrence of the Markov chain \( \xi_n \) corresponds to (3.37) holding with probability 1, while transience of \( \xi_n \) corresponds to (3.38) holding with probability 1. \( \square \)
The concepts of irreducibility and regeneration also play an important role in the discussion of positive- and null-recurrence. For a countable state-space $S$, define for $x \in S$ the occupation time (also called local time)

$$L_n(x) := \sum_{m=0}^{n} 1\{X_m = x\}, \text{ for } n \in \mathbb{N}.$$ 

Also define the occupation times during the $k$th excursion for $x \in S$ by

$$\ell_k(x) := \sum_{m=\nu_k-1}^{\nu_k-1} 1\{X_m = x\}, \text{ for } k \in \mathbb{N}.$$ 

The following ergodic theorem shows that our regeneration and irreducibility assumptions imply that normalized occupation times converge; if the limit is zero, the process is null, otherwise it is positive.

**Theorem 3.6.9.** Suppose that (I) and (R) hold for a countable $S$ with $0 \in S$. Then, for any $x \in S$,

$$\lim_{n \to \infty} \frac{1}{n} L_n(x) = \pi_x := \begin{cases} \frac{\mathbb{E}\ell_1(x)}{\mathbb{E}\kappa_1} & \text{if } \mathbb{E}\kappa_1 < \infty; \\ 0 & \text{if } \mathbb{E}\kappa_1 = \infty; \end{cases}$$ 

the limit statement holding a.s. and in $L^q$ for any $q \geq 1$. In the case $\mathbb{E}\kappa_1 < \infty$, we have $\sum_{x \in S} \pi_x = 1$.

If $X_n$ is a positive-recurrent, irreducible, time-homogeneous Markov process, then the $\pi_x$ in Theorem 3.6.9 is the usual (unique) stationary distribution; Theorem 3.6.9 thus extends (the Cesàro version of) Theorem 2.1.6.

The proof of Theorem 3.6.9 will be accomplished by a sequence of lemmas. The first simple lemma will also be used frequently later on. Recall that $\tau_y = \min\{n \geq 0 : X_n = y\}$.

**Lemma 3.6.10.** Suppose that (I) and (Ra) hold for a countable $S$ with $0 \in S$. Then $\mathbb{P}[\kappa_1 > 1] > 0$, and, for any $y \in S \setminus \{0\}$, there exists $c(y) > 0$ so that

$$\mathbb{P}[\tau_y < \kappa_1 \mid \mathcal{F}_0] = c(y), \text{ a.s.}$$

**Proof.** First suppose, for the purpose of deriving a contradiction, that $\mathbb{P}[\kappa_1 = 1] = 1$. Then it follows from an induction on (3.39) that $\mathbb{P}[\kappa_k = 1] = 1$ for all $k$, which implies that $X_n = 0$ for all $n$, a.s. But this contradicts Lemma 3.6.3. Hence $\mathbb{P}[\kappa_1 > 1] > 0$. 

Next the irreducibility assumption (3.36) implies that for any $k$,
\[
P[X_{\nu_k} + M_{\nu_k}(y) = y \mid F_{\nu_k}] \geq \varphi(0, y), \text{ on } \{\nu_k < \infty\}. \tag{3.40}
\]
By the regenerative assumption (Ra), $P$ [hit $y$ before returning to $0 \mid F_{\nu_k}$] is constant on $\{\nu_k < \infty\}$; call this probability $c(y)$. Then, an upper bound on the probability that $X_n$ ever visits $y$ is
\[
P\left[\bigcup_{k=0}^{\infty} (\{\nu_k < \infty\} \cap \{\text{hit } y \text{ between } \nu_k \text{ and } \nu_{k+1}\})]\leq \sum_{k=0}^{\infty} c(y).
\]
Thus if $c(y) = 0$, the probability of eventually hitting $y$ is also 0, which contradicts (3.40). Hence $c(y) > 0$. \hspace{1cm} \square

Lemma 3.6.10 shows that any $x \in S \setminus \{0\}$ is visited at least once during an excursion with positive probability. The next key result shows that the expected number of visits is finite.

**Lemma 3.6.11.** Suppose that (I) and (R) hold for a countable $S$ with $0 \in S$. Suppose that $P[k_1 < \infty] = 1$. Then, for any $x \in S$, $\ell_k(x)$, $k \geq 1$, are i.i.d. with $E\ell_1(x) \in (0, \infty)$.

**Proof.** Assuming $P[k_1 < \infty] = 1$, the regenerative assumption (R) shows that $X_n = 0$ i.o., and hence Lemma 3.6.3 shows that $X_n = x$ i.o. also for any $x \in S$. Also, (R) shows that the $\ell_k(x)$ are i.i.d.

The fact that $E\ell_1(x) < \infty$ will follow from irreducibility. Indeed, fix $x \in S$. Write $m = m(x, 0)$ and $\varphi = \varphi(x, 0)$, the constants given by (I). Set $\tau_1 = \min\{n \geq 0 : X_n = x\}$ and for $k \geq 2$ set $\tau_k = \min\{n > \tau_{k-1} + m : X_n = x\}$, so that $\tau_1, \tau_2, \ldots$ are a subsequence of the times of successive visits to $x$. (Note that $\tau_k < \infty$ for all $k$, since $X_n = x$ i.o.) Set
\[
J_x = \min\{k \geq 1 : X_{\tau_k + M_{\tau_k}(0)} = 0\}.
\]
Recall that $M_{\tau_k}(0) \leq m$ a.s. Then $X_n$ must visit $0$ by time $\tau_{J_x} + m$, and by that time $X_n$ has visited $x$ at most $(m + 1)J_x$ times, since for each visit to $x$ recorded as some $\tau_k$, other (unrecorded) visits may occur anywhere in the interval $[\tau_k, \tau_k + m]$. Hence $\ell_1(x) \leq (m + 1)J_x$, a.s. However, by (I) applied at time $\tau_1$,
\[
P[J_x \geq 2 \mid \mathcal{F}_{\tau_1}] \leq P[X_{\tau_1 + M_{\tau_1}(0)} \neq 0 \mid \mathcal{F}_{\tau_1}] \leq 1 - \varphi, \text{ a.s.,}
\]
and, since $\{J_x \geq 2\} \in \mathcal{F}_{\tau_2}$ by the fact that $\tau_1 + M_{\tau_1}(0) \leq \tau_2$, we have
\[
P[J_x \geq 3 \mid \mathcal{F}_{\tau_1}] = E[1\{J_x \geq 2\} P[J_x \geq 3 \mid \mathcal{F}_{\tau_2}] \mid \mathcal{F}_{\tau_1}]
\]
by (I) applied at time $\tau_2$. Iterating this argument gives $\mathbb{P}[J_x > k] \leq (1 - \varphi)^k$ for any $k \geq 0$, so in particular $\mathbb{E} \ell_1(x) \leq (m + 1) \mathbb{E} J_x < \infty$.

Finally, we have $\mathbb{E} \ell_1(x) > 0$ since $\ell_1(0) \geq 1$, a.s., and, for $x \in \mathcal{S} \setminus \{0\}$, $\mathbb{E} \ell_1(x) \geq \mathbb{P}[\tau_x < \kappa_1]$, which is positive by Lemma 3.6.10.

Let $N_n := \max\{k \geq 1 : \nu_k \leq n\}$ denote the number of completed excursions by time $n \geq 0$.

**Lemma 3.6.12.** Suppose that (I) and (R) hold for a countable $\mathcal{S}$ with $0 \in \mathcal{S}$. Then, a.s.,

$$\lim_{n \to \infty} \frac{1}{n} N_n = \begin{cases} \frac{1}{\mathbb{E} \kappa_1} \in (0, 1) & \text{if } \mathbb{E} \kappa_1 < \infty; \\ 0 & \text{if } \mathbb{E} \kappa_1 = \infty. \end{cases}$$

**Proof.** If $\mathbb{P}[\kappa_1 = \infty] > 0$, then $\lim_{n \to \infty} N_n = N - 1 < \infty$, a.s., and $n^{-1} N_n \to 0$, trivially. So suppose that $\mathbb{P}[\kappa_1 < \infty] = 1$.

The result is essentially an inversion of the strong law of large numbers for $\nu_k = \nu_k - \nu_0 = \sum_{k=1}^k \kappa_\ell$, where $\kappa_1, \kappa_2, \ldots$ are i.i.d., by (R).

Suppose first that $\mathbb{E} \kappa_1 < \infty$, then, since $\mathbb{P}[\kappa_1 > 1] > 0$ (see Lemma 3.6.10) this means $\mathbb{E} \kappa_1 \in (1, \infty)$. Then the strong law of large numbers shows that $k^{-1} \nu_k \to \mathbb{E} \kappa_1$, a.s. Hence for any $\varepsilon \in (0, \mathbb{E} \kappa_1)$, there exists an a.s.-finite $K_\varepsilon$ such that $\nu_k \geq k(\mathbb{E} \kappa_1 - \varepsilon)$ for all $k \geq K_\varepsilon$. Then if $k \in \mathbb{N}$ is such that $k(\mathbb{E} \kappa_1 - \varepsilon) > n$ for an integer $n$ with $n > K_\varepsilon(\mathbb{E} \kappa_1 - \varepsilon)$, we have $\nu_k > n$; hence $N_n \leq \frac{n}{\mathbb{E} \kappa_1 - \varepsilon}$ for all $n$ sufficiently large. Since $\varepsilon > 0$ was arbitrary, we obtain

$$\limsup_{n \to \infty} n^{-1} N_n \leq \frac{1}{\mathbb{E} \kappa_1} \in (0, 1).$$

A similar argument in the other direction gives the corresponding lim inf statement.

On the other hand, suppose that $\mathbb{E} \kappa_1 = \infty$; now the strong law implies that $k^{-1} \nu_k \to \infty$, a.s. So, for any $\varepsilon > 0$, $\nu_k > k/\varepsilon$ for all $k$ sufficiently large, which yields $\limsup_{n \to \infty} n^{-1} N_n \leq \varepsilon$ by a similar argument to above. Since $\varepsilon > 0$ was arbitrary, the result follows.

Now we can complete the proof of Theorem 3.6.9.

**Proof of Theorem 3.6.9.** Suppose that $\mathbb{P}[\kappa_1 = \infty] > 0$. Then $\lim_{n \to \infty} L_n(x) = \sum_{m=0}^\infty 1\{X_m = x\} < \infty$, a.s., by Lemma 3.6.7, so that $n^{-1} L_n(x) \to 0$. 

So suppose that $\mathbb{P}[\kappa_1 < \infty] = 1$. Lemma 3.6.11 and the strong law of large numbers shows that

$$
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} \ell_k(x) = \mathbb{E} \ell_1(x), \text{ a.s.} \quad (3.41)
$$

First note that, by definition of $N_n$ and $\ell_k(x)$, a.s.,

$$
\sum_{k=1}^{N_n} \ell_k(x) \leq L_n(x) = \sum_{k=1}^{N_n} \ell_k(x) + \sum_{m=\nu N_n}^{n} \mathbf{1}\{X_m = x\} \leq \sum_{k=1}^{N_n+1} \ell_k(x), \quad (3.42)
$$

since $\nu N_n \leq n < \nu N_n + 1$. In particular, the upper bound in (3.42) gives

$$
\frac{1}{n} L_n(x) \leq \frac{N_n + 1}{n} \cdot \frac{1}{N_n + 1} \sum_{k=1}^{N_n+1} \ell_k(x).
$$

Hence, since $N_n \to \infty$ a.s., it follows from (3.41) that

$$
\limsup_{n \to \infty} \frac{1}{n} L_n(x) \leq \mathbb{E} \ell_1(x) \limsup_{n \to \infty} \frac{N_n + 1}{n}, \text{ a.s.} \quad (3.43)
$$

If $\mathbb{E} \kappa_1 = \infty$, Lemma 3.6.12 shows that the limit on the right-hand side of (3.43) is zero, so that $n^{-1} L_n(x) \to 0$, a.s. Suppose now that $\mathbb{E} \kappa_1 < \infty$. In this case, (3.43) with Lemma 3.6.12 shows that

$$
\limsup_{n \to \infty} \frac{1}{n} L_n(x) \leq \frac{\mathbb{E} \ell_1(x)}{\mathbb{E} \kappa_1}, \text{ a.s.}
$$

In the other direction, the lower bound in (3.42) gives

$$
\frac{1}{n} L_n(x) \geq \frac{N_n}{n} \cdot \frac{1}{N_n} \sum_{k=1}^{N_n} \ell_k(x),
$$

so that by Lemma 3.6.12 and (3.41),

$$
\liminf_{n \to \infty} \frac{1}{n} L_n(x) \geq \frac{\mathbb{E} \ell_1(x)}{\mathbb{E} \kappa_1}, \text{ a.s.}
$$

This completes the proof.

To end this section, we state a technical lemma on almost-sure bounds for sums and maxima of i.i.d. random variables; under the regenerative assumption, these results will be useful for our analysis of excursions of Lamperti-type processes presented later in this chapter.
Lemma 3.6.13. Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. $\mathbb{R}_+$-valued random variables.

(i) Suppose that, for some $\theta \in (0, 1)$ and $\varphi \in \mathbb{R},$
\[
\limsup_{x \to \infty} x^\theta (\log x)^{-\varphi} \mathbb{P}[\zeta_1 \geq x] < \infty. \tag{3.44}
\]
Then, for any $\varepsilon > 0$, a.s., for all but finitely many $n$,
\[
\sum_{i=1}^{n} \zeta_i \leq n^{\frac{1}{\theta}} (\log n)^{\frac{\varphi+1}{\theta} + \varepsilon}.
\]

(ii) Suppose that, for some $\theta \in (0, \infty)$ and $\varphi \in \mathbb{R},$
\[
\liminf_{x \to \infty} x^\theta (\log x)^{-\varphi} \mathbb{P}[\zeta_1 \geq x] > 0. \tag{3.45}
\]
Then, for any $\varepsilon > 0$, a.s., for all but finitely many $n$,
\[
\max_{1 \leq i \leq n} \zeta_i \geq n^{\frac{1}{\theta}} (\log n)^{\frac{\varphi-1}{\theta} - \varepsilon}.
\]

Proof. Part (i) belongs to a family of classical results related to the strong laws of large numbers of Marcinkiewicz and Zygmund (see e.g. [150, p. 73]): it follows from Theorem 2 of Feller [97] (see also [200, p. 253] for a more general result).

For part (ii), we have from (3.45) that for some $c > 0$ and all $x$ large enough, $\mathbb{P}[\zeta_1 \geq x] \geq c x^{-\theta} (\log x)^{\varphi}$, so that
\[
\mathbb{P} \left[ \max_{1 \leq i \leq n} \zeta_i < x \right] \leq \left( 1 - c x^{-\theta} (\log x)^{\varphi} \right)^n.
\]
Taking $x = n^{1/\theta} (\log n)^q$ we obtain
\[
\mathbb{P} \left[ \max_{1 \leq i \leq n} \zeta_i < n^{1/\theta} (\log n)^q \right] \leq \left( 1 - cn^{-1} (\log n)^{\varphi-\theta q (1 + o(1))} \right)^n = O \left( \exp \left( -c (\log n)^{\varphi-\theta q (1 + o(1))} \right) \right),
\]
which is summable over $n \geq 2$ if $q < (\varphi - 1)/\theta$; now use the Borel–Cantelli lemma. \qed
3.7 Moments and tails of passage times

Recall that for \( x \geq 0 \), \( \lambda_x = \min\{n \geq 0 : X_n \leq x\} \). In the recurrent cases identified in Section 3.5, we have that \( \lambda_x < \infty \) a.s., at least for all \( x \) sufficiently large. To quantify recurrence, it is natural to ask for more information about these hitting times, such as tail or moments bounds. In this section we use the general results from Section 2.7 to investigate this question. We start with a result on existence of moments.

**Theorem 3.7.1.** Suppose that (L0) holds. Suppose that, for some \( \alpha > 0 \), (L1) holds with \( p > \max\{2, 2\alpha\} \), and that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). Suppose also that one of the following two conditions holds.

(i) We have \( \alpha \in (0, \frac{1}{2}] \) and \( \limsup_{x \to \infty} \left( 2x\bar{\mu}_1(x) - (1 - 2\alpha)\mu_2(x) \right) < 0 \).

(ii) We have \( \alpha \in (\frac{1}{2}, \infty) \) and \( \limsup_{x \to \infty} \left( 2x\bar{\mu}_1(x) + (2\alpha - 1)\mu_2(x) \right) < 0 \).

Then there exists \( \gamma_0 \in (2\alpha, p) \) and \( x_0 \in \mathbb{R}_+ \) for which, for any \( \gamma \in (2\alpha, \gamma_0) \), any \( s \in (0, \gamma/2) \), and any \( x \geq x_0 \), \( \mathbb{E}[\lambda^s_x | \mathcal{F}_0] \leq CX_0^\gamma \), where \( C = C(\gamma, s) < \infty \). In particular, for any \( x \geq x_0 \), \( \mathbb{E}[\lambda^x_x] < \infty \).

Now we turn to non-existence of moments. The next result also gives a lower tail bound.

**Theorem 3.7.2.** Suppose that (L0) and (L2) hold. Suppose that, for some \( \beta > 0 \), (L1) holds with \( p > \max\{2, 2\beta\} \), and that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). Suppose also that one of the following two conditions holds.

(i) We have \( \beta \in (0, \frac{1}{2}] \) and \( \liminf_{x \to \infty} \left( 2x\mu_1(x) - (1 - 2\beta)\mu_2(x) \right) > 0 \).

(ii) We have \( \beta \in (\frac{1}{2}, \infty) \) and \( \liminf_{x \to \infty} \left( 2x\mu_1(x) + (2\beta - 1)\mu_2(x) \right) > 0 \).

Then there exists \( x_0 \in \mathbb{R}_+ \) such that for any \( x \geq x_0 \), \( \mathbb{E}[\lambda^x_x | \mathcal{F}_0] = \infty \) on \( \{X_0 > x\} \). Moreover, for any \( x_1 > x_0 \) and any \( x < x_1 \), there is a constant \( c = c(x_0, x_1, x, \beta) > 0 \) such that, for all \( n \) sufficiently large,

\[
\mathbb{P}[\lambda_x \geq n | \mathcal{F}_0] \geq cn^{-\beta}, \text{ on } \{X_0 = x_1\}.
\]

The following more specific result gives a sufficient condition to ensure that \( \mathbb{E}\lambda_x = \infty \) which is sharper than the \( \beta = 1 \) case of Theorem 3.7.2.

**Theorem 3.7.3.** Suppose that (L0) and (L2) hold. Suppose that (L1) holds with \( p > 2 \), and that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). Suppose also that there exist...
3.7. Moments and tails of passage times

$x_0 \in (0, \infty)$, $v \in (0, \infty)$, and $\theta \in (0, 1)$ such that for all $x \geq x_0$, $\mu(x) \geq v$ and

$$2x\mu(x) + \left( 1 + \frac{1-\theta}{\log x} \right) \mu(x) \geq 0.$$ 

Then there exists $x_0 \in \mathbb{R}_+$ such that, for any $x \leq x_0$, $E[\lambda_x | F_0] = \infty$ on $\{X_0 > x\}$. Moreover, for any $\nu > 1 - \theta$ there exists $x_0 \in \mathbb{R}_+$ such that, for any $x_1 > x_0$ and any $x < x_1$, there is a constant $c = c(x_0, x_1, x, \nu, \theta) > 0$ such that, for all $n$ sufficiently large,

$$\mathbb{P}[\lambda_x \geq n | F_0] \geq cn^{-1}(\log n)^{-\nu}, \text{ on } \{X_0 = x_1\}.$$ 

Note that Theorem 3.7.3 is a considerably stronger result than one can obtain from Theorem 2.6.10; one would like to apply the latter result to $X_n$ satisfying the conditions of Theorem 3.7.3, but the moments condition in Theorem 2.6.10 will not be satisfied.

Example 3.7.4. As in Example 3.5.4, let $(\xi_n, n \geq 0)$ be a time-homogeneous Markov chain on state space $\Sigma \subseteq \mathbb{R}^2$ with increments $\theta_n := \xi_{n+1} - \xi_n$ such that $\|\theta_n\| \leq B$, a.s., for some $B \in \mathbb{R}_+$, and, for all $x \in \Sigma$,

$$\mathbb{E}[\theta_n | \xi_n = x] = 0, \text{ and } \mathbb{E}[\theta_n^\top | \xi_n = x] = I_2,$$

the 2 by 2 identity matrix. In Example 3.5.4, we saw that $\xi_n$ is recurrent. Moreover, for $X_n = \|\xi_n\|$, 

$$\liminf_{x \to \infty} \left( 2x\mu(x) - (1 - 2\beta)\bar{\mu}(x) \right) = 1 - (1 - 2\beta) > 0,$$

for any $\beta > 0$, so Theorem 3.7.2 shows that, for suitable $x_1 > 0$, $E[\lambda^\beta_x] = \infty$ for any $\|\xi_0\| > x_1 > x$. Example 2.7.9 gave a version of this conclusion for symmetric SRW on $\mathbb{Z}^2$.

The rest of this section is devoted to the proofs of the preceding theorems. The first step is to apply the results of Section 2.7 with suitable Lyapunov functions. First we work towards a proof of Theorem 3.7.1.

Lemma 3.7.5. Suppose that (L0) holds. Suppose that, for some $\alpha > 0$, (L1) holds with $p \geq \max\{2, 2\alpha\}$, and that $\limsup_{x \to \infty} \bar{\mu}(x) < \infty$. Suppose also that one of the following two conditions holds.

(i) We have $\alpha \in (0, \frac{1}{2}]$ and $\limsup_{x \to \infty} \left( 2x\bar{\mu}(x) - (1 - 2\alpha)\bar{\mu}(x) \right) < 0.$

(ii) We have $\alpha \in (\frac{1}{2}, \infty)$ and $\limsup_{x \to \infty} \left( 2x\bar{\mu}(x) + (2\alpha - 1)\bar{\mu}(x) \right) < 0.$
There exist \( \gamma_0 \in (2\alpha, p) \), \( x_0 \in \mathbb{R}_+ \), and \( \varepsilon_0 > 0 \) such that, for any \( \gamma \in (2\alpha, \gamma_0) \),
\[
\mathbb{E}[X_{n+1}^\gamma - X_n^\gamma \mid F_n] \leq -\varepsilon_0 X_n^{\gamma-2}, \quad \text{on } \{ X_n \geq x_0 \}.
\]

**Proof.** In this case, for any \( \gamma \in (0, p) \), (3.18) gives
\[
\mathbb{E}[X_{n+1}^\gamma - X_n^\gamma \mid F_n] = \frac{\gamma}{2} (2X_n \mathbb{E}[\Delta_n \mid F_n] + (\gamma - 1) \mathbb{E}[\Delta_n^2 \mid F_n]) X_n^{-2} + o_{X_n}(X_n^{-2}),
\]
using the fact that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). First suppose condition (i) holds, so that for any \( \varepsilon_1 > 0 \) sufficiently small there exists \( x_1 \in \mathbb{R}_+ \) such that \( 2x\bar{\mu}_1(x) - (1 - 2\alpha)\bar{\mu}_2(x) < -\varepsilon_1 \) for all \( x \geq x_1 \). Take \( \gamma_0 = 2\alpha + \varepsilon_2 \in (2\alpha, p) \). Then, for \( \gamma \in (2\alpha, \gamma_0) \),
\[
2X_n \mathbb{E}[\Delta_n \mid F_n] + (\gamma - 1) \mathbb{E}[\Delta_n^2 \mid F_n] \\
\leq 2X_n\bar{\mu}_1(X_n) - (1 - 2\alpha)\bar{\mu}_2(X_n) + \varepsilon_2\bar{\mu}_2(X_n) \\
\leq -\varepsilon_1 + \varepsilon_2\bar{\mu}_2(X_n),
\]
on \( \{ X_n \geq x_1 \} \). Since \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \), we may choose \( \varepsilon_2 > 0 \) small enough so that this last display is less than \(-\varepsilon_1/2\), say, on \( \{ X_n \geq x_1 \} \), uniformly for \( \gamma \in (2\alpha, \gamma_0) \). Hence
\[
\mathbb{E}[X_{n+1}^\gamma - X_n^\gamma \mid F_n] \leq -\frac{\varepsilon_1}{4}\gamma X_n^{\gamma-2} + o_{X_n}(X_n^{\gamma-2}),
\]
on \( \{ X_n \geq x_1 \} \). Since \( \gamma \) is bounded uniformly away from 0, there exist \( \varepsilon_0 > 0 \) and \( x_0 < \infty \), not depending on \( \gamma \in (2\alpha, \gamma_0) \), such that
\[
\mathbb{E}[X_{n+1}^\gamma - X_n^\gamma \mid F_n] \leq -\varepsilon_0 X_n^{\gamma-2}, \quad \text{on } \{ X_n \geq x_0 \}.
\]
A similar argument applies if condition (ii) holds.

**Proof of Theorem 3.7.1.** Let \( \gamma_0 \), \( x_0 \), and \( \varepsilon_0 \) be the constants in Lemma 3.7.5, and let \( \gamma \in (2\alpha, \gamma_0) \). Then Lemma 3.7.5 shows that we may apply Theorem 2.7.1 with \( s_0 = \gamma/2 \) to conclude that \( \mathbb{E}[\lambda_0^s \mid F_0] \leq CX_0^\gamma \) for any \( s < \gamma/2 \) and any \( x \geq x_0 \). In particular, since \( \gamma > 2\alpha \), we may take \( s = \alpha \) to conclude that \( \mathbb{E}[\lambda_0^\alpha] < \infty \).

Next we turn to the non-existence of moments results. We will obtain Theorems 3.7.2 and 3.7.3 from applications of Lemma 2.7.7. Thus we need lower bounds on the extent of an excursion away from a bounded set. The following result is required for Theorems 3.7.2, and will also be useful in Section 3.8 below.
Lemma 3.7.6. Suppose that (L0) and (L2) hold. Suppose that, for some $\beta > 0$, (L1) holds with $p > \max\{2, 2\beta\}$, and that $\limsup_{x \to \infty} \tilde{\mu}_2(x) < \infty$. Suppose also that one of the following two conditions holds.

(i) We have $\beta \in (0, \frac{1}{2}]$ and $\liminf_{x \to \infty} (2x \tilde{\mu}_1(x) - (1 - 2\beta) \tilde{\mu}_2(x)) > 0$.

(ii) We have $\beta \in (\frac{1}{2}, \infty)$ and $\liminf_{x \to \infty} (2x \tilde{\mu}_1(x) + (2\beta - 1) \tilde{\mu}_2(x)) > 0$.

Then there exist $x_0 \in \mathbb{R}_+$ and a constant $c = c(\beta) > 0$ such that, for any $x \geq x_0$, for all $y > x$,

$$\mathbb{P}\left[ \max_{0 \leq m \leq \lambda_x} X_m \geq y \mid \mathcal{F}_0 \right] \geq c(X_0^{2\beta} - x^{2\beta})y^{-2\beta}, \text{ on } \{x < X_0 < y\}.$$ 

Proof. Write $f = f_{2\beta}$ and $f^y = f_{2\beta,0}^y$, where $f_{2\beta,0}$ is the Lyapunov function defined at (3.16) and $f_{2\beta,0}^y$ is the truncated version defined at (3.27). Lemma 3.4.3 shows that, for any $y > e$,

$$\mathbb{E}[f^y(X_{n+1}) - f^y(X_n) \mid \mathcal{F}_n] \geq \beta \left(2X_n \mathbb{E}[\Delta_n \mid \mathcal{F}_n] + (2\beta - 1) \mathbb{E}[\Delta_n^2 \mid \mathcal{F}_n]\right) X_n^{2\beta-2} + \sigma_{X_n}^2(X_n^{2\beta-2}),$$

on $\{X_n \leq y\}$, where the implicit constant in the error term does not depend on $y$. Hence under either condition (i) or condition (ii) in the lemma, we have that there exists $x_1 \in \mathbb{R}_+$ such that, for any $y > x_1$,

$$\mathbb{E}[f^y(X_{n+1}) - f^y(X_n) \mid \mathcal{F}_n] \geq 0, \text{ on } \{x_1 \leq X_n \leq y\}.$$ 

Let $x_1 \leq x < y$, and set $Y_n = f^y(X_n \wedge \lambda_x \wedge \sigma_y)$. Then $Y_n$ is a non-negative submartingale, which is uniformly bounded by $f(2y)$, and hence uniformly integrable; moreover, (L2) implies that $\lambda_x \wedge \sigma_y < \infty$, a.s. Hence, by optional stopping (Theorem 2.3.7) we have, on $\{x < X_0 < y\}$,

$$f(X_0) = Y_0 \leq \mathbb{E}[Y_{\lambda_x \wedge \sigma_y} \mid \mathcal{F}_0] \leq f(2y) \mathbb{P}[\sigma_y < \lambda_x \mid \mathcal{F}_0] + f(x) \mathbb{P}[\lambda_x < \sigma_y \mid \mathcal{F}_0],$$

since $f^y$ is non-decreasing and bounded above by $f(2y)$. Hence

$$\mathbb{P}[\sigma_y < \lambda_x \mid \mathcal{F}_0] \geq \frac{f(X_0) - f(x)}{f(2y) - f(x)} \geq \frac{f(X_0) - f(x)}{f(2y)},$$

which yields the statement in the lemma. \qed
Proof of Theorem 3.7.2. Let \( x_0 \) be the constant from Lemma 3.7.6. We apply Lemma 2.7.7 with the excursion lower bound from Lemma 3.7.6 to obtain, for any \( x \geq x_0 \),
\[
P[\lambda_x \geq n \mid F_0] \geq \frac{1}{2} P\left[ \max_{0 \leq m \leq \lambda_x} X_m \geq Cn^{1/2} \mid F_0 \right] \geq c(x_0^{2\beta} - x^{2\beta})n^{-\beta}1\{x < X_0 < Cn^{1/2}\},
\]
for constants \( c > 0 \) (depending on \( \beta \)) and \( C < \infty \). It follows that, for some \( c > 0 \) depending on \( \beta \),
\[
E[\lambda_x^\beta \mid F_0] \geq c(x_0^{2\beta} - x^{2\beta}) \sum_{n > X_0/C} n^{-1} = \infty, \text{ on } \{X_0 > x\}.
\]
Moreover, for any \( x_1 > x_0 \), for any \( x < x_1 \), on \( \{X_0 = x_1\} \),
\[
P[\lambda_x \geq n \mid F_0] \geq P[\lambda_{x \vee x_0} \geq n \mid F_0] \geq c(x_1^{2\beta} - (x \vee x_0)^{2\beta})n^{-\beta},
\]
for all \( n \geq n_0 \) sufficiently large. This yields the result. \( \square \)

Now we work towards a proof of Theorem 3.7.3. First we need an excursion bound analogous to Lemma 3.7.6.

Lemma 3.7.7. Suppose that (L0) and (L2) hold. Suppose that (L1) holds with \( p > 2 \), and that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). Suppose also that there exist \( x_0 \in (0, \infty) \), \( v \in (0, \infty) \), and \( \theta \in (0, 1) \) such that for all \( x \geq x_0 \),
\[
\mu_2(x) \geq v, \quad \text{and} \quad 2x\mu_1(x) + \left(1 + \frac{1 - \theta}{\log x}\right) \mu_2(x) \geq 0.
\]
Then for any \( \nu > 1 - \theta \) there exist \( x_1 \in \mathbb{R}_+ \) and a constant \( c = c(\nu, \theta) > 0 \) such that, for any \( x \geq x_1 \), for all \( y > x \), on \( \{x < X_0 < y\} \),
\[
P\left[ \max_{0 \leq m \leq \lambda_x} X_m \geq y \mid F_0 \right] \geq c(X_0^2 \log^\nu x - x^2 \log^\nu x)y^{-2}(\log y)^{-\nu}.
\]

Proof. Write \( f = f_{2,\nu} \) and \( f^y = f_{2,\nu}^y \), where \( f_{2,\nu} \) and \( f_{2,\nu}^y \) are defined at (3.16) and (3.27) respectively. Take \( \nu > 1 - \theta \). Lemma 3.4.3 shows that, for any \( y > 2e \),
\[
\begin{align*}
\mathbb{E}[f^y(X_{n+1}) - f^y(X_n) \mid F_n] \\
\geq (2X_n\mathbb{E}[\Delta_n \mid F_n] + \mathbb{E}[\Delta_n^2 \mid F_n]) \log^\nu X_n \\
+ \frac{\nu}{2} (2X_n\mathbb{E}[\Delta_n \mid F_n] + 3\mathbb{E}[\Delta_n^2 \mid F_n]) \log^{\nu-1} X_n
\end{align*}
\]
3.7. Moments and tails of passage times

\[ + \sigma_{X_n}^2(\log^{\nu - 1} X_n), \]
on \{X_n \leq y\}, where the implicit constant in the error term does not depend
on \(y\). Using the fact that on \(\{X_n \geq x_0\},\)

\[ 2X_n \mathbb{E}[\Delta_n | F_n] + \mathbb{E}[\Delta_n^2 | F_n] \geq -\frac{1 - \theta}{\log X_n} \mu_2(X_n), \]

we thus obtain

\[ \mathbb{E}[f^y(X_{n+1}) - f^y(X_n) | F_n] \geq (\nu - (1 - \theta) + \sigma_{X_n}^2(1)) \mu_2(X_n) \log^{\nu - 1} X_n, \]

which is non-negative for all \(X_n\) sufficiently large, by the choice of \(\nu > 1 - \theta\)
and the fact that \(\mu_2\) is eventually positive. In other words, there exists
\(x_1 \in \mathbb{R}_+\) such that, for any \(y > x_1,\)

\[ \mathbb{E}[f^y(X_{n+1}) - f^y(X_n) | F_n] \geq 0, \text{ on } \{1 \leq X_n \leq y\}. \]
The remainder of the proof now follows the same lines as the proof of
Lemma 3.7.6.

\[ \square \]

Proof of Theorem 3.7.3. Choose \(\nu \in (1 - \theta, 1),\) and let \(x_0\) be as in the state-
ment of Lemma 3.7.7. We apply Lemma 2.7.7 with the excursion lower
bound from Lemma 3.7.7 to obtain, for any \(x \geq x_0,\) on \(\{x < X_0 < Cn^{1/2}\},\)

\[ P[\lambda_x \geq n | F_0] \geq \frac{1}{2} \mathbb{P} \left[ \max_{0 \leq m \leq \lambda_x} X_m \geq Cn^{1/2} | F_0 \right] \geq c(X_0^2 \log^{-\nu} X_0 - x^2 \log^{-\nu} x)n^{-1} \log^{-\nu} n, \]

for constants \(c > 0\) (depending on \(\nu\) and \(\theta\)) and \(C < \infty\). It follows that, for
some \(c > 0,\) on \(\{X_0 > x\},\)

\[ \mathbb{E}[\lambda_x | F_0] \geq c(X_0^2 \log^{-\nu} X_0 - x^2 \log^{-\nu} x) \sum_{n > X_0/C} n^{-1} \log^{-\nu} n = \infty, \]
as claimed. Moreover, for any \(x_1 > x_0,\) on \(\{X_0 = x_1\},\)

\[ P[\lambda_x \geq n | F_0] \geq P[\lambda_{x \vee x_0} \geq n | F_0] \geq cn^{-1} \log^{-\nu} n, \]

for some constant \(c = c(x_0, x_1, x, \theta, \nu) > 0\) and all \(n \geq n_0\) sufficiently large.

\[ \square \]
3.8 Excursion durations and maxima

The results in Section 3.7 deal with bounds for passage times into bounded intervals, and the extent of trajectories up until such a passage time, for general Lamperti processes started far enough away from the origin. In this section we impose the additional structure provided by the regeneration and irreducibility assumptions of Section 3.6, and obtain analogous results for full excursions in that setting.

First we consider tail bounds for the excursion duration $\kappa_1$. The first result is an analogue of the upper tail bound result for passage times into bounded intervals, Theorem 3.7.1.

**Theorem 3.8.1.** Suppose that (L0), (I), and (R) hold for a locally finite $S \subset \mathbb{R}_+$. Suppose that, for some $\alpha > 0$, (L1) holds with $p > \max\{2, 2\alpha\}$, and that $\limsup_{x \to \infty} \tilde{\mu}_2(x) < \infty$. Suppose also that one of the following two conditions holds.

(i) We have $\alpha \in (0, \frac{1}{2}]$ and $\limsup_{x \to \infty} (2x\tilde{\mu}_1(x) - (1 - 2\alpha)\mu_2(x)) < 0$.

(ii) We have $\alpha \in (\frac{1}{2}, \infty)$ and $\limsup_{x \to \infty} (2x\tilde{\mu}_1(x) + (2\alpha - 1)\mu_2(x)) < 0$.

Then there exists $\varepsilon > 0$ for which $\mathbb{P}[\kappa_1 \geq n] = O(n^{-\alpha - \varepsilon})$. In particular, $\mathbb{E}[\kappa_1^\alpha] < \infty$.

Note that the case $\alpha = 1$ of Theorem 3.8.1 generalizes the positive-recurrence result Theorem 3.2.3(iii). The analogue of the lower tail bound result for passage times into bounded intervals, Theorem 3.7.2, is the following.

**Theorem 3.8.2.** Suppose that (L0), (I), and (R) hold for a locally finite $S \subset \mathbb{R}_+$. Suppose that, for some $\beta > 0$, (L1) holds with $p > \max\{2, 2\beta\}$, and that $\limsup_{x \to \infty} \tilde{\mu}_2(x) < \infty$. Suppose also that one of the following two conditions holds.

(i) We have $\beta \in (0, \frac{1}{2}]$ and $\liminf_{x \to \infty} (2x\mu_1(x) - (1 - 2\beta)\mu_2(x)) > 0$.

(ii) We have $\beta \in (\frac{1}{2}, \infty)$ and $\liminf_{x \to \infty} (2x\mu_1(x) + (2\beta - 1)\mu_2(x)) > 0$.

Then there exists $c = c(\beta) > 0$ such that $\mathbb{P}[\kappa_1 \geq n] \geq cn^{-\beta}$ for all $n \geq 1$. In particular, $\mathbb{E}[\kappa_1^\beta] = \infty$.

We also present the following analogue of Theorem 3.7.3, which sharpens the $\beta = 1$ case of Theorem 3.8.2.
3.8. Excursion durations and maxima

**Theorem 3.8.3.** Suppose that (L0), (I), and (R) hold for a locally finite \( S \subset \mathbb{R}_+ \). Suppose that (L1) holds with \( p > 2 \), and that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). Suppose also that there exist \( x_0 \in (0, \infty) \) and \( \theta \in (0, 1) \) such that for all \( x \geq x_0 \),

\[
2x\mu_1(x) + \left(1 + \frac{1 - \theta}{\log x}\right)\mu_2(x) \geq 0.
\]

Then for any \( \nu > 1 - \theta \), there exists \( c = c(\nu) > 0 \) such that \( \mathbb{P}[\kappa_1 \geq n] \geq cn^{-1}(\log n)^{-\nu} \) for all \( n \geq 2 \). In particular, \( \mathbb{E}\kappa_1 = \infty \).

In this section we also present an almost-sure lower bound on the maximum of the process after a large number of excursions; the basis for this result is Lemma 3.7.6.

**Theorem 3.8.4.** Suppose that (L0), (I), and (R) hold for a locally finite \( S \subset \mathbb{R}_+ \). Suppose that, for some \( \beta > 0 \), (L1) holds with \( p > \max\{2, 2\beta\} \), and that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \). Suppose also that one of the following two conditions holds.

(i) We have \( \beta \in (0, \frac{1}{2}] \) and \( \liminf_{x \to \infty} (2x\mu_1(x) - (2\beta)\bar{\mu}_2(x)) > 0 \).

(ii) We have \( \beta \in (\frac{1}{2}, \infty) \) and \( \liminf_{x \to \infty} (2x\mu_1(x) + (2\beta - 1)\bar{\mu}_2(x)) > 0 \).

Then for any \( \varepsilon > 0 \), a.s., for all but finitely many \( k \),

\[
\max_{0 \leq m \leq \kappa_1} X_m \geq k^{\frac{1}{2\beta}}(\log k)^{-\frac{1}{2\beta} - \varepsilon}.
\]

The proofs of the preceding theorems occupy the rest of this section. We first need a technical result that says that, with uniformly positive probability, the process will reach the origin before exceeding a fixed level, provided it is not too close to the starting point.

**Lemma 3.8.5.** Suppose that (L0), (I), and (R) hold for a locally finite \( S \subset \mathbb{R}_+ \). Suppose that \( \mathbb{E}[\Delta_n \mid F_n] \leq B \), a.s., for some \( B \in \mathbb{R}_+ \). Then for any \( x \in \mathbb{R}_+ \) there exists \( y \in (x, \infty) \) such that, for all \( n \geq 0 \),

\[
\mathbb{P}\left[ \max_{n \leq m \leq \kappa_1} X_m < y \ \bigg| \ F_n \right] \geq \delta, \text{ on } \{X_n \leq x\}.
\]

**Proof.** Fix \( x \in \mathbb{R}_+ \). Since \( S \) is locally finite and \( \inf S = 0, S \cap [0, x] \) is finite and non-empty; by (I), \( m := \max_{y \in S \cap [0, x]} m(y, 0) \) and \( \varphi := \min_{y \in S \cap [0, x]} \varphi(y, 0) \) satisfy \( m < \infty \) and \( \varphi > 0 \). Hence \( \mathbb{P}[\kappa_1 \leq n + m \mid F_n] \geq \varphi \) on \( \{X_n \leq x\} \). In
addition, the first moment bound in the lemma with the maximal inequality in Theorem 2.4.7 gives
\[
P\left[ \max_{n \leq k \leq n+m} X_k \geq y \mid \mathcal{F}_n \right] \leq \frac{Bm + x}{y},
\]
on \{X_n \leq x\}. Choosing \( y = 2(Bm + x)/\varphi \) it follows that
\[
P\left[ \{\kappa_1 \leq m\} \cap \left\{ \max_{n \leq k \leq n+m} X_k < y \right\} \right] \geq \varphi - \frac{Bm + x}{y} \geq \frac{\varphi}{2}, \quad \text{on} \ \{X_n \leq x\},
\]
which gives the result. \( \square \)

We can now give the proof of Theorem 3.8.1.

**Proof of Theorem 3.8.1.** Lemma 3.7.5 shows that there exist \( \gamma_0 \in (2\alpha, p) \), \( C \in \mathbb{R}_+ \), and \( x_0 < \infty \) such that, for any \( \gamma \in (2\alpha, \gamma_0) \),
\[
\mathbb{E}[X_{n+1}^\gamma - X_n^\gamma \mid \mathcal{F}_n] \leq C\mathbb{1}\{X_n < x_0\}.
\]
It follows from Theorem 2.4.8 that, for any \( x \in \mathbb{R}_+ \),
\[
\mathbb{E}[X_{n\wedge \sigma_x}^\gamma \mid \mathcal{F}_0] \leq X_0^\gamma + C\mathbb{E}[\sigma_x \mid \mathcal{F}_0].
\]
Moreover, Lemma 3.6.4 shows that (L0), (I), and the fact that \( \mathcal{S} \) is locally finite imply that (L3) holds; hence it follows from Proposition 3.3.4 that \( \mathbb{E}[\sigma_x \mid \mathcal{F}_0] \leq K_x \), a.s., for some \( K_x < \infty \) depending on \( x \). Hence
\[
\mathbb{E}[X_{n\wedge \sigma_x}^\gamma \mid \mathcal{F}_0] \leq X_0^\gamma + K_x, \ a.s., \quad (3.46)
\]
for some \( K_x < \infty \) depending on \( x \).

Now take \( x_1 \in (x_0, \infty) \). Define stopping times \( \lambda_k \) and \( \rho_k \) recursively by \( \rho_0 = 1 \) and, for \( k \in \mathbb{N} \),
\[
\lambda_k = \min\{n \geq \rho_{k-1} : X_n \leq x_0\}, \quad \text{and} \quad \rho_k = \min\{n \geq \lambda_k : X_n \geq x_1\}.
\]
Lemma 3.8.5 shows that we may choose \( x_1 \) sufficiently large so that, for a constant \( \delta > 0 \),
\[
P[\kappa_1 < \rho_k \mid \mathcal{F}_{\lambda_k}] \geq \delta, \ a.s. \quad (3.47)
\]
An application of Theorem 3.7.1 shows that, for all \( k \geq 0 \),
\[
\mathbb{E}[(\lambda_{k+1} - \rho_k)^\gamma \mid \mathcal{F}_{\rho_k}] \leq CX_{\rho_k}^\gamma, \ a.s.,
\]
for any $s \in (0, \gamma/2)$. Here, it follows from (3.46) and Fatou’s lemma that, for all $k \geq 0$,

$$
\mathbb{E}[X_{\rho_k}^\gamma | \mathcal{F}_{\lambda_k}] \leq x_0^\gamma + K_{x_1}, \text{ a.s.,}
$$

so that, for all $k \geq 0$, $\mathbb{E}[(\lambda_{k+1} - \rho_k)^s] \leq C$ for some $C < \infty$ depending on $s, \gamma, x_0, \text{ and } x_1$.

Another application of Proposition 3.3.4 shows that, for all $k \geq 1$, $\mathbb{E}[(\lambda_{k+1} - \rho_k)^s] \leq C$ for some $C < \infty$ depending on $\gamma$ and $x_1$. Hence, by Minkowski’s inequality, for all $n \geq 0$ and all $k \geq 1$,

$$
P[\lambda_{k+1} - \lambda_k \geq n] \leq C n^{-s}.
$$

Let $N = \min\{k \geq 0 : \rho_k > \kappa_1\}$. Then $\mathbb{P}[N > 1] = \mathbb{E}[\mathbb{P}[\kappa_1 > \rho_1 | \mathcal{F}_{\lambda_1}]] \leq 1 - \delta$, by (3.47). Similarly, two applications of (3.47) show that

$$
P[N > 2] = \mathbb{E}[\mathbb{P}[\kappa_1 > \rho_2 | \mathcal{F}_{\lambda_2}| \{\kappa_1 > \rho_1\}]] \leq (1 - \delta)^2;
$$

iterating this argument gives $\mathbb{P}[N > k] \leq (1 - \delta)^k \leq e^{-\delta k}$ for all $k \geq 0$.

Moreover, by definition of $N$, we have $\kappa_1 < \rho_N < \lambda_{N+1}$, so that $\kappa_1 \leq \sum_{k=1}^{N}(\lambda_{k+1} - \lambda_k)$. Hence

$$
P[\kappa_1 \geq n] \leq \mathbb{P}[N \geq n^\varepsilon] + \mathbb{P} \left[ \sum_{k=1}^{\lfloor n^\varepsilon \rfloor} (\lambda_{k+1} - \lambda_k) \geq n \right]
$$

$$
\leq e^{-\delta n^\varepsilon} + n^\varepsilon \sup_k \mathbb{P}[\lambda_{k+1} - \lambda_k \geq n^{1-\varepsilon}].
$$

Thus for any $\varepsilon > 0$ and any $s \in (0, \gamma/2)$, $\mathbb{P}[\kappa_1 \geq n] = O(n^{\varepsilon-(1-\varepsilon)s})$. Since $\gamma > 2\alpha$, we may choose $s > \alpha$ and then choose $\varepsilon > 0$ small enough to see that $\mathbb{P}[\kappa_1 \geq n] = O(n^{-\alpha-\varepsilon})$.

The proofs of Theorems 3.8.2 and 3.8.3 are accomplished by converting lower bounds on the return time to a bounded interval starting far enough away from the origin (given in Theorems 3.7.2 and 3.7.3 respectively) into lower bounds for the duration of excursions away from the origin, with the aid of Lemma 3.6.10.

**Proof of Theorem 3.8.2.** Recall that $\tau_y = \min\{n \geq 0 : X_n = y\}$. Theorem 3.7.2 shows that there exists $x_0 \in \mathbb{R}_+$ such that, for any $x_1 > x_0$, there is a constant $c = c(x_0, x_1, \beta) > 0$ such that, for all $n \geq n_0$ sufficiently large,

$$
P[\lambda_{x_0} \geq n | \mathcal{F}_0] \geq cn^{-\beta}, \text{ on } \{X_0 = x_1\}.
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It follows that, for \( n \geq n_0 \),
\[
P[\kappa_1 \geq n] \geq \mathbb{E}[P[\kappa_1 \geq n \mid \mathcal{F}_{\tau_1}] | \{\tau_1 < \kappa_1\}] \\
\geq cn^{-\beta} P[\tau_1 < \kappa_1]; \tag{3.48}
\]
the result now follows from Lemma 3.6.10. \( \square \)

Proof of Theorem 3.8.3. Let \( \nu > 1 - \theta \). Theorem 3.7.3 shows that there exists \( x_0 \in \mathbb{R}_+ \) such that, for any \( x_1 > x_0 \), there is a constant \( c > 0 \) such that, for all \( n \) sufficiently large,
\[
P[\lambda_{x_0} \geq n \mid \mathcal{F}_0] \geq cn^{-1} \log^{-\nu} n, \text{ on } \{X_0 = x_1\}.
\]
Another application of (3.48) and Lemma 3.6.10 now completes the proof. \( \square \)

Finally, we can complete the proof of Theorem 3.8.4.

Proof of Theorem 3.8.4. Under the stated conditions, Lemma 3.7.6 applies; let \( x_0 \in \mathbb{R}_+ \) be the constant in the statement of Lemma 3.7.6. Recall that, assuming (R), we have \( X_0 = 0 \). Define for \( k \in \mathbb{Z}_+ \),
\[
M_k := \max_{\nu_k \leq m < \nu_{k+1}} X_m,
\]
the maximum extent of the \( k \)th excursion. Then choose (and fix) \( x = \inf(S \cap (x_0, \infty)) \); since \( S \) is locally finite, \( x > x_0 \).
\[
P[M_0 \geq y] \geq \mathbb{P}\left[\{\tau_x < \kappa_1\} \cap \left\{\max_{\tau_x \leq m \leq \kappa_1} X_m \geq y\right\}\right] \\
\geq \mathbb{E}\left[1\{\tau_x < \kappa_1\} \mathbb{P}\left[\max_{\tau_x \leq m \leq \kappa_1} X_m \geq y \mid \mathcal{F}_{\tau_x}\right]\right].
\]
Here, by Lemma 3.7.6, for all \( y > x \),
\[
P\left[\max_{\tau_x \leq m \leq \kappa_1} X_m \geq y \mid \mathcal{F}_{\tau_x}\right] \geq c_1 y^{-2\beta}, \text{ a.s.,}
\]
for some constant \( c_1 > 0 \) depending only on \( \beta \) and \( x \), but not on \( y \). Hence, \( P[M_0 \geq y] \geq c_1 y^{-2\beta} P[\tau_x < \kappa_1] \). Also, Lemma 3.6.10 shows that \( P[\tau_x < \kappa_1] = c_2 \) for a constant \( c_2 > 0 \) depending only on \( x \). Hence there exists \( c = c(\beta, x) > 0 \) such that
\[
P[M_0 \geq y] \geq cy^{-2\beta}, \text{ for all } y > x. \tag{3.49}
\]
3.9. Almost-sure bounds on trajectories

By the regenerative assumption (R), the random variables $M_0, M_1, M_2, \ldots$ are i.i.d. and satisfy the tail bound (3.49). Hence an application of the $\theta = 2\beta, \varphi = 0$ case of Lemma 3.6.13(ii) shows that, for any $\varepsilon > 0$, a.s., for all but finitely many $k$,

$$\max_{0 \leq t \leq k} M_t \geq k^{1/\beta} (\log k)^{1/2} - \frac{1}{2} \varepsilon.$$

The result now follows, since $\max_{0 \leq m \leq m_k} X_m \geq \max_{0 \leq t \leq k-1} M_t$. $\Box$

3.9 Almost-sure bounds on trajectories

In this section we present almost-sure upper and lower bounds on the running maximum $\max_{0 \leq m \leq n} X_m$ for a Lamperti process $X_n$. We start with the upper bounds, which we deduce from the results of Section 2.8. The first result gives a general diffusive upper bound.

**Theorem 3.9.1.** Suppose that (L0) holds and that for some constant $C \in \mathbb{R}^+$, $E[\Delta_n^2 \mid F_n] \leq C$, a.s. Suppose also that $\limsup_{x \to \infty} (x \bar{\mu}_1(x)) < \infty$. Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \geq 0$,

$$\max_{0 \leq m \leq n} X_m \leq n^{1/2} (\log n)^{1/2+\varepsilon}.$$

**Proof.** Given that $E[\Delta_n^2 \mid F_n] \leq C$, $\bar{\mu}_2(x)$ and $\bar{\mu}_1(x)$ are well defined and uniformly bounded. We will apply Theorem 2.8.1 with $f(x) = x^2$. Here

$$E[f(X_{n+1}) - f(X_n) \mid F_n] = 2X_n E[\Delta_n \mid F_n] + E[\Delta_n^2 \mid F_n] \leq 2X_n \bar{\mu}_1(X_n) + \bar{\mu}_2(X_n), \text{ a.s.,}$$

which is bounded above by a finite constant, a.s., by assumption. Then Theorem 2.8.1 yields the result. $\Box$

The bound in Theorem 3.9.1 is not always sharp. In some cases, a stronger sub-diffusive upper bound holds.

**Theorem 3.9.2.** Let $\alpha > 1$. Suppose that (L0) holds and that (L1) holds with $p > 2\alpha$. Suppose also that $\limsup_{x \to \infty} (x \bar{\mu}_2(x)) < \infty$ and

$$\limsup_{x \to \infty} (2x \bar{\mu}_1(x) + (2\alpha - 1) \bar{\mu}_2(x)) < 0.$$

There exists $\varepsilon > 0$ such that, a.s., for all but finitely many $n \geq 0$,

$$\max_{0 \leq m \leq n} X_m \leq n^{1/2\alpha} - \varepsilon.$$
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**Remark 3.9.3.** In the Markov chain setting, Theorem 3.9.2 applies in the case of positive-recurrence. Indeed, the condition for positive-recurrence given in Theorem 3.2.3(iii) is

$$\limsup_{x \to \infty} \left( 2x \mu_1(x) + \mu_2(x) \right) < 0;$$

since \( \limsup_{x \to \infty} \mu_2(x) < \infty \), it follows that for some \( \alpha > 1 \),

$$\limsup_{x \to \infty} \left( 2x \mu_1(x) + (2\alpha - 1) \mu_2(x) \right) < 0.$$

**Proof of Theorem 3.9.2.** Let \( f(x) = x^\gamma \). Under the assumptions of the theorem, Lemma 3.7.5 shows that there exists \( \gamma > \frac{2}{\alpha} \) for which \( f(X_n) \) is integrable and

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] < 0,$$

for some \( x_0 \in \mathbb{R}_+ \). Thus we may apply Theorem 2.8.1 to give the result. □

With additional regularity assumptions on the increment moment functions, a consequence of the preceding results in the following corollary, which uses log-log scaling to identify (upper bounds on) scaling exponents.

**Corollary 3.9.4.** Suppose that (L0) holds, and that (L1) holds with \( p > 2 \). Suppose also that, for some \( a, b \in \mathbb{R} \) with \( b > 0 \),

$$\lim_{x \to \infty} \mu_2(x) = \lim_{x \to \infty} \tilde{\mu}_2(x) = b;$$

$$\lim_{x \to \infty} x \mu_1(x) = \lim_{x \to \infty} x\tilde{\mu}_1(x) = a.$$

(i) If \( 2a + b \geq 0 \), then, a.s.,

$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} \leq \frac{1}{2};$$

(ii) If \( 2a + b < 0 \) and (L1) holds with \( p > \frac{b - 2a}{b} \), then, a.s.,

$$\limsup_{n \to \infty} \frac{\log X_n}{\log n} \leq \frac{b}{b - 2a} \in \left( 0, \frac{1}{2} \right).$$

**Proof.** First of all, for all \( a \) and \( b \), Theorem 3.9.1 implies that

$$\max_{0 \leq m \leq n} \log X_m \leq \frac{1}{2} \log n + O(\log \log n), \text{ a.s.},$$
which gives part (i). For part (ii), suppose that \(2a + b < 0\). Then, since \(b > 0\), we have \(a < 0\) and \(b - 2a > 2b > 0\). Moreover, for \(\alpha > 0\),

\[
\limsup_{x \to \infty} (2x\bar{\mu}_1(x) + (2\alpha - 1)\bar{\mu}_2(x)) = 2a + (2\alpha - 1)b < 0,
\]

provided \(\alpha < \frac{b - 2a}{2b}\). Hence Theorem 3.9.2 shows that, a.s., for all but finitely many \(n\), \(\max_{0 \leq m \leq n} X_m \leq n^{\frac{1}{2\alpha}}\), so that

\[
\limsup_{n \to \infty} \frac{\log X_n}{\log n} \leq \frac{1}{2\alpha}.
\]

Since \(\alpha \in (0, \frac{b - 2a}{2b})\) was arbitrary, we obtain part (ii).

Obtaining lower bounds requires stronger assumptions, and it is most convenient to use the concepts of irreducibility and regeneration introduced in Section 3.6. Under these assumptions, we can complement the upper bounds in Corollary 3.9.4 by corresponding lower bounds, to fully identify scaling exponents.

**Theorem 3.9.5.** Suppose that (L0), (I), and (R) hold for a locally finite \(S \subset \mathbb{R}_+\), and that (L1) holds with \(p > 2\). Suppose that, for some \(a, b \in \mathbb{R}\) with \(b > 0\),

\[
\begin{align*}
\lim_{x \to \infty} \mu_2(x) &= \lim_{x \to \infty} \bar{\mu}_2(x) = b; \\
\lim_{x \to \infty} x\mu_1(x) &= \lim_{x \to \infty} x\bar{\mu}_1(x) = a.
\end{align*}
\]

(i) If \(2a + b \geq 0\), then, a.s.,

\[
\limsup_{n \to \infty} \frac{\log X_n}{\log n} = \frac{1}{2}.
\]

(ii) If \(2a + b < 0\) and (L1) holds with \(p > \frac{b - 2a}{b}\), then, a.s.,

\[
\limsup_{n \to \infty} \frac{\log X_n}{\log n} = \frac{b}{b - 2a} \in \left(0, \frac{1}{2}\right).
\]

**Proof.** To prove the equalities in parts (i) and (ii), it suffices to prove only the corresponding \(\geq\) statements, since the \(\leq\) statements are given in Corollary 3.9.4 (under weaker conditions).

For part (i), suppose that \(2a + b \geq 0\). Then, for \(\alpha > 0\),

\[
\limsup_{x \to \infty} (2x\bar{\mu}_1(x) + (2\alpha - 1)\bar{\mu}_2(x)) = \limsup_{x \to \infty} (2x\bar{\mu}_1(x) + (1 - 2\alpha)\mu_2(x))
\]
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\[ = 2a + (2\alpha - 1)b < 0, \]

provided \( \alpha \in (0, \frac{b - 2a}{2b}) \); fix such an \( \alpha \) (so \( \alpha < 1 \), since \( 2a + b \geq 0 \)). Theorem 3.8.1 bounds applies, and shows that \( \mathbb{P}[\kappa_1 \geq n] = O(n^{-\alpha - \varepsilon}) \). An application of Lemma 3.6.13(i) shows that \( \nu_k \leq k^{1/\alpha} \), a.s., for all but finitely many \( k \). In other words, \( \nu_{\lfloor k^{\alpha} \rfloor} \leq k \) for all but finitely many \( k \).

On the other hand, for \( \beta > 0 \),

\[
\liminf_{x \to \infty} (2x\mu_1(x) + (2\beta - 1)\mu_2(x)) = \liminf_{x \to \infty} (2x\mu_1(x) - (1 - 2\beta)\mu_2(x)) = 2a + (2\beta - 1)b > 0,
\]

provided \( \beta > \frac{b - 2a}{2b} \); fix such a \( \beta \). Then, by Theorem 3.8.4(i), for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \),

\[
\max_{0 \leq m \leq n} X_m \geq \max_{0 \leq m \leq \nu_{\lfloor n^{1/\alpha} \rfloor}} \geq n^{\frac{\alpha - 2\beta}{2\beta}}.
\]

It follows that, for any \( \varepsilon > 0 \), a.s.,

\[
\limsup_{n \to \infty} \frac{\log X_n}{\log n} \geq \frac{\alpha}{2\beta} - \varepsilon,
\]

which yields the ‘\( \geq \)’ half of part (i), since \( \varepsilon > 0 \), \( \alpha < \frac{b - 2a}{2b} \), and \( \beta > \frac{b - 2a}{2b} \) were arbitrary.

For part (ii), suppose that \( 2a + b < 0 \). Then, for \( \alpha > 0 \),

\[
\limsup_{x \to \infty} (2x\tilde{\mu}_1(x) + (2\alpha - 1)\tilde{\mu}_2(x)) = 2a + (2\alpha - 1)b < 0,
\]

provided \( \alpha < \frac{b - 2a}{2b} \). Since \( 2a + b < 0 \), we may choose \( \alpha \in (1, \frac{b - 2a}{2b}) \); fix such an \( \alpha \). Theorem 3.8.1 bounds applies, and shows that \( \mathbb{E}\kappa_1 < \infty \). Then the strong law of large numbers shows that \( k^{-1}\nu_k \to \mathbb{E}\kappa_1 \), a.s. Choosing \( \delta > 0 \) so that \( \delta \mathbb{E}\kappa_1 < 1 \), we have that \( \nu_{\lfloor \delta k \rfloor} \leq k \), a.s., for all but finitely many \( k \).

On the other hand, for \( \beta > 0 \),

\[
\liminf_{x \to \infty} (2x\mu_1(x) + (2\beta - 1)\mu_2(x)) = 2a + (2\beta - 1)b > 0,
\]

provided \( \beta > \frac{b - 2a}{2b} \); fix such a \( \beta \) (necessarily, \( \beta > \alpha > 1 \)). Then, by Theorem 3.8.4(ii), for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \),

\[
\max_{0 \leq m \leq n} X_m \geq \max_{0 \leq m \leq \nu_{\lfloor \delta n \rfloor}} \geq n^{\frac{1}{2\beta} - \varepsilon}.
\]

It follows that, a.s.,

\[
\limsup_{n \to \infty} \frac{\log X_n}{\log n} \geq \frac{1}{2\beta} - \varepsilon,
\]

giving the ‘\( \geq \)’ half of part (ii), since \( \varepsilon > 0 \) and \( \beta > \frac{b - 2a}{2b} \) were arbitrary. □
3.10 Transient theory in the critical case

The main result of this section is the following diffusive lower bound on the rate of escape in the transient case.

**Theorem 3.10.1.** Suppose that \((L0)\) and \((L3)\) hold, and that \((L1)\) holds with \(p > 2\) and \(\delta \in (0, p - 2]\). Suppose also that there exists \(\theta \in (0, \delta)\) such that

\[
\limsup_{x \to \infty} \bar{\varphi}_2(x) < \infty, \text{ and } \liminf_{x \to \infty} \left(2x\varphi_1(x) - (1 + \theta)\bar{\varphi}_2(x)\right) > 0. \tag{3.50}
\]

Then for any \(\varepsilon > 0\), a.s., for all but finitely many \(n \geq 0\),

\[
X_n \geq n^{\frac{1}{2}}(\log n)^{-\frac{1}{2} - \frac{1}{2}d - \varepsilon}.
\]

**Remark 3.10.2.** The conditions in Theorem 3.10.1 are the same as those in the transience classification result, Theorem 3.5.1, except that the non-confinement condition \((L2)\) is replaced by the stronger local escape condition \((L3)\), which ensures that the process does not spend too much time near zero: see also Remark 3.10.9 below. Note that if the Lamperti condition (3.29) for transience from Theorem 3.5.1 holds, then (3.50) holds for some \(\theta > 0\).

We give two examples.

**Example 3.10.3.** Consider again \(S_n\) symmetric SRW on \(\mathbb{Z}^d\), and let \(X_n = \|S_n\|\) and \(\mathcal{F}_n = \sigma(S_0, \ldots, S_n)\). Take \(d \geq 3\) (the transient case). The calculations in Example 3.5.3 show that

\[
\lim_{x \to \infty} \left(2x\varphi_1(x) - (1 + \theta)\bar{\varphi}_2(x)\right) = 1 - \frac{2 + \theta}{d},
\]

which is strictly positive if \(\theta \in (0, d - 2)\). Since in this case \((L1)\) holds with \(\delta = p - 2\) for all \(p > 2\), Theorem 3.10.1 shows that, for any \(\varepsilon > 0\), a.s., all but finitely often,

\[
\|S_n\| \geq n^{\frac{1}{2}}(\log n)^{-\frac{1}{2} - \frac{1}{d} - \varepsilon}.
\]

This result is not quite sharp. A theorem of Dvoretzky and Erdős [84] says that the sharp bound is, for any \(\varepsilon > 0\), a.s., all but finitely often,

\[
\|S_n\| \geq n^{\frac{1}{2}}(\log n)^{-\frac{1}{d} - \varepsilon},
\]

and this inequality fails infinitely often for \(\varepsilon = 0\). △
Example 3.10.4. Continuing from Example 3.2.5, consider the simple random walk on $\mathbb{Z}_+$ with

$$ p(x, x-1) = p_x = 1 - p(x, x+1) = \frac{1}{2} \left( 1 + \frac{c}{x} \right) + o(x^{-1}). $$

Then $\mu_1(x) = 1 - 2p_x = -(c + o(1))x^{-1}$ and $\mu_2(x) = 1$. In the transient case $c < -1/2$, we thus have that (3.50) holds for any $\theta \in (0, -1 - 2c)$, and hence Theorem 3.10.1 shows that, for any $\varepsilon > 0$, a.s., all but finitely often,

$$ X_n \geq n^{\frac{1}{2}}(\log n)^{-\frac{1}{2} + \frac{1}{1+2c}} - \varepsilon. $$

Again, this result is not quite sharp: a result of Csáki et al. [65] shows that the $-\frac{1}{2}$ in the exponent of the logarithm can be dropped. $\triangle$

The rest of this section is devoted to the proof of Theorem 3.10.1. The starting point is the following. Recall that $\sigma_y = \min\{n \geq 0 : X_n \geq y\}$ is the first passage time into $[y, \infty)$. To quantify the transience of the process $X_n$, we study

$$ \eta_x := \max\{n \geq 0 : X_n \leq x\}, \quad (3.51) $$

the last exit time from $[0, x]$. Note that $\eta_x$ is not a stopping time. For $y > x$, let $\lambda_{y,x} := \min\{n \geq \sigma_y : X_n \leq x\}$ be the return time to $[0, x]$ having reached $[y, \infty)$. Then we have, for any $y > x$,

$$ P[\eta_x > n] = P[\eta_x > n, \sigma_y \leq n] + P[\eta_x > n, \sigma_y > n]. $$

Here

$$ P[\eta_x > n, \sigma_y \leq n] \leq P[\eta_x > \sigma_y, \sigma_y \leq n] \leq P[\sigma_y \leq n, \lambda_{y,x} < \infty]. $$

Hence we obtain the simple upper bound, valid for any $y > x$,

$$ P[\eta_x > n] \leq P[\lambda_{y,x} < \infty] + P[\sigma_y > n]. \quad (3.52) $$

Theorem 3.10.1 will follow from an analysis of the last exit estimate (3.52).

A decreasing Lyapunov function that gives a supermartingale outside a set provides a bound on the probability of returning to that set having reached a more distant region. An increasing Lyapunov function that gives a strict submartingale outside a bounded set provides an upper bound on the hitting time of the distant region. Together these estimates can be used to estimate the rate of escape for a transient process.
3.10. Transient theory in the critical case

To bound $\mathbb{P}[\sigma_y > n]$ we use an extension of the idea of the ‘reverse Foster’ Theorem 2.4.11. The main shortcoming of Theorem 2.4.11 is the fact that (2.19) does not permit an exceptional set. Also, to get a suitable bound on the right-hand side of (2.20) we previously imposed a uniform upper bound on the increments (see Theorem 2.8.3). In this section we address these points; in fact, the second necessitates dealing with the first, which is the purpose of the next result.

**Lemma 3.10.5.** Suppose that (L0) holds. Suppose that there exist a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, and constants $x_0 \geq 0$ and $\varepsilon > 0$, for which, for all $n \geq 0,$

$$E[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \geq \varepsilon, \text{ on } \{X_n \geq x_0\}. \quad (3.53)$$

Then for any $x > 0,$

$$E\sigma_x \leq \varepsilon^{-1} E f(X_{\sigma_x}) + (1 + \varepsilon^{-1} f(x_0)) E \sum_{n=0}^{\sigma_x} 1\{X_n \leq x_0\}.$$ 

**Proof.** Let $Y_n = X_{n \wedge \sigma_x}$. Note first that, on $\{X_m \leq x_0\} \cap \{m < \sigma_x\},$

$$E[f(Y_{m+1}) - f(Y_m) \mid \mathcal{F}_m] \geq -f(Y_m) = -f(X_m) \geq -f(x_0),$$

since $f$ is non-negative and non-decreasing. So, with $K = \varepsilon + f(x_0),$ we may express (3.53) as

$$E[f(Y_{m+1}) - f(Y_m) \mid \mathcal{F}_m] \geq 1\{m < \sigma_x\}(\varepsilon - K 1\{X_m \leq x_0\}). \quad (3.54)$$

The rest of the proof now follows closely the proof of Theorem 2.4.11; taking expectations in (3.54) and summing for $m$ from 0 up to $n - 1$ we get

$$E f(Y_n) \geq \varepsilon \sum_{m=0}^{n-1} \mathbb{P}[\sigma_x > m] - K E \sum_{m=0}^{\infty} 1\{m < \sigma_x, X_m \leq x_0\}.$$ 

Rearranging this inequality and using the fact that $f(Y_n) \leq f(X_{\sigma_x})$ we get

$$\sum_{m=0}^{n-1} \mathbb{P}[\sigma_x > m] \leq \frac{1}{\varepsilon} E f(X_{\sigma_x}) + \frac{K}{\varepsilon} E \sum_{m=0}^{\sigma_x} 1\{X_m \leq x_0\}.$$ 

The statement in the lemma follows on taking $n \to \infty.$
To apply Lemma 3.10.5, one must also bound $E f(X_{\sigma_x})$. This may be done by assuming an upper bound on jumps as in Theorem 2.8.3, but such an assumption is often too restrictive. The next result imposes a weaker \lq\text{uniform } f\text{-integrability}\rq\ condition on the increments instead.

**Lemma 3.10.6.** Suppose that (L0) holds. Suppose that there exist a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, and constants $x_0 \geq 0$ and $\varepsilon > 0$, for which (3.53) holds for all $n \geq 0$. Let $\delta > 0$. Suppose that for any $\nu > 0$ there exists $y \in \mathbb{R}_+$ such that, for all $n \in \mathbb{Z}_+$,

\[
E[f((1 + \delta^{-1})\Delta_n^+)1\{\Delta_n^+ > y\} \mid \mathcal{F}_n] \leq \nu, \ a.s. \tag{3.55}
\]

Then there is a constant $x_1 \geq x_0$ such that for all $x > x_1$,

\[
E_{\sigma_x} \leq 2\varepsilon^{-1}f((1 + \delta)x) + 2(1 + \varepsilon^{-1})f(x_1)E \sum_{n=0}^{\sigma_x} 1\{X_n \leq x_1\}.
\]

**Remark 3.10.7.** For example, if $f(x) = x^p$ for some $p > 0$, then a sufficient condition for (3.55) is that there exist $q > p$ and $C < \infty$ for which

\[
E[(\Delta_n^+)^q \mid \mathcal{F}_n] \leq C, \ a.s., \tag{3.56}
\]

for all $n \geq 0$. Indeed, in this case,

\[
f((1 + \delta^{-1})\Delta_n^+)1\{\Delta_n^+ > y\} = (1 + \delta^{-1})^p(\Delta_n^+)^q(\Delta_n^+)^{p-q}1\{\Delta_n^+ > y\}
\]

\[
\leq (1 + \delta^{-1})^p(\Delta_n^+)^q y^{p-q},
\]

since $p - q > 0$, so that

\[
E[f((1 + \delta^{-1})\Delta_n^+)1\{\Delta_n^+ > y\} \mid \mathcal{F}_n] \leq C(1 + \delta^{-1})^p y^{p-q},
\]

by (3.56). Hence (3.55) is satisfied.

**Proof of Lemma 3.10.6.** We use a truncation argument. Let

\[
f_x(y) := \min\{f(y), f((1 + \delta)x)\},
\]

where $\delta > 0$ is as in the statement of the lemma. We will show that a version of (3.53) holds with $f_x$ instead of $f$, up to modification of the constants $\varepsilon$ and $x_0$. Specifically, we claim that there is some $x_1 > x_0$ such that

\[
E[f_x(X_{n+1}) - f_x(X_n) \mid \mathcal{F}_n] \geq \varepsilon/2, \text{ on } \{x_1 \leq X_n < x\}, \tag{3.57}
\]
for all \( x > x_1 \) and all \( n \geq 0 \).

Assuming the claim (3.57), we may apply Lemma 3.10.5, since (3.57) implies that (3.54) now holds with \( f \) replaced by \( f_x \), \( x_0 \) replaced by \( x_1 \), and \( \varepsilon \) replaced by \( \varepsilon/2 \). Then we conclude from Lemma 3.10.5 that, for all \( x > x_1 \),

\[
\mathbb{E} \sigma_x \leq 2\varepsilon^{-1} \mathbb{E} f_x(X_{\sigma_x}) + 2(1 + \varepsilon^{-1})f_x(x_1) \mathbb{E} \sum_{n=0}^{\sigma_x} \mathbf{1}\{X_n \leq x_1\}.
\]

The statement of the lemma now follows since, by definition of \( f_x \),

\[
f_x(X_{n+1}) - f_x(X_n) = f((1 + \delta)x) - f(x + \Delta_n) \mathbf{1}\{\Delta_n > \delta x\},
\]

using the fact that \( \{X_n < x\} \cap \{X_n+1 > (1 + \delta)x\} \) implies that \( \Delta_n > \delta x \) and \( f(x + \Delta_n) \leq f(x + \Delta_n) \). Hence, again using the fact that \( f \) is non-decreasing, on \( \{X_n < x\} \),

\[
f_x(X_{n+1}) - f_x(X_n) \geq f(X_{n+1}) - f(X_n) - f((1 + \delta^{-1})\Delta_n) \mathbf{1}\{\Delta_n > \delta x\}.
\]

Here we have \( \mathbb{E}[f(X_{n+1}) - f(X_n) \mid F_n] \geq \varepsilon \) on \( \{X_n \geq x_0\} \). Also

\[
\mathbb{E}[f((1 + \delta^{-1})\Delta_n) \mathbf{1}\{\Delta_n > \delta x\} \mid F_n] \leq \varepsilon/2,
\]

for all \( x > x_1 \) sufficiently large, by (3.55). So we may choose \( x_1 > x_0 \) such that (3.57) holds, as claimed. \( \square \)

Denote the renewal function associated to \( X_n \) by

\[
H(x) := \mathbb{E} \sum_{n=0}^{\infty} \mathbf{1}\{X_n \leq x\} = \sum_{n=0}^{\infty} \mathbb{P}[X_n \leq x]. \tag{3.58}
\]

Note that \( H(x) \leq 1 + \mathbb{E} \eta_x \). It is important for application of Lemma 3.10.5 or Lemma 3.10.6 to show that \( H(x_1) < \infty \). The next result establishes this in the present setting.

**Lemma 3.10.8.** Suppose that (L0) and (L3) hold, and that (L1) holds with \( p > 2 \). Suppose also that (3.29) holds. Then, for any \( x \in \mathbb{R}_+ \), \( \mathbb{E} \eta_x < \infty \).
Remark 3.10.9. This is the only point in the proof of Theorem 3.10.1 where the local escape condition (L3) (rather than just the fact that \( \limsup_{n \to \infty} X_n = \infty \)) is used; presumably this condition is stronger than is necessary, but it is not too restrictive.

Proof of Lemma 3.10.8. Fix \( x \in \mathbb{R}_+ \). Define \( E(n) = \{ \eta_x \leq n \} \), the event that the process never visits \([0, x] \) after time \( n \). For the duration of this proof, write \( f = f_{0, \nu} \) as defined at (3.16). It was shown in the proof of Theorem 3.5.1 that (L0) and the \( p > 2 \) case of (L1) imply that for some \( \nu < 0 \) and some \( x_1 \in \mathbb{R}_+ \),

\[
E[f(X_{n+1})] - f(X_n) \leq 0, \quad \text{on} \quad \{ X_n \geq x_1 \}.
\]

An application of Lemma 3.5.7 shows that there exists \( y > x \) such that

\[
P[E(n) \mid F_n] \geq \frac{1}{2}, \quad \text{on} \quad \{ X_n \geq y \}.
\]

For this choice of \( y \), write \( r = r_y \) and \( \delta = \delta_y \) for the constants in (L3). Then, on \( \{ X_n < y \} \), if \( \kappa_{n,y} = \min\{ k \geq n : X_k \geq y \} \),

\[
P[E(n + r) \mid F_n] \geq P[\{ \kappa_{n,y} \leq n + r \} \cap E(\kappa_{n,y}) \mid F_n]
\]

\[
= E[1\{ \kappa_{n,y} \leq n + r \} P[E(\kappa_{n,y}) \mid F_{\kappa_{n,y}}] \mid F_n]
\]

\[
\geq \frac{1}{2} P[\kappa_{n,y} \leq n + r \mid F_n] \geq \frac{\delta}{2}, \quad \text{a.s.,}
\]

by (L3). On the other hand, on \( \{ X_n \geq y \} \), \( P[E(n + r) \mid F_n] \geq P[E(n) \mid F_n] \geq 1/2 \). It follows that, for all \( n \geq 0 \),

\[
P[E(n + r) \mid F_n] \geq \delta/2, \quad \text{a.s.} \quad (3.59)
\]

For \( n \in \mathbb{N} \), let \( A(n) = \cup_{2r(n-1) \leq k \leq 2rn} \{ X_k \leq x \} \) and \( G_n = F_{2rn} \). Then \( A(n) \in G_n \). For \( n \geq 0 \), let \( \tau_1 = \min\{ n \geq 1 : A(n) \text{ occurs} \} \), and for \( k \geq 2 \) set \( \tau_k = \min\{ n > \tau_{k-1} : A(n) \text{ occurs} \} \). Then, on \( \{ \tau_k < \infty \} \),

\[
P[\tau_{k+2} = \infty \mid G_{\tau_k}] \geq P[E(2r\tau_k + r) \mid F_{2r\tau_k}] \geq \delta/2 =: q,
\]

by (3.59), while \( \tau_k = \infty \) implies that \( \tau_{k+2} = \infty \). Hence

\[
P[\tau_{k+2} < \infty] = E[P[\tau_{k+2} < \infty \mid G_{\tau_k}] 1\{ \tau_k < \infty \}]
\]

\[
\leq (1 - q) P[\tau_k < \infty].
\]
In other words, if \( N = \max\{k \geq 1 : \tau_k < \infty\} \) denotes the number of times \( A(n) \) occurs, we have, setting \( \tau_0 = 0 \),

\[
P[N \geq k + 2 \mid N \geq k] = P[\tau_{k+2} < \infty \mid \tau_k < \infty] \leq 1 - q < 1,
\]

for all \( k \geq 0 \). It follows that \( P[N \geq k] \leq (1 - q)^{k/2} \) for all \( k \geq 0 \), so in particular \( \E \eta_x \leq 2r \E N < \infty \).

Now we can complete the proof of Theorem 3.10.1.

**Proof of Theorem 3.10.1.** Let \( f(x) = x^2 \). Then

\[
\E[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] = 2X_n \E[\Delta_n \mid \mathcal{F}_n] + \E[\Delta_n^2 \mid \mathcal{F}_n] \\
\geq 2X_n \mu_1(X_n) \geq \varepsilon, \text{ on } \{X_n \geq x_0\},
\]

for some constants \( \varepsilon > 0 \) and \( x_0 \in \mathbb{R}_+ \), by assumption. Thus we can apply Lemma 3.10.6 with \( f(x) = x^2 \) and Lemma 3.10.8 to see that there exists a constant \( C \in \mathbb{R}_+ \) such that

\[
\E \sigma_y \leq C(1 + y^2), \text{ for all } y \geq 0.
\]

Hence, by Markov’s inequality,

\[
P[\sigma_y > n] \leq \frac{C(1 + y^2)}{n}, \text{ for all } y > 0.
\]

Now we turn to the other term on the right-hand side of (3.52). Let \( f = f_{-\theta,0} \). Provided \( \theta \in (0, \delta) \), we may apply the \( \gamma = -\theta \) case of Lemma 3.4.1 to obtain, using the fact that \( \limsup_{x \to \infty} \bar{\mu}_2(x) < \infty \),

\[
\E[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq -\frac{\theta}{2} \left( 2X_n \mu_1(X_n) - (1 + \theta) \bar{\mu}_2(x) \right) X_n^{-2-\theta} + \sigma_{\rho_n}(X_n^{-2-\theta}).
\]

By the hypothesis of the theorem, there exist \( \varepsilon > 0 \) and \( x_0 \in \mathbb{R}_+ \) such that, for \( x \geq x_0, \mu_1(x) \geq 0 \) and \( 2x \mu_1(x) - (1 + \theta) \bar{\mu}_2(x) > \varepsilon \). Hence

\[
\E[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq -\frac{\theta}{2} \varepsilon X_n^{-2-\theta} + \sigma_{\rho_n}(X_n^{-2-\theta}).
\]

So there exists \( x_1 \in \mathbb{R}_+ \) such that

\[
\E[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq 0, \text{ on } \{X_n \geq x_1\}.
\]

Then an application of Lemma 2.5.10 shows that, for any \( x > x_1 \),

\[
P[\lambda_{y,x} < \infty \mid \mathcal{F}_{\sigma_y}] \leq \frac{f(X_{\sigma_y})}{\inf_{z \leq x} f(z)} \leq \frac{f(y)}{f(x)},
\]
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since \( z \mapsto f(z) \) is non-increasing.

Thus we have from (3.52) that, for all \( x \) sufficiently large,

\[
\mathbb{P}[\eta_x > n] \leq \left( \frac{x}{y} \right)^\theta + \frac{C(1+y)^2}{n}.
\]  

(3.60)

Fix \( \varepsilon > 0 \). Choose

\[
y = y(x) = x(\log x)^{\frac{1}{2} + \varepsilon}, \quad \text{and} \quad n = n(x) = x^2(\log x)^{1+\frac{2}{\theta}+4\varepsilon}.
\]

Then, for all \( x \) sufficiently large,

\[
\mathbb{P}[\eta_x > n(x)] \leq (\log x)^{-1-\theta \varepsilon} + (\log x)^{-1-\varepsilon}.
\]

So \( \sum_{k \in \mathbb{Z}_+} \mathbb{P}[\eta_{2k} > n(2^k)] < \infty \). Hence the Borel–Cantelli lemma shows that, a.s., \( \eta_{2k} \leq n(2^k) \) for all but finitely many \( k \in \mathbb{Z}_+ \).

Any \( x \in \mathbb{R}_+ \) has \( 2^{k(x)} \leq x < 2^{k(x)+1} \) for some \( k(x) \in \mathbb{Z}_+ \) with \( k(x) \to \infty \) as \( x \to \infty \). Then for all \( x \) sufficiently large, a.s.,

\[
\eta_x \leq \eta_{2^{k(x)+1}} \leq n(2 \cdot 2^{k(x)}) \leq n(2x) \leq 5n(x),
\]

since \( n(\cdot) \) is eventually increasing. So we get that for any \( \varepsilon > 0 \), a.s., for all \( x \) sufficiently large,

\[
\eta_x \leq x^2(\log x)^{1+\frac{2}{\theta}+\varepsilon}.
\]

In other words, since (by Theorem 3.5.1) \( X_n \to \infty \), we have that, a.s., for all but finitely many \( n \),

\[
n \leq \eta_{X_n} \leq X_n^2(\log X_n)^{1+\frac{2}{\theta}+\varepsilon}.
\]

The statement of the theorem follows. \( \square \)

3.11 Nullity and weak limits

For this section, we restrict attention to the Markov case as described in Section 3.2. We are interested here in the null case, when, roughly speaking, the probability mass of the process escapes to infinity. The following result gives a sufficient condition for the null property, formalized by recalling from Section 3.6 the notation for occupation times \( L_n(x) = \sum_{m=0}^{n} 1\{X_m = x\} \).
Theorem 3.11.1. Suppose that (M0) and (M1) hold. Suppose that $0 < \liminf_{x \to \infty} \mu_2(x) \leq \limsup_{x \to \infty} \mu_2(x) < \infty$, and, for some constants $\theta > 0$ and $x_0 \in \mathbb{R}_+$, for all $x \geq x_0$,

$$2x\mu_1(x) + \left(1 + \frac{1 - \theta}{\log x}\right)\mu_2(x) \geq 0.$$ 

Then for any $x \in S$, $\lim_{n \to \infty} n^{-1}L_n(x) = 0$ a.s. and in $L^q$ for any $q \geq 1$.

Proof. Under the conditions of the theorem, Theorem 3.7.3 applies, and shows that there exists $x \in S$ such that $E[\lambda_x \mid F_0] = \infty$ on $\{X_0 > x\}$. Choose $y \in S$ with $y > x$; then, on $\{\tau_y < \tau_0^+\}$, we have

$$E[\tau_0^+ \mid F_{\tau_y}] \geq E[\min\{n \geq 0 : X_{\tau_y+n} \leq x\} \mid F_{\tau_y}] = \infty.$$ 

Moreover, by Lemma 3.6.10, $P_0[\tau_y < \tau_0^+] > 0$, and so

$$E_0[\tau_0^+] \geq E_0[E[\tau_0^+ \mid F_{\tau_y}]1\{\tau_y < \tau_0^+\}] = \infty.$$ 

In the notation of Section 3.6, we have shown that $E\kappa_1 = \infty$, and then the statement follows from Theorem 3.6.9. \qed

To proceed further, we impose the additional assumption (M2); Theorem 3.11.1 applies when $b > 0$ and $2a + b \geq 0$. To obtain a non-trivial distributional limit theorem, we must scale $X_n$. The diffusive bounds given in Theorem 3.2.7 suggest $n^{-1/2}X_n$ as the most likely candidate for a non-trivial limit; we will see that this is indeed the case. The limit statement will involve the distribution function $F_{\alpha,\theta}$ defined for parameters $\alpha > 0$ and $\theta > 0$ as the (normalized) lower incomplete Gamma function,

$$F_{\alpha,\theta}(x) = \frac{\gamma(\alpha, x^2/\theta)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^{x^2/\theta} z^{\alpha-1} e^{-z} dz, \quad (x \geq 0). \quad (3.61)$$

Note that, if $Z \sim \Gamma(\alpha, \theta)$ is a Gamma random variable with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$, then $P[\sqrt{Z} \leq x] = F_{\alpha,\theta}(x)$. (In the special case with $\alpha = 1/2$ and $\theta = 2$, $F_{\alpha,\theta}$ is the distribution of the square-root of a $\chi^2$ random variable with one degree of freedom, i.e., the absolute value of a standard normal random variable.)

Theorem 3.11.2. Suppose that (M0) and (M1) hold, and that (M2) holds for $b > 0$ and $2a + b > 0$. Set

$$\alpha = \frac{b + 2a}{2b}, \quad \text{and} \quad \theta = 2b. \quad (3.62)$$

Then, for any $x \geq 0$,

$$\lim_{n \to \infty} P[n^{-1/2}X_n \leq x] = F_{\alpha,\theta}(x).$$
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The remainder of this section is devoted to the proof of Theorem 3.11.2. Let \( m \in \mathbb{N} \). We define an auxiliary process \((\tilde{X}_{m,n}, n \geq 0)\) coupled to \((X_n, n \geq 0)\) via truncation of increments. As usual, we write \( \Delta_n = X_{n+1} - X_n \); also write \( \Delta_{m,n} = \tilde{X}_{m,n+1} - \tilde{X}_{m,n} \). For some \( \gamma \in (0, 1) \) to be specified later, let

\[
b_m(x) := (\max\{x, m^{1/2}\})^\gamma.
\]

We will define our truncation via the function

\[
T_m(x, h) := \begin{cases}
  h & \text{if } |h| \leq b_m(x) \\
  b_m(x) & \text{if } h > b_m(x) \\
  -b_m(x) & \text{if } h < -b_m(x).
\end{cases}
\]

The pair \((X_n, \tilde{X}_{m,n})\) for \( n \geq 0 \) will be a time-homogeneous Markov process on \( \mathbb{R}^2_+ \). We take \( \tilde{X}_{m,0} = X_0 \in \mathbb{R}_+ \). The transition law of \((X_n, \tilde{X}_{m,n})\) is as follows. Given \( X_n = \tilde{X}_{m,n} \in \mathbb{R}_+ \), we take \( X_{n+1} = X_n + \Delta_n \), and \( \tilde{X}_{m,n+1} = X_n + T_m(X_n, \Delta_n) \).

Given \( X_n \neq \tilde{X}_{m,n} \), we take \( \Delta_{m,n} \) to be independent of \( \Delta_n \) but with the appropriate truncated distribution, i.e., for any Borel subsets \( B_1, B_2 \) of \( \mathbb{R} \),

\[
\mathbb{P}[\Delta_n, \tilde{\Delta}_{m,n} \in B_1 \times B_2 \mid (X_n, \tilde{X}_{m,n} = (x, y))] = \mathbb{P}[\Delta_n \in B_1 \mid X_n = x] \mathbb{P}[T_m(y, \Delta_n) \in B_2 \mid X_n = y], \text{ for all } x \neq y.
\]

With this coupling, the process \((X_n, n \geq 0)\) is our original Markov process, and \((\tilde{X}_{m,n}, n \geq 0)\) is also time-homogeneous and Markov. Note that \( |\tilde{\Delta}_{m,n}| \leq b_m(\tilde{X}_{m,n}), \text{ a.s.} \) by construction. We write

\[
\tilde{\mu}_k(m, x) = \mathbb{E}[\tilde{\Delta}_{m,n}^k \mid \tilde{X}_{m,n} = x],
\]

for the moments of the increments of \( \tilde{X}_{m,n} \).

By construction of the coupled Markov processes,

\[
\mathbb{P}[\tilde{X}_{m,n+1} \neq X_{n+1} \mid X_n = \tilde{X}_{m,n}] \leq \mathbb{P}[|\Delta_n| > b_m(X_n) \mid X_n = \tilde{X}_{m,n}]
\leq \mathbb{P}[|\Delta_n| > m^{\gamma/2} \mid X_n]
\leq Cm^{-p\gamma/2},
\]

by Markov’s inequality and (M1). From this point onwards, we choose (and fix) \( \gamma \in (2/p, 1) \), which is possible since \( p > 2 \). Then

\[
\mathbb{P}\left[ \bigcup_{0 \leq n \leq m} \{X_n \neq \tilde{X}_{m,n}\} \right] \leq C \sum_{n=0}^{m} m^{-p\gamma/2} = o(1),
\]
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as \( m \to \infty \). Hence, for any \( \varepsilon > 0 \),

\[
\lim_{m \to \infty} \mathbb{P}[|X_m - \tilde{X}_{m,m}| > \varepsilon] = 0.
\]

Thus to complete the proof of the theorem it suffices to show that

\[
\lim_{m \to \infty} \mathbb{P}[m^{-1/2} \tilde{X}_{m,m} \leq x] = F_{\alpha,\theta}(x). \quad (3.63)
\]

Note from (M2) that \( 2x\mu_1(x) + \mu_2(x) = 2a + b + o(1) \), and so \( X_n \) satisfies the conditions of Theorem 3.11.1. In particular, since \( \Sigma \) is locally finite,

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \mathbb{P}[X_{\ell} \leq x] = 0,
\]

for any \( x \in \mathbb{R}_+ \). Moreover,

\[
\left| \frac{1}{m} \sum_{\ell=0}^{m-1} \mathbb{P}[X_{\ell} \leq x] - \frac{1}{m} \sum_{\ell=0}^{m-1} \mathbb{P}[\tilde{X}_{m,\ell} \leq x] \right| \leq \mathbb{P}\left[ \bigcup_{0 \leq n \leq m} \{X_n \neq \tilde{X}_{m,n}\} \right],
\]

which tends to 0, so that

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \mathbb{P}[\tilde{X}_{m,\ell} \leq x] = 0. \quad (3.64)
\]

By definition of \( \tilde{\Delta}_{m,n} \), we have that

\[
|\tilde{\mu}_k(m, x) - \mu_k(x)| \leq \mathbb{E}[|\Delta_n|^{k \{ \Delta_n > x^\gamma \}} \mid X_n = x],
\]

so that for \( k \in \{1, 2\} \), by (M1) and a similar argument to Lemma 3.4.2,

\[
|\tilde{\mu}_k(m, x) - \mu_k(x)| \leq Cx^{\gamma(k-p)} = o_x(x^{k-2}), \quad (3.65)
\]

since \( \gamma > \frac{2}{p} > \frac{1}{p-1} \), where the notation \( o_x \) emphasizes that the error term is uniform in \( m \) as \( x \to \infty \). Moreover, for integer \( k \geq 3 \),

\[
\mathbb{E}[|\tilde{\Delta}_{m,n}|^k \mid \tilde{X}_{m,n} = x] \leq \mathbb{E}[|\Delta_{m,n}|^2 | \tilde{\Delta}_{m,n}|^{k-2} \mid \tilde{X}_{m,n} = x] \\
\leq b_n(x)^{k-2} \mathbb{E}[\tilde{\Delta}_{m,n}^2 \mid \tilde{X}_{m,n} = x] \\
\leq Cb_n(x)^{k-2}. \quad (3.66)
\]

The distributional convergence of \( \tilde{X}_{m,m} \) will be deduced by the method of moments, based on the following result.
Lemma 3.11.3. For any $j \in \mathbb{N}$,

\[
\lim_{m \to \infty} \left( m^{-j} \sup_{0 \leq n \leq m} \left| \mathbb{E}[\tilde{X}_{m,n}^{2j}] - n^j \prod_{i=1}^{j} (2a + (2i-1)b) \right| \right) = 0. \tag{3.67}
\]

Proof. We use induction on $j$. First, for $j = 1$, for $\ell \in \mathbb{Z}_+$,

\[
\mathbb{E}[\tilde{X}_{m,\ell+1}^2 - \tilde{X}_{m,\ell}^2 | \tilde{X}_{m,\ell} = x] = \mathbb{E}[(x + \tilde{\Delta}_{m,\ell})^2 - x^2 | \tilde{X}_{m,\ell} = x] = 2x\tilde{\mu}_1(m, x) + \tilde{\mu}_2(m, x). \tag{3.68}
\]

By (3.65) and our assumptions on $\mu_1$ and $\mu_2$,

\[
\lim_{x \to \infty} \sup_{m \in \mathbb{N}} |x\tilde{\mu}_1(m, x) - a| = 0, \text{ and } \lim_{x \to \infty} \sup_{m \in \mathbb{N}} |\tilde{\mu}_2(m, x) - b| = 0. \tag{3.69}
\]

For compactness, we write these last expressions as

\[
x\tilde{\mu}_1(m, x) = a + o_1(x) \quad \text{and} \quad \tilde{\mu}_2(m, x) = b + o_1(x),
\]

where again the subscript $x$ on the error term indicates that it is uniform in $m$ as $x \to \infty$. Hence for any $\varepsilon > 0$ there exist $x \in \mathbb{R}_+$ and $C \in \mathbb{R}_+$ such that

\[
\left| \mathbb{E}[\tilde{X}_{m,\ell+1}^2 - \tilde{X}_{m,\ell}^2 | \tilde{X}_{m,\ell} = x] - (2a + b) \right| \leq \varepsilon + C \mathbb{P}[\tilde{X}_{m,\ell} \leq x].
\]

Hence, taking expectations,

\[
\left| \mathbb{E}[\tilde{X}_{m,\ell+1}^2] - \mathbb{E}[\tilde{X}_{m,\ell}^2] - (2a + b) \right| \leq \varepsilon + C \mathbb{P}[\tilde{X}_{m,\ell} \leq x].
\]

Summing from $\ell = 0$ to $\ell = n - 1$ we obtain

\[
\sup_{0 \leq \ell \leq m} \left| \mathbb{E}[\tilde{X}_{m,\ell}^2] - \mathbb{E}[\tilde{X}_{m,0}^2] - n(2a + b) \right| \leq m\varepsilon + C \sum_{\ell=0}^{m-1} \mathbb{P}[\tilde{X}_{m,\ell} \leq x] \leq 2m\varepsilon,
\]

for all $m \geq m_0$ sufficiently large, by (3.64). In other words, since $\varepsilon > 0$ was arbitrary,

\[
m^{-1} \sup_{0 \leq \ell \leq m} \left| \mathbb{E}[\tilde{X}_{m,\ell}^2] - n(2a + b) \right| \to 0,
\]

as $m \to \infty$, which is the $j = 1$ case of (3.67).

For the inductive step, suppose that (3.67) holds for all $j \in \{1, 2, \ldots, k-1\}$ for some $k \geq 2$. Consider

\[
\mathbb{E}[\tilde{X}_{m,\ell+1}^{2k} - \tilde{X}_{m,\ell}^{2k} | \tilde{X}_{m,\ell} = x] = \mathbb{E}[(x + \tilde{\Delta}_{m,\ell})^{2k} - x^{2k} | \tilde{X}_{m,\ell} = x]
\]
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\[= \sum_{i=1}^{2k} \binom{2k}{i} x^{2k-i} \tilde{\mu}_i(m, x),\]

by the binomial formula. Hence, using (3.69) to estimate the terms \(i = 1\) and \(i = 2\) in the sum, we obtain

\[
\mathbb{E}[\tilde{X}^{2k}_{m,\ell+1} - \tilde{X}^{2k}_{m,\ell} | \tilde{X}_{m,\ell} = x]
\]

\[= k(2a + (2k - 1)b + o_x(1)) x^{2k-2} + \sum_{i=3}^{2k} \binom{2k}{i} x^{2k-i} \tilde{\mu}_i(m, x). \quad (3.70)\]

Also, we have from (3.66) that, for all \(i \geq 3,\)

\[|\tilde{\mu}_i(m, x)| \leq C x^{\gamma(i-2)} + C m^{\gamma(i-2)/2} 1\{x^2 < m\}\]

\[\leq C x^{\gamma(i-2)} + C m^{(i-2)/2} m^{-(1-\gamma)/2} 1\{x^2 < m\}\]

\[\leq C x^{\gamma(i-2)} + C m^{(i-2)/2} x^{-(1-\gamma)/4}.\]

Here, since \(\gamma < 1,\)

\[\sum_{i=3}^{2k} \binom{2k}{i} x^{2k-i} x^{\gamma(i-2)} = O_x(x^{2k+\gamma-3}) = o_x(x^{2k-2}),\]

so we have from (3.70) that

\[|\mathbb{E}[\tilde{X}^{2k}_{m,\ell+1} - \tilde{X}^{2k}_{m,\ell} | \tilde{X}_{m,\ell} = x] - k(2a + (2k - 1)b + o_x(1)) x^{2k-2}|
\]

\[\leq o_x(1) \sum_{i=3}^{2k} \binom{2k}{i} x^{2k-i} m^{(i-2)/2}.\]

In other words, for any \(\varepsilon > 0\) there exist constants \(x \in \mathbb{R}_+\) and \(C \in \mathbb{R}_+\) (depending on \(k\) but not on \(\ell\) or \(m\)) so that

\[|\mathbb{E}[\tilde{X}^{2k}_{m,\ell+1} - \tilde{X}^{2k}_{m,\ell} | \tilde{X}_{m,\ell} = x] - k(2a + (2k - 1)b) \tilde{X}^{2k-2}_{m,\ell}|
\]

\[\leq \varepsilon \sum_{i=2}^{2k} \binom{2k}{i} \tilde{X}^{2k-i} m^{(i-2)/2} + C m^{k-1} 1\{\tilde{X}_{m,\ell} \leq x\}.
\]

By the inductive hypothesis (3.67) and Lyapunov’s inequality, we have that for all \(i \in \{2, 3, \ldots, 2k\}\) and all \(\ell \leq m,\)

\[\mathbb{E}[\tilde{X}^{2k-i}_{m,\ell}] \leq \left(\mathbb{E}[\tilde{X}^{2k-2}_{m,\ell}]\right)^{\frac{2k-i}{2k-2}} \leq C \ell^{k-(i/2)} \leq C m^{k-(i/2)},\]

\[\mathbb{E}[\tilde{X}^{2k-2}_{m,\ell}] \leq \left(\mathbb{E}[\tilde{X}^{2k-2}_{m,\ell}]\right)^{\frac{2k-1}{2k-2}} \leq C \ell^{k-(i/2)} \leq C m^{k-(i/2)},\]

\[\mathbb{E}[\tilde{X}^{2k-2}_{m,\ell}] \leq \left(\mathbb{E}[\tilde{X}^{2k-2}_{m,\ell}]\right)^{\frac{2k-1}{2k-2}} \leq C \ell^{k-(i/2)} \leq C m^{k-(i/2)},\]
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for some $C \in \mathbb{R}_+$ (depending on $k$ but not on $m$). Thus, for all $\ell \leq m$,

$$
\sum_{i=2}^{2k} \binom{2k}{i} E[\tilde{X}_{m,\ell}^{2k-i}] m^{(i-2)/2} \leq C m^{k-1},
$$

where, again, $C \in \mathbb{R}_+$ depends on $k$ but not on $m$. Together with the $j = k - 1$ case of (3.67), it follows that for any $\varepsilon > 0$, for all $0 \leq \ell \leq m$ with $m \geq m_0$ sufficiently large,

$$
\left| E[\tilde{X}_{m,\ell+1}^{2k}] - E[\tilde{X}_{m,\ell}^{2k}] - k\ell^{k-1} \prod_{i=1}^{k} (2a + (2i - b)) \right|
\leq \varepsilon m^{k-1} + C m^{k-1} \mathbb{P}[\tilde{X}_{m,\ell} \leq x].
$$

(3.71)

Hence summing from $\ell = 0$ to $\ell = n - 1$ in (3.71) we see that, for any $\varepsilon > 0$, for all $m$ sufficiently large,

$$
\sup_{0 \leq n \leq m} \left| E[\tilde{X}_{m,n}^{2k}] - E[\tilde{X}_{m,0}^{2k}] - n^{k} \prod_{i=1}^{k} (2a + (2i - b)) \right|
\leq \varepsilon m^{k} + C m^{k} \left( \frac{1}{m} \sum_{\ell=0}^{m-1} \mathbb{P}[\tilde{X}_{m,\ell} \leq x] \right) \leq 2\varepsilon m^{k},
$$

by (3.64). This verifies the $j = k$ case of (3.67), and completes the induction.

Proof of Theorem 3.11.2. We appeal to the method of moments (see e.g. [123, §8.4]): Lemma 3.11.3 shows that, for all $j \in \mathbb{N}$,

$$
\lim_{m \to \infty} E \left[ (m^{-1} \tilde{X}_{m,m}^{2})^j \right] = (2b)^j \frac{\Gamma \left( \frac{q}{b} + \frac{1}{2} + j \right)}{\Gamma \left( \frac{q}{b} + \frac{1}{2} \right)} = E[Y^j],
$$

where $Y$ is a gamma-(\(a, \theta\)) random variable with $a = \frac{q}{b} + \frac{1}{2}$ and $\theta = 2b$. Since the law of $Y$ is determined uniquely by its moments, it follows that $m^{-1} \tilde{X}_{m,m}^{2}$ converges in distribution to $Y$, and hence, as $\tilde{X}_{m,m} \geq 0$, we get $m^{-1/2} \tilde{X}_{m,m}$ converges to $\sqrt{Y}$. This establishes (3.63) and hence proves the theorem.

\[\square\]
3.12 Supercritical case

In this section we turn to the supercritical case where $\bar{\mu}_1(x)$ and $\mu_1(x)$ are positive and of order $x^{-\beta}$ for some $\beta \in (0, 1)$, in the general setting of Section 3.3. In this case, under mild conditions, transience is assured: our primary interest is to quantify this transience by studying the rate of escape and accompanying second-order behaviour.

As before, we need to assume a moments condition on the increments $\Delta_n = X_{n+1} - X_n$.

(L5) Suppose that for some $C \in \mathbb{R}_+$, $\mathbb{E}[|\Delta_n|^{p} | \mathcal{F}_n] \leq C$, a.s.

If (L5) holds for $p \geq 1$, then $\bar{\mu}_1$ and $\mu_1$ given by (3.5) and (3.6) exist as $\mathbb{R}$-valued functions. The next result gives transience of $X_n$.

**Theorem 3.12.1.** Suppose that (L0) and (L2) hold, and that there exists $\beta \in [0, 1)$ such that (L5) holds for some $p > 1 + \beta$, and

$$\liminf_{x \to \infty} (x^\beta \bar{\mu}_1(x)) > 0.$$  

Then $X_n \to \infty$ a.s. as $n \to \infty$.

The main topic of this section is the quantitative asymptotic behaviour of $X_n$. Our results involve the constants $\lambda(a, \beta)$ defined by

$$\lambda(a, \beta) := (a(1 + \beta))^{\frac{1}{1+\beta}}. \quad (3.72)$$

The next result gives sharp almost-sure bounds on $X_n$, and shows that for $\beta \in (0, 1)$ the transience given in Theorem 3.12.1 is super-diffusive but sub-ballistic, since the exponent $\frac{1}{1+\beta}$ is greater than $\frac{1}{2}$ but less than 1.

**Theorem 3.12.2.** Suppose that (L0) and (L2) hold. Suppose that there exist $\beta \in [0, 1)$ and $a, A$ with $0 < a \leq A < \infty$ such that

$$a = \liminf_{x \to \infty} (x^\beta \mu_1(x)) \leq \limsup_{x \to \infty} (x^\beta \bar{\mu}_1(x)) = A, \quad (3.73)$$

and (L5) holds for some $p > 2 + 2\beta$. Then, a.s.,

$$\lambda(a, \beta) \leq \liminf_{n \to \infty} n^{-\frac{1}{1+\beta}} X_n \leq \limsup_{n \to \infty} n^{-\frac{1}{1+\beta}} X_n \leq \lambda(A, \beta).$$

**Remark 3.12.3.** The proof of the upper bound on $X_n$ given by Theorem 3.12.2 only uses the condition on $\bar{\mu}_1$ in (3.73), and not the condition on $\mu_1$ there.
A corollary of Theorem 3.12.2, obtained on taking $a = A = \rho$, is the following strong law of large numbers.

**Theorem 3.12.4.** Suppose that (L0) and (L2) hold. Suppose that there exists $\beta \in [0,1)$ such that (L5) holds for some $p > 2 + 2\beta$, and

$$
\lim_{x \to \infty} x^\beta \bar{\mu}_1(x) = \lim_{x \to \infty} x^\beta \mu_1(x) = \rho \in (0, \infty).
$$

(3.74)

Then

$$
\lim_{n \to \infty} n^{-1/\beta} X_n = \lambda(\rho, \beta), \text{ a.s.}
$$

(3.75)

The next result shows that there is a central limit theorem to accompany the law of large numbers in Theorem 3.12.4, provided that we impose a somewhat stronger version of (3.74) and an asymptotic stability condition on the second moments of the increments. Unlike the preceding results in this section, the case $\beta = 0$ is excluded from the following theorem.

**Theorem 3.12.5.** Suppose that (L0) and (L2) hold. Suppose that there exist $\beta \in (0,1)$ and $\rho \in (0, \infty)$ such that (L5) holds for some $p > \max\{2 + 2\beta, \frac{3 + \beta}{1 + \beta}\}$, and

$$
\mu_1(x) = \rho x^{-\beta} + o(x^{-\beta - \frac{1}{2}}); \quad \bar{\mu}_1(x) = \rho x^{-\beta} + o(x^{-\beta - \frac{1}{2}}).
$$

(3.76)

Suppose also that there exists $\sigma^2 \in (0, \infty)$ such that

$$
E[\Delta_n^2 \mid F_n] \to \sigma^2, \text{ a.s.}
$$

(3.77)

Then, as $n \to \infty$,

$$
n^{-1/2} \left( X_n - \lambda(\rho, \beta)n^{1/\beta} \right) \xrightarrow{d} Z\sqrt{\frac{1 + \beta}{1 + 3\beta}},
$$

where $Z$ is a standard normal random variable.

**Example 3.12.6.** We consider a nearest-neighbour walk that generalizes slightly the model of Section 2.2 by allowing transitions whereby the walk stays at the same site. Specifically, suppose that there exist sequences $a_x, b_x, c_x$ ($x \in \mathbb{N}$) with $a_x > 0$, $b_x \geq 0$, $c_x > 0$ and $a_x + b_x + c_x = 1$. Define the transition law of $X_n$ as follows: for $x \in \mathbb{N}$,

$$
\mathbb{P}[X_{n+1} = x + 1 \mid X_n = x] = a_x,
$$

$$
\mathbb{P}[X_{n+1} = x \mid X_n = x] = b_x,
$$

and

$$
\mathbb{P}[X_{n+1} = x - 1 \mid X_n = x] = c_x.
$$

(3.78)
3.12. Supercritical case

\[ \mathbb{P}[X_{n+1} = x - 1 \mid X_n = x] = c_x, \]

and with reflection from 0 governed by \( \mathbb{P}[X_{n+1} = 1 \mid X_n = 0] = a_0 \in (0, 1) \).

For \( x \in \mathbb{N} \), with \( \mu_1(x) \) and \( \mu_2(x) \) defined by (3.1) we have that

\[ \mu_1(x) = a_x - c_x, \quad \text{and} \quad \mu_2(x) = 1 - b_x > 0. \]

The asymptotically zero drift case is the case where \( \lim_{x \to \infty} (a_x - c_x) = 0 \).

We are in the supercritical case if, for \( \beta \in (0, 1) \),

\[ \lim_{x \to \infty} x^\beta (a_x - c_x) = \rho \in (0, \infty). \quad (3.78) \]

In this case Theorem 3.12.4 implies that (3.75) holds. Moreover, if we suppose that

\[ a_x - c_x = \rho x^{-\beta} + o(x^{-\beta - \frac{1 - \beta}{2}}); \quad \lim_{x \to \infty} b_x = b \in [0, 1), \]

then Theorem 3.12.5 yields the central limit theorem

\[ \frac{X_n - \lambda(\rho, \beta)n^{1/(1+\beta)}}{n^{1/2}} \xrightarrow{d} Z \sqrt{\frac{(1-b)(1+\beta)}{1+3\beta}}. \]

In the rest of this section we give the proofs of the theorems stated above. We start with a technical result. Again we use the notation \( E_\varepsilon(n) = \{|\Delta_n| \leq (1 + X_n)^{1-\varepsilon}\} \) as at (3.19). The next truncation result is a relative of Lemma 3.4.2.

**Lemma 3.12.7.** Suppose that (L0) holds and that (L5) holds for \( p > 0 \). For any \( q \in [0, p] \) and \( \varepsilon \in (0, 1) \) there exists \( C \in \mathbb{R}_+ \) for which, for all \( n \geq 0 \),

\[ \mathbb{E}[|\Delta_n|^q 1(E_\varepsilon(n)) \mid \mathcal{F}_n] \leq C(1 + X_n)^{-(q-p)(1-\varepsilon)}, \quad \text{a.s.} \]

**Proof.** For any \( n \geq 0 \),

\[ |\Delta_n|^q 1(E_\varepsilon(n)) = |\Delta_n|^p |\Delta_n|^{q-p} 1(E_\varepsilon(n)) \leq |\Delta_n|^p (1 + X_n)^{(q-p)(1-\varepsilon)}. \]

Taking expectations and using (L5) gives the result.

Next we give the proof of the transience result, Theorem 3.12.1.
Proof of Theorem 3.12.1. We use the Lyapunov function $f(x) = (1 + x)^{-\gamma}$, \( \gamma > 0 \). We claim that there exist \( \gamma > 0 \) and \( x_0 \in \mathbb{R}_+ \) such that for all \( n \geq 0 \),
\[
\mathbb{P}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq 0, \quad \text{on } \{X_n \geq x_0\}. \tag{3.79}
\]
Note that our assumption on \( \mu_1 \) implies that for some \( c > 0 \) and \( x_1 \in \mathbb{R}_+ \),
\[
\mathbb{E}[\Delta_n \mid \mathcal{F}_n] \geq c X_n^{-\beta} \quad \text{on } \{X_n \geq x_1\}. \]
Let \( \varepsilon \in (0, 1) \) be such that \( p\varepsilon < p - 1 - \beta \). Since \( f \) is decreasing,
\[
f(X_{n+1}) - f(X_n) \leq (f(X_{n+1}) - f(X_n)) \mathbb{1}(E_\varepsilon(n))
+ (f(X_{n+1}) - f(X_n)) \mathbb{1}\{\Delta_n \leq -(1 + X_n)^{1-\varepsilon}\}. \tag{3.80}
\]
Taylor’s formula with Lagrange remainder shows that
\[
(f(X_{n+1}) - f(X_n)) \mathbb{1}(E_\varepsilon(n))
= -\gamma (1 + X_n)^{-\gamma-1} (1 + O_{X_n}^\varepsilon(X_n^{-\varepsilon})) \Delta_n \mathbb{1}(E_\varepsilon(n)). \tag{3.81}
\]
Since (L5) holds for \( p > 1 + \beta \geq 1 \), we may apply the \( q = 1 \) case of Lemma 3.12.7 to obtain
\[
\mathbb{E}[\Delta_n \mathbb{1}(E_\varepsilon(n)) \mid \mathcal{F}_n] = \mathbb{E}[\Delta_n \mid \mathcal{F}_n] + O_{X_n}^\varepsilon(X_n^{-(p-1)+p\varepsilon}), \tag{3.82}
\]
and this last error term is \( o_{X_n}^\varepsilon(X_n^{-\beta}) \) by choice of \( \varepsilon \). Hence taking expectations in (3.81) and using (3.82) we obtain
\[
\mathbb{E}[(f(X_{n+1}) - f(X_n)) \mathbb{1}(E_\varepsilon(n)) \mid \mathcal{F}_n] \leq -\gamma X_n^{-\gamma-1-\beta} (c + o_{X_n}^\varepsilon(1)). \tag{3.83}
\]
On the other hand, since \( f(x) \in [0, 1] \), the \( q = 0 \) case of Lemma 3.12.7 yields
\[
\mathbb{E}[\mathbb{1}\{f(X_{n+1}) - f(X_n) \leq -(1 + X_n)^{1-\varepsilon}\} \mid \mathcal{F}_n]
\leq \mathbb{P}[E_\varepsilon^n \mid \mathcal{F}_n] = O_{X_n}^\varepsilon(X_n^{-(p-1)+\varepsilon}) = o_{X_n}^\varepsilon(X_n^{-\gamma-1-\beta}), \tag{3.84}
\]
provided we take \( \gamma > 0 \) with \( \gamma < p - 1 - \beta - p\varepsilon \), which is possible by choice of \( \varepsilon \). From (3.80) with (3.83) and (3.84), we therefore conclude that, for \( \gamma > 0 \) sufficiently small,
\[
\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathcal{F}_n] \leq -(c\gamma + o_{X_n}^\varepsilon(1)) X_n^{-\gamma-1-\beta},
\]
which yields the claim (3.79). Transience now follows from an application of Theorem 3.5.6. \( \square \)

Now we turn to our main aim, and work toward the proof of Theorem 3.12.2. The following technical result will allow us to work with the process \( X_n^{1+\beta} \).
3.12. Supercritical case

Lemma 3.12.8. Suppose that (L0) holds, and that there exists $\beta \in [0, 1)$ such that (L5) holds for some $p > 1 + \beta$.

(i) If $\limsup_{x \to \infty} (x^\beta \bar{\mu}_1(x)) < \infty$, then there exists $C \in \mathbb{R}_+$ such that,
\[ E[X_{n+1}^{1+\beta} - X_n^{1+\beta} | \mathcal{F}_n] \leq C, \text{ a.s.} \quad (3.85) \]

(ii) Suppose that for some $A \in (0, \infty)$, $\limsup_{x \to \infty} (x^\beta \bar{\mu}_1(x)) \leq A$. Then for any $\varepsilon > 0$ there exists $x \in \mathbb{R}_+$ such that for all $n \geq 0$,
\[ E[X_{n+1}^{1+\beta} - X_n^{1+\beta} | \mathcal{F}_n] \leq A(1 + \beta) + \varepsilon, \text{ on } \{X_n \geq x\}. \quad (3.86) \]

(iii) Suppose that for some $a \in (0, \infty)$, $\liminf_{x \to \infty} (x^\beta \bar{\mu}_1(x)) \geq a$. Then for any $\varepsilon > 0$ there exists $x \in \mathbb{R}_+$ such that for all $n \geq 0$,
\[ E[X_{n+1}^{1+\beta} - X_n^{1+\beta} | \mathcal{F}_n] \geq a(1 + \beta) - \varepsilon, \text{ on } \{X_n \geq x\}. \quad (3.87) \]

(iv) Let $r \in [1, \frac{p}{1+\beta})$. Then there exists $C \in \mathbb{R}_+$ such that for all $n \geq 0$,
\[ E[|X_{n+1}^{1+\beta} - X_n^{1+\beta}|^r | \mathcal{F}_n] \leq C(1 + X_n^{\beta r}), \text{ a.s.} \quad (3.88) \]

Proof. Let $\varepsilon \in (0, 1)$ be such that $p \varepsilon < p - 1 - \beta$. By a Taylor’s formula argument that should now be familiar, we have that
\[ X_{n+1}^{1+\beta} - X_n^{1+\beta} = (1 + \beta + O_{X_n}(X_n^{-\varepsilon}))X_n^\beta \Delta_n \mathbf{1}(E_\varepsilon(n)) + R_n, \quad (3.89) \]
where $R_n = (X_{n+1}^{1+\beta} - X_n^{1+\beta}) \mathbf{1}(E_\varepsilon^c(n))$. Since on $E_\varepsilon(n)$ we have $X_n \leq |\Delta_n|^{1/(1-\varepsilon)}$, it follows that, by choice of $\varepsilon$,
\[ |R_n| \leq C|\Delta_n|^{1+\beta+pe} \mathbf{1}(E_\varepsilon^c(n)), \text{ a.s.,} \quad (3.90) \]
for some absolute constant $C \in \mathbb{R}_+$. Then taking expectations in (3.90) and using the $q = 1 + \beta + p \varepsilon < p$ case of Lemma 3.12.7 we have that for $n \geq 0$,
\[ E[|R_n| | \mathcal{F}_n] \leq C(1 + X_n)^{(p-1-\beta-pe)(1-\varepsilon)} = o_{X_n}(1), \]
by choice of $\varepsilon$. Also, as at (3.82), we have that
\[ E[\Delta_n \mathbf{1}(E_\varepsilon(n)) | \mathcal{F}_n] = E[\Delta_n | \mathcal{F}_n] + o_{X_n}(X_n^{-\beta}), \]
by choice of $\varepsilon$. It follows from (3.89) that, a.s., for all $n \geq 0$,
\[ E[X_{n+1}^{1+\beta} - X_n^{1+\beta} | \mathcal{F}_n] = (1 + \beta + o_{X_n}(1))X_n^\beta E[\Delta_n | \mathcal{F}_n] + o_{X_n}(1). \]
Chapter 3. Lamperti’s problem

Under the conditions of part (ii) of the lemma we have that $E[\Delta_n \mid \mathcal{F}_n] \leq (A + o^\mathcal{F}_n(1))X_n^{-\beta}$, and (3.86) follows. Similarly (3.87) follows under the conditions of part (iii) of the lemma.

Next we prove part (iv) of the lemma. From (3.89) and (3.90), for any $\varepsilon > 0$ small enough,

$$|X_{n+1}^{1+\beta} - X_n^{1+\beta}|^r \leq C(1 + X_n)^{\beta r}|\Delta_n|^r + C|\Delta_n|^{(1+\beta+p\varepsilon)r},$$

for a constant $C \in \mathbb{R}_+$. Given $r < \frac{p}{1+\beta}$, we may choose $\varepsilon > 0$ small enough so that $(1 + \beta + p\varepsilon)r \leq p$. Then (L5) shows that both $E[|\Delta_n|^r \mid \mathcal{F}_n]$ and $E[|\Delta_n|^{(1+\beta+p\varepsilon)r} \mid \mathcal{F}_n]$ are uniformly bounded above. Thus (3.88) follows.

Finally, part (i) of the lemma follows on combining part (ii) with the $r = 1$ case of part (iv). \qed

Under conditions of comparable strength to those in Theorem 3.12.1, we have the following upper bound, which, while relatively crude, will be useful in the proofs of the sharper results of this section.

Lemma 3.12.9. Suppose that (L0) and (L2) hold, and that there exists $\beta \in (0, 1)$ such that (L5) holds for some $p > 1 + \beta$, and $\limsup_{x \to \infty} (x^{\beta} \mu_1(x)) < \infty$. Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \geq 0$,

$$\max_{0 \leq k \leq n} X_k \leq n^{\frac{1}{1+\beta}} (\log n)^{\frac{1}{1+\beta} + \varepsilon}.$$ 

(3.91)

Proof. Lemma 3.12.8(i) shows that under the conditions of the lemma, we may apply Theorem 2.8.1 with $f(x) = x^{1+\beta}$ and $a(x) = x(\log x)^{1+\varepsilon}$, which yields the result. \qed

For notational convenience, for the rest of this section we write

$$Y_n := X_n^{1+\beta}, \quad \text{and} \quad D_n := E[Y_{n+1} - Y_n \mid \mathcal{F}_n].$$

The Doob decomposition for $Y_n$ (see Theorem 2.3.1) is

$$Y_n = M_n + A_n,$$ 

(3.92)

where $A_0 := 0$, $A_n := \sum_{k=0}^{n-1} D_k$ for $n \geq 1$, and $M_n$ is an $\mathcal{F}_n$-adapted martingale. The next result gives conditions for $A_n$ to be a good approximation for $X_n^{1+\beta}$.

Lemma 3.12.10. Suppose that (L0) holds, and that there exists $\beta \in (0, 1)$ such that (L5) holds for some $p > 2 + 2\beta$. If $\limsup_{x \to \infty} (x^{\beta} \mu_1(x)) < \infty$, then

$$\lim_{n \to \infty} n^{-1}|X_n^{1+\beta} - A_n| = 0, \quad \text{a.s.}$$

Proof. \qed
Proof. Since \( M_n = Y_n - \sum_{k=0}^{n-1} D_k \) is a martingale, we have that
\[
\mathbb{E}[M_{n+1}^2 - M_n^2 | \mathcal{F}_n] = \mathbb{E}[(M_{n+1} - M_n)^2 | \mathcal{F}_n] = \mathbb{E}[(Y_{n+1} - Y_n - D_n)^2 | \mathcal{F}_n] = \mathbb{E}[(Y_{n+1} - Y_n)^2 | \mathcal{F}_n] - D_n^2,
\]
where we have expanded the term \((Y_{n+1} - Y_n - D_n)^2\) and used the fact that \(D_n\) is \(\mathcal{F}_n\)-measurable. Now by (3.93) and the \(r = 2\) case of (3.88) (which is valid since \(p > 2 + 2\beta\)) we have that for all \(n \geq 0\),
\[
\mathbb{E}[M_{n+1}^2 - M_n^2 | \mathcal{F}_0] = \mathbb{E}[\mathbb{E}[M_{n+1}^2 - M_n^2 | \mathcal{F}_n] | \mathcal{F}_0] \leq C + C \mathbb{E}[X_n^{2\beta} | \mathcal{F}_0].
\]
Now since \(\beta \in [0, 1)\), the (conditional) Jensen inequality implies that
\[
\mathbb{E}[X_n^{2\beta} | \mathcal{F}_0] \leq (\mathbb{E}[X_n^{1+\beta} | \mathcal{F}_0])^{2\beta} \leq (X_0^{1+\beta} + Cn)^{2\beta},
\]
by (3.85). Thus \(M_n^2\) is a non-negative submartingale with
\[
\mathbb{E}[M_n^2 | \mathcal{F}_0] \leq Y_0^2 + \sum_{k=0}^{n-1} \mathbb{E}[M_{k+1}^2 - M_k^2 | \mathcal{F}_0] \leq X_0^{2+2\beta} + n(X_0^{1+\beta} + Cn)^{2\beta}.
\]
Doob’s inequality (Theorem 2.3.14) then implies that for any \(\varepsilon > 0\),
\[
\mathbb{P}\left[ \max_{0 \leq k \leq n} M_k^2 > n^{1+\frac{3\beta}{1+\beta}} + \varepsilon \right] \leq n^{-\frac{1+\beta}{1+\beta} - \varepsilon} \mathbb{E}[M_n^2] = O(n^{-\varepsilon}).
\]
Hence the Borel–Cantelli lemma implies that for any \(\varepsilon > 0\), a.s.,
\[
\max_{0 \leq k \leq 2^m} |M_k| \leq (2^m)^{\frac{1+3\beta}{2+2\beta} + \varepsilon},
\]
for all but finitely many \(m \in \mathbb{Z}_+\). Since for any \(n \in \mathbb{N}\) we have \(2^m(n) \leq n < 2^{m(n)+1}\) for some \(m(n) \in \mathbb{Z}_+\) with \(m(n) \to \infty\) as \(n \to \infty\), we have that for any \(\varepsilon > 0\), a.s., for all but finitely many \(n \geq 0\),
\[
\max_{0 \leq k \leq 2^m} |M_k| \leq \max_{0 \leq k \leq 2^{m(n)+1}} |M_k| \leq (2^{m(n)+1})^{\frac{1+3\beta}{2+2\beta} + \varepsilon} \leq Cn^{\frac{1+3\beta}{2+2\beta} + \varepsilon},
\]
for some \(C \in \mathbb{R}_+\) not depending on \(n\). Since \(\beta \in [0, 1)\), we may take \(\varepsilon\) small enough so that \(\frac{1+3\beta}{2+2\beta} + \varepsilon \leq 1 - \varepsilon\). Then we have that \(|A_n - Y_n| \leq Cn^{1-\varepsilon}\) for all but finitely many \(n \geq 0\).
Proof of Theorem 3.12.2. Under the conditions of the theorem, we have from Lemma 3.12.8 that (3.86) and (3.87) hold. Moreover, we know from Theorem 3.12.1 that \( X_n \to \infty \) as \( n \to \infty \), a.s. Hence for any \( \varepsilon > 0 \), a.s.,

\[
a(1 + \beta) - \varepsilon \leq D_n \leq A(1 + \beta) + \varepsilon,
\]

for all but finitely many \( n \geq 0 \). Hence with \( A_n \) given in (3.92) we have that for any \( \varepsilon > 0 \), a.s.,

\[
a(1 + \beta) - \varepsilon \leq n^{-1}A_n \leq A(1 + \beta) + \varepsilon,
\]

for all but finitely many \( n \geq 0 \). Since \( \varepsilon > 0 \) was arbitrary, this means

\[
a(1 + \beta) \leq \liminf_{n \to \infty} n^{-1}A_n \leq \limsup_{n \to \infty} n^{-1}A_n \leq A(1 + \beta), \text{ a.s.}
\]

Together with Lemma 3.12.10, this implies

\[
a(1 + \beta) \leq \liminf_{n \to \infty} n^{-1}X_n^1 + \beta \leq \limsup_{n \to \infty} n^{-1}X_n^1 + \beta \leq A(1 + \beta), \text{ a.s.}
\]

and the statement in the theorem follows.

The basic ingredients of the proof of Theorem 3.12.5 are already in place in the decomposition used in the proof of Lemma 3.12.10, but we need to revisit some of our earlier estimates and obtain sharper bounds under the conditions of Theorem 3.12.5. First we have the following refinement of Lemma 3.12.8 in this case.

**Lemma 3.12.11.** Suppose that (L0) holds, and that for \( \beta \in (0,1) \) and \( \rho \in (0,\infty) \), (3.76) holds. Suppose that (L5) holds for \( p > \max\{2 + 2\beta, \frac{3 + \beta}{1 + \beta}\} \). Then, as \( n \to \infty \),

\[
n^{1+2\beta} \left( \mathbb{E}[X_{n+1}^{1+\beta} - X_n^{1+\beta} | \mathcal{F}_n] - \rho(1 + \beta) \right) \to 0, \text{ a.s.} \quad (3.94)
\]

If in addition (3.77) holds for some \( \sigma^2 \in (0,\infty) \), then

\[
n^{-2\beta} \mathbb{E}[(X_{n+1}^{1+\beta} - X_n^{1+\beta})^2 | \mathcal{F}_n] \to \sigma^2(1 + \beta)^2 \lambda(\rho, \beta) 2\beta^2, \text{ a.s.} \quad (3.95)
\]

**Proof.** We need to obtain better estimates for the error terms in (3.89) than we did in the proof of Lemma 3.12.8. For this reason the \( \varepsilon \) there will not be arbitrarily small, so we cannot use (3.90). Instead, from the argument leading to (3.90) we have

\[
|R_n| \leq C|\Delta_n|^{1+\beta} 1(E^\varepsilon(n)),
\]
so that from the \( q = \frac{1 + \beta}{1 - \varepsilon} \) case of Lemma 3.12.7, which is valid provided \( p \varepsilon \leq p - 1 - \beta \), we have
\[
\mathbb{E}[|R_n| \mid \mathcal{F}_n] \leq C(1 + X_n)^{-(p - \frac{1 + \beta}{1 - \varepsilon})(1 - \varepsilon)} = C(1 + X_n)^{-(p - 1 - \beta - p \varepsilon)}.
\]

Now choose \( \varepsilon > \frac{1 - \beta}{2} \) such that \( p - 1 - \beta - p \varepsilon > \frac{1 - \beta}{2} \); this is possible provided \( p > \frac{3 + \beta}{1 - \beta} \). Then \( \mathbb{E}[|R_n| \mid \mathcal{F}_n] = o_{X_n}(X_n^{-\frac{1 - \beta}{2}}) \). Moreover, we have that from the \( q = 1 \) case of Lemma 3.12.7,
\[
X_n^\beta \mathbb{E}[\Delta_n 1(E^c_n(n)) \mid \mathcal{F}_n] \leq C(1 + X_n)^{-(p - 1 - \beta - p \varepsilon)} = o_{X_n}(X_n^{-\frac{1 - \beta}{2}}),
\]
by choice of \( \varepsilon \). Hence
\[
X_n^\beta \mathbb{E}[\Delta_n 1(E_n^c(n)) \mid \mathcal{F}_n] = X_n^\beta \mathbb{E}[\Delta_n \mid \mathcal{F}_n] + o_{X_n}(X_n^{-\frac{1 - \beta}{2}}) = \rho + o_{X_n}(X_n^{-\frac{1 - \beta}{2}}),
\]
by (3.76). With these sharper bounds, from (3.89) and the present choice of \( \varepsilon \) we obtain
\[
\mathbb{E}[X_{n+1}^{1 + \beta} - X_n^{1 + \beta} \mid \mathcal{F}_n] = (1 + \beta) \rho + o_{X_n}(X_n^{-\frac{1 - \beta}{2}}).
\]

Under the conditions of the lemma, Theorem 3.12.4 applies so that \( X_n \sim \lambda(\rho, \beta)n^{\frac{1}{1 + \beta}} \). Thus we obtain (3.94). The argument for (3.95) is similar, starting by squaring (3.89) and then taking \( \varepsilon > 0 \) small enough, so we omit the details.

Lemma 3.12.12. Suppose that (L0) holds, and that for \( \beta \in (0, 1) \) and \( \rho \in (0, \infty) \), (3.76) holds. Suppose that (L5) holds for \( p > \max\{2 + 2\beta, \frac{3 + \beta}{1 + \beta}\} \) and (3.77) holds for \( \sigma^2 \in (0, \infty) \). Then with \( M_n \) as defined at (3.92), we have that as \( n \to \infty \),
\[
n^{-\frac{1 + 3\beta}{2 + 3\beta}} M_n \overset{d}{\to} Z \sigma \lambda(\rho, \beta)^\beta \sqrt{\frac{(1 + \beta)^3}{(1 + 3\beta)}},
\]
where \( Z \) is a standard normal random variable.

Proof. We will apply a standard martingale central limit theorem. Set \( M_{n,k} = n^{-\frac{1 + 3\beta}{2 + 3\beta}}(M_k - M_{k-1}) \) for \( 1 \leq k \leq n \). For fixed \( n \), \( M_{n,k} \) is a martingale difference sequence with \( \mathbb{E}[M_{n,k} \mid \mathcal{F}_{k-1}] = 0 \). Moreover,
\[
\sum_{k=1}^{n} \mathbb{E}[M_{n,k}^2 \mid \mathcal{F}_{k-1}] = n^{-\frac{1 + 3\beta}{1 + \beta}} \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2 \mid \mathcal{F}_{k-1}]
\]
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\[ n^{(1+3\beta)/(1+\beta)} \sum_{k=1}^{n} \left( \mathbb{E}[|X_k^{1+\beta} - X_{k-1}^{1+\beta}|^2 | \mathcal{F}_{k-1}] + O(1) \right), \text{ a.s.,} \]

by (3.93), using the fact that $|D_n| = O(1)$, which is easily seen by combining part (iii) of Lemma 3.12.8 with the $r = 1$ case of part (iv). Then by (3.95),

\[ \sum_{k=1}^{n} \mathbb{E}[M_{n,k}^2 | \mathcal{F}_{k-1}] \sim n^{(1+3\beta)/(1+\beta)} \sigma^2 (1 + \beta)^2 \sum_{k=1}^{n} \frac{k^{2\beta}}{1+3\beta} \lambda(\rho, \beta)^2 \beta, \text{ a.s.} \]  

(3.96)

We also need to verify a form of the conditional Lindeberg condition. We claim that

\[ \sum_{k=1}^{n} \mathbb{E}[M_{n,k}^q 1\{|M_{n,k}| > \delta \} | \mathcal{F}_{k-1}] \to 0, \text{ a.s.,} \]  

(3.97)

for any $\delta > 0$. To see this, take $q \in (2, \frac{2}{1+\beta})$. By the elementary inequality

\[ |M_{n,k}|^q 1\{|M_{n,k}| > \delta \} \leq \delta^{2-q} |M_{n,k}|^q \]

we have that, for any $\delta > 0$,

\[ \mathbb{E}[M_{n,k}^2 1\{|M_{n,k}| > \delta \} | \mathcal{F}_{k-1}] \leq \delta^{2-q} \mathbb{E}[|M_{n,k}|^q | \mathcal{F}_{k-1}] \]

Then we have that

\[ \mathbb{E}[|M_{n,k}|^q | \mathcal{F}_{k-1}] = n^{(1+3\beta)/(2+2\beta)} \mathbb{E}[(M_k - M_{k-1})^q | \mathcal{F}_{k-1}] \]

\[ = n^{(1+3\beta)/(2+2\beta)} \mathbb{E}[(X_k^{1+\beta} - X_{k-1}^{1+\beta} - D_{k-1})^q | \mathcal{F}_{k-1}], \]

where $|D_{k-1}| = O(1)$, and by the $r = q$ case of (3.88),

\[ \mathbb{E}[(X_k^{1+\beta} - X_{k-1}^{1+\beta})^q | \mathcal{F}_{k-1}] \leq C(1 + X_{k-1})^{\beta q}. \]

Hence by the (conditional) Minkowski inequality,

\[ \mathbb{E}[|M_{n,k}|^q | \mathcal{F}_{k-1}] \leq Cn^{(1+3\beta)/2+2\beta} (1 + X_{k-1})^{\beta q}. \]

Here, by Theorem 3.12.4, we have $X_{k-1} \leq Cn^{1/(1+\beta)}$ for some (random) $C < \infty$ and all $k \leq n$. Thus we obtain

\[ \sum_{k=1}^{n} \mathbb{E}[M_{n,k}^2 1\{|M_{n,k}| > \delta \} | \mathcal{F}_{k-1}] \leq Cn \frac{1}{1+\beta} \frac{(1+3\beta)q}{2+2\beta}, \text{ a.s.,} \]

for some (random) $C < \infty$. From this last bound we verify (3.97) since $p > 2$. Given (3.96) and (3.97), we can apply a standard central limit theorem for martingale differences (e.g. [25, Theorem 35.12, p. 476]) to complete the proof. \qed
Proof of Theorem 3.12.5. Recall the decomposition at (3.92). Throughout this proof, \( \varepsilon_n \) will denote a sequence such that \( \varepsilon_n \to 0, \text{a.s.} \). Under the conditions of the theorem, Theorem 3.12.4 and the proof of Lemma 3.12.10 imply that \( |M_n| \leq \varepsilon_n A_n, \text{a.s.} \), so that

\[
X_n = (A_n + M_n)^{\frac{1}{1+\beta}} = A_n^{\frac{1}{1+\beta}} + \frac{1}{1+\beta} M_n A_n^{-\frac{\beta}{1+\beta}} (1 + \varepsilon_n).
\]

Here we have from (3.94) that, a.s., \( A_n = \rho (1+\beta)n + \varepsilon_n n^{\frac{1+3\beta}{2+2\beta}} \). It follows that, a.s.,

\[
X_n = \lambda (\rho, \beta) n^{\frac{1}{1+\beta}} + n^{1/2} \varepsilon_n + \frac{1}{1+\beta} \lambda (\rho, \beta)^{-\beta} n^{-\frac{\beta}{1+\beta}} M_n (1 + \varepsilon_n).
\]

Rearranging we obtain, a.s.,

\[
\frac{X_n - \lambda (\rho, \beta) n^{\frac{1}{1+\beta}}}{n^{1/2}} = \frac{1}{1+\beta} \lambda (\rho, \beta)^{-\beta} n^{-\frac{1+3\beta}{2+2\beta}} M_n (1 + \varepsilon_n) + \varepsilon_n.
\]

Now, on letting \( n \to \infty \), Lemma 3.12.12 completes the proof. \( \square \)

3.13 Proofs for the Markovian case

We now record the proofs of the results stated in Section 3.2; this is mostly simply collecting results from the previous sections. We start with the recurrence classification.

Proof of Theorem 3.2.3. The transience condition in part (i) is a consequence of Theorem 3.5.1; Theorem 3.6.8 demonstrates the equivalence of the two notions of transience. The positive-recurrence condition in part (iii) is a consequence of the \( \alpha = 1 \) case of Theorem 3.8.1. In part (ii), recurrence follows from Theorem 3.5.2; Theorem 3.6.8 demonstrates the equivalence of the two notions of recurrence. Finally, the null property in part (ii) follows from Theorem 3.7.3. \( \square \)

Next we give the proof of the result on moments of return times.

Proof of Theorem 3.2.6. The existence of moments statement in part (i) follows from Theorem 3.8.1, while the non-existence of moments in part (ii) follows from Theorem 3.8.2. \( \square \)

Finally, the almost-sure bounds on trajectories.
Proof of Theorem 3.2.7. This is a combination of the bounds from Theorem 3.9.5 with the lower bound from Theorem 3.10.1 in the transient case for the ‘lim inf’ half of part (i).

\[ \square \]

Bibliographical notes

Section 3.1

The foundations for the material in this chapter were laid in the early 1960s by J. Lamperti in a seminal series of papers [190, 191, 192]. Lamperti systematically studied conditions under which the asymptotic behaviour of a non-negative real-valued discrete-time stochastic process is governed by the (first two) moment functions of its increments. These three papers have provided a rich source of material for subsequent theoretical work, as well as providing key evidence of the power of the semimartingale approach for studying multidimensional processes via Lyapunov functions. Some of the chapters later in this book describe examples of the application of this method to near-critical processes, often in cases where it is hard to find any serious methodological competitor.

Example 3.1.1 gives a hint to the ubiquity of Lamperti-type situations. As remarked in the text, if \( \xi_n \) is Markov then typically \( X_n = f(\xi_n) \) is not: see Section 1.3 and the accompanying bibliographical notes for Chapter 1. Many dimensional zero-drift random walks satisfying the conditions of Example 3.1.1 may be either recurrent or transient, as we will see in Chapter 4; this contrasts with the one-dimensional case where recurrence is assured under similar conditions, as shown in Theorem 2.5.7.

Section 3.2

The Markovian recurrence classification, Theorem 3.2.3, improves a little on Lamperti’s original results from [190, 192] in the Markov case. Theorem 3.2.3 is usually sufficiently sharp for any application, but can be sharpened near the phase boundaries; an ultimate refinement is given by Menshikov et al. [228]. Theorem 3.2.3 parts (i) and (ii) are contained in Theorem 3.1 of [190] and Theorem 2.1 of [192] respectively; part (iii) is sharper than Lamperti’s results, and is contained in the results of [228].

Nearest neighbour walks in the vein of Example 3.2.5 have been extensively studied for a long time; these models also go under the name \textit{birth-and-death chains}. In the context, the observation that the regime where \( x\mu_1(x) = O(1) \) is critical from the point of view of the recurrence clas-
sification goes back at least to Harris [127]; another early contribution is [129]. In certain cases, nearest-neighbour models are amenable to explicit calculation, facilitated by reversibility and associated algebraic structure. An intricate apparatus was introduced by Karlin and McGregor [152, 153], using orthogonal polynomials to provide a spectral representation, to give a deep analysis of models such as

$$\mathbb{P}[X_{n+1} = X_n \pm 1 \mid X_n = x] = \frac{1}{2} \mp \frac{\delta}{4x + 2\delta},$$

for \(x > 0\) and a parameter \(\delta\), which is an instance of Example 3.2.5 with \(c = -\delta/2\). This and closely related models were considered by many subsequent authors, including for instance [275, 91, 92, 93, 299, 112, 310] and, more recently [65, 64, 67, 140, 2, 182]. Alexander [2] calls such nearest-neighbour random walks with drift \(O(x^{-1})\) at \(x\) ‘Bessel-like’. See e.g. [59] for a survey of methods based on orthogonal polynomials.

Theorem 3.2.6 on moments of return times in the Markov case is a consequence of the more general results in Section 3.7; see the notes below on that section for bibliographical details. Likewise, Theorem 3.2.7 is a consequence of the more general results on almost-sure bounds in Sections 3.9 and 3.10; see the notes below on the relevant sections.

One possible extension to the models of this section is to move from the half-line \(\mathbb{R}_+\) to a Markov process on a half strip \(A \times \mathbb{Z}_+\), where \(A\) is a finite set, in which the process has different Lamperti-type characteristics on each copy of \(\mathbb{Z}_+\), and is close to being Markov on \(A\): see [115].

**Section 3.3**

For all the reasons outlined in the text, the importance of a setting more general than the Markovian one of Section 3.2 was emphasized by Lamperti [190]. The formulation (3.5) and (3.6) builds on the original formulation of [190], and rectifies an omission in a similar earlier version in [236].

The representation (3.7) is a very general *stochastic difference equation*; stochastic recursions of this form have received much attention in various contexts over the years, particularly when \(\psi_{1,n} \equiv 0\), including stochastic growth models (see e.g. [155]) and *stochastic approximation* (see e.g. [255]).

The natural non-confinement condition (L2) was used by Lamperti [190], and avoids technicalities involving irreducibility in general spaces. The uniform ellipticity assumption discussed in Example 3.3.5 is common in the literature on multidimensional random walks.
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Section 3.4

The Lyapunov function computations presented in this section are variations on a well-established theme. The closest presentation in the literature is that of [135]. Instead of the truncation near zero in defining $f_{\gamma,\nu}$ at (3.16), an alternative is to take say $(e + x)^\gamma \log\nu(e + x)$, but this makes some of the computations a little more cumbersome.

Section 3.5

The transience criterion, Theorem 3.5.1, is the transience half of Theorem 3.2 of [190]. The recurrence criterion, Theorem 3.5.2, improves on the recurrence half of Theorem 3.2 of [190], which admits $\theta = 1$ but not $\theta \in (0, 1)$. Formally, Theorem 3.5.2 is not contained in the results of [228], which are stated in the Markovian case, although it is likely that these sharper results can be extended to the more general setting.

The fact that symmetric SRW on $\mathbb{Z}^d$ is recurrent if and only if $d \leq 2$ is a seminal result of Pólya [259]; the fact that the statement extends to non-degenerate homogeneous random walks whose increments have finite second moments is a consequence of the equally famous Chung–Fuchs theorem (see [47] or Chapter 9 of [150]). Pólya’s original proof is based on path-counting and generating functions, and is still the most widely-presented proof of the SRW result. The proof of the Chung–Fuchs theorem is by Fourier analysis.

The approach via Lyapunov functions, as presented in Examples 3.5.3 and 3.5.4, is due to Lamperti [190], and represents a significant breakthrough, not only in clarifying the probabilistic intuition behind Pólya’s result, but also in providing a powerful methodology that is applicable in greater generality than combinatorial or analytical approaches.

The fundamental semimartingale results Theorems 3.5.6 and 3.5.8 build on ideas of [190], as well as being relatives of the results of Foster presented in Chapter 2. Lamperti [190] worked under the non-confinement assumption (L2), and his formulations are somewhat different. The formulation here, which permits cases where (L2) does not hold, is closely related to that contained in Kersting [159].

The analysis of branching processes via Lamperti-type methods, as in Example 3.5.9, goes back to Lamperti himself in a different context [193]. A reference for the classical theory of Galton–Watson processes is [11].
Section 3.6

The material on irreducibility and regeneration and their consequences on countable state-spaces is standard, although no single reference covers the presentation in this section. A useful reference for regenerative processes is Chapter VI of [6].

The ergodic theorem for occupation times (Theorem 3.6.9) is a standard result, and the regenerative structure and irreducibility combine to give an easy and natural proof. The statement of the result given here is not always apparent in the standard references, which often focus on the (deeper) result \( \lim_{n \to \infty} P[X_n = x] = \pi_x \), which requires aperiodicity (cf. Theorem 2.1.6). Under suitable conditions, there is a central limit theorem to accompany Theorem 3.6.9, namely,

\[
\frac{L_n(x) - n\pi_x}{\sqrt{n}}
\]

converges in distribution to a centred normal distribution with finite positive variance: see e.g. §6.9 of [252] or §VI.3 of [6] for further details.

More generally than Theorem 3.6.9, similar ergodicity results hold for time-averages of other functionals \( f : S \to \mathbb{R} \), provided they are integrable with respect to the limiting measure \( \pi_x \). Specifically, if (I) and (R) hold for a countable \( S \) with \( 0 \in S \), and if \( \mathbb{E} k_1 < \infty \) then, a.s.,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n-1} f(X_m) = \lim_{n \to \infty} \frac{1}{n} \sum_{x \in S} L_n(x) f(x) = \sum_{x \in S} \pi_x f(x),
\]

provided \( \sum_{x \in S} \pi_x |f(x)| < \infty \); the proof is similar to that given for Theorem 3.6.9. Compare e.g. [6, Theorem VI.3.1, p. 178].

Much of the discussion in this section can be extended to more general state-spaces, at the expense of technical complications: in that context, detailed accounts of the relevant concepts may be found in the books [247, 270, 246, 239]. Roughly speaking, to obtain analogous results to those presented here, one needs a concept that fulfils not only the role of irreducibility, but also that of the local finiteness of the state-space.

Although we do not discuss it in this book, it is an interesting problem to determine the \( x \)-asymptotics of the stationary distribution \( \pi_x \) for a positive-recurrent Lamperti process on a locally finite subset of \( \mathbb{R}^+ \), for which the conditions (M0), (M1) and (M2) hold with \( 2a < -b \), say. In the case of a Markov process on \( \mathbb{Z}^+ \) with uniformly bounded increments, Menshikov and Popov [230] showed that

\[
\pi_x = x^{-(2a/b) + o(1)}, \quad \text{as } x \to \infty;
\]
related results appear in \[8\].

We sketch how neighbouring results may be obtained for the associated tail \(\overline{\pi}_x := \sum_{y \geq x} \pi_y\) from the ingredients in this chapter. Note that

\[
\overline{\pi}_x = \frac{1}{E \kappa_1} \sum_{y \geq x} E \ell_1(y) = \frac{1}{E \kappa_1} E \sum_{m=0}^{\kappa_1-1} 1\{X_m \geq x\}.
\]

For a lower bound on \(\overline{\pi}_x\), the idea of Example 2.7.8 gives, for some \(c > 0\) and \(\varepsilon > 0\),

\[
\overline{\pi}_y \geq cy^2 P\left[\max_{0 \leq m \leq \lambda_x} X_m \geq 2y\right] \geq cy^{1+(2a/b)-\varepsilon},
\]

by Lemma 3.7.6. On the other hand, an upper bound is

\[
\overline{\pi}_y \leq E \left[\kappa_1 1\{\max_{0 \leq m \leq \kappa_1} X_m \geq y\}\right],
\]

so we have from Hölder’s inequality that

\[
\overline{\pi}_y \leq \left(E[\kappa_1^{p'}]\right)^{1/p} \left(P\left[\max_{0 \leq m \leq \kappa_1} X_m \geq y\right]\right)^{1/p}.
\]

By Theorem 3.2.6, for example, the optimal choice is to take \(p < \frac{b-2a}{2a}\) as big as possible, which will give \(\overline{\pi}_y \leq cy^{1+(2a/b)+\varepsilon}\). Denisov et al. [72] obtain sharper asymptotics for the tail under certain conditions; see also [180].

The results of Menshikov and Popov [230] also address the rate of convergence to stationarity; see also [182] in the case of a nearest-neighbour random walk and [308, 309] for a similar approach to some queueing models. This is one sense in which Lamperti processes exhibit only polynomial ergodicity rather than geometrical ergodicity.

Sections 3.7 and 3.8

As described in the bibliographical notes for Section 2.7, results on the existence and non-existence of moments of passage times originate with Lamperti [192] (for positive integer moments) and were extended in [10] and subsequently in [7, 9].

Theorem 3.7.1 is closely related to Proposition 1 of [10], which was stated in the Markovian case. This extended Theorem 2.2 of [192], which applied only to integer moments, and was again stated in the Markovian case, although Lamperti indicated how it held more generally.
Theorems 3.7.2 and 3.7.3 on lower tail bounds and non-existence of moments extend Proposition 2 of [10] and Theorem 3.2 of [192], which were again stated in the Markovian case. The results of [10] did not provide a tail bound, while [192] applied only to integer moments. The tail bound in Theorem 3.7.2 can be derived, under more restrictive assumptions (including uniformly bounded increments), from Corollary 1 of [7]. The proofs of the lower tail bounds given here, which use Lemma 2.7.7 and lower tail bounds on excursions, are related to the arguments in [135], where related excursion bounds are given under slightly different conditions.

The fact that tails of passage times for Lamperti processes satisfy upper and lower bounds of polynomial order is typical of near-critical systems, which exhibit polynomial behaviour; in the case of a positive-recurrent Markov chain, this is polynomial ergodicity in another sense (closely related to that described above for stationary distributions).

In the special case of nearest neighbour random walks, Fal’ [91] gives asymptotics for excursion times and the number of excursions.

Section 3.9

Related almost-bounds in a general Lamperti-type setting are given in [232, Section 4]; there a uniform bound on the increments was imposed. The upper bounds in Theorems 3.9.1 and 3.9.2 are variations on Theorems 4.1(i) and 4.3(i) from [232]; see also Section 6 of [53].

In [232], lower bounds were also given, using a version of Theorem 2.8.3, but this required a somewhat unnatural ejection condition. The approach to lower bounds given here, building on the excursion developments of Sections 3.7 and 3.8, follows similar ideas to [135], where slightly sharper bounds are given under somewhat different assumptions.

In various special cases of certain nearest-neighbour random walks on \( \mathbb{Z}^+ \), various results on almost-sure bounds have been obtained; often, using methods restricted to the nearest-neighbour case, sharper results are available. The diffusive case (cf Theorem 3.9.1) has received most attention, and iterated-logarithm results are available [275, 93, 299, 112, 310, 140]; of these, only [140] treats the sub-diffusive case (cf Theorem 3.9.2).

Section 3.10

Theorem 3.10.1 is a substantial improvement of Theorem 4.2 of [232], which gave the first general almost-sure lower bounds for the transient Lamperti problem. Theorem 4.2 of [232] gave a lower bound \( X_n \geq n^{1/2} (\log n)^{-D} \), for
some constant $D$ not given explicitly, under much more stringent conditions, including the Markov assumption, a uniform bound on the increments, and an unnatural ‘re-entry’ condition to overcome the technicality addressed here by Lemmas 3.10.5 and 3.10.8.

The proof of Theorem 3.10.1 given here improves the proof in [232]; the connection to last exit times via the inequality (3.52) simplifies the argument. Lemmas 3.10.5 and 3.10.6 extend the idea of Lemma 3.2 of [232] to permit an exceptional set in the Lyapunov function condition, and to replace a uniform increments bound by a moments condition. Related ideas appear in Section 3 of [72].

As mentioned in the text of Example 3.10.3, the rate of escape for transient SRW on $\mathbb{Z}^d \ (d \geq 3)$ was studied by Dvoretzky and Erdős [84]. Essentially the same result holds for spatially homogeneous, zero-drift random walks under suitable moments assumptions: see e.g. [121, 261] and references therein.

As mentioned in the text of Example 3.10.4, in the much more restricted setting of nearest-neighbour random walks, a sharp version of Theorem 3.10.1 is given by Csáki et al. [64]. Specifically, under the conditions of Example 3.10.4, Theorem 6.1 of [64] says that for any $\varepsilon > 0$, a.s., for all but finitely many $n$,

$$X_n \geq n^{\frac{1}{2}} (\log n)^{\frac{1}{1+2\varepsilon}} - \varepsilon,$$

and that this result is sharp in that it fails with $\varepsilon = 0$. The proof in [64] proceeds via strong approximation to an appropriate transient Bessel process, and then appeals to results on the latter.

Another way to quantify transience is via the rate of growth of the renewal function $H(x)$ defined at (3.58). Under conditions similar to those in Theorem 3.10.1, Denisov et al. [72] show that $H(x)$ is of order $x^2$ as $x \to \infty$.

### Section 3.11

The weak convergence result Theorem 3.11.2 was first obtained by Lamperti under stronger conditions [191]. Although already implicit in Lamperti’s work, the fact that his results could be extended to processes with different scaling behaviour was not appreciated more widely until much later: Klebaner [173] and Kersting [160] provided extended versions of Theorem 3.11.2, although both [173] and [160] admit only the transient case, where $X_n \to \infty$. The version of Theorem 3.11.2 presented here follows Denisov et al. [72]. For other related recent work, see [24]. A version of Theorem 3.11.2 for Lamperti-type processes on half strips is given in [115].
In [191], Lamperti extended his version of the marginal weak limit result Theorem 3.11.2 to a full *invariance principle*, i.e., weak convergence on the appropriate sample-path space of the scaled Markov chain to a time-homogeneous diffusion process on \( \mathbb{R}_+ \). The diffusions that arise in the limit are, up to a deterministic scale factor, *Bessel processes*. The parameter (‘dimension’) of the process is given by \( 1 + \frac{2a}{b} \), where \( a \) and \( b \) are the constants in the statement of Theorem 3.11.2.

### Section 3.12

The presentation and results in this section are based on [236]. Theorem 3.12.1 is Theorem 2.1 of [236]. The strong law of large numbers Theorem 3.12.4 and accompanying central limit theorem (Theorem 3.12.5) are Theorems 2.4 and 2.5 of [236], respectively. The version of Theorem 3.12.5 stated here corrects a minor error in the moments assumption stated in Theorem 2.5 of [236]. A version of Theorem 3.12.4 in the case where the moments Markov property (3.15) holds is contained in Keller *et al.* [155], among results that permit more general growth rates. The first result in this direction was a *weak* law of large numbers obtained by Lamperti [191]. Specifically, if \( X_n \) is a time-homogeneous Markov process with \( \lim_{x \to \infty} x^\beta \mu_1(x) = \rho \) and \( \sup_x |\mu_k(x)| < \infty \) for all \( k \), where \( \mu_k \) is given by (3.1), then Theorem 7.1 of [191] says that (3.75) holds with convergence in probability (only). The question of a central limit theorem in this context was raised by Lamperti [191, p. 768], and seems to have remained open until [236] even for the case of a birth-and-death chain. The paper [155] (a reference unknown to the authors of [236] when that paper was written) builds on earlier work of Küster [189], and includes certain second order (distributional limit) results, but none pertinent to Theorem 3.12.5.

The application to the nearest-neighbour random walk of Example 3.12.6 is taken from Section 3.1 of [236]. In the setting of Example 3.12.6, Theorem 3.12.4 generalizes a result of Voit [311, Theorem 2.11], who imposed the additional assumption that

\[
\lim_{x \to \infty} a_x \quad \text{and} \quad \lim_{x \to \infty} c_x \ \text{exist in} \ (0, 1),
\]

(in which case the two limits must take the same value). Note that there is a misprint in the limiting constant in the statement of Theorem 2.11 of [311] (the proof there does yield the correct constant): the \( 1/(1 + \alpha) \) power should be applied to the entire limiting expression, not just the \( \mu \) there; this typo persists into [65, Theorem D]. The central limit theorem
Chapter 3. Lamperti’s problem

presented in Example 3.12.6 is from [236]. Note that the assumption that 
\( \lim_{x \to \infty} b_x = b \) implies that (3.98) holds (with limit \( \frac{1-b}{2} \) for \( a_x \) and \( c_x \)), so this central limit theorem can be seen as the natural companion to Voit’s law of large numbers [311, Theorem 2.11].

We make some final remarks on the case, of secondary interest to us here, where \( \beta = 0 \) in Example 3.12.6. Theorem 3.12.4 applies to the case \( \beta = 0 \), i.e., where \( a_x - c_x \to \rho \in (0,1] \) as \( x \to \infty \), in which case the result says that \( n^{-1}X_n \to \rho \) a.s. as \( n \to \infty \). This particular result was obtained by Pakes [251, Proposition 4], under some more restrictive conditions, including \( \rho = 1 \) and \( b_x \equiv 0 \), and also, for general \( \rho \) but again under conditions more restrictive than ours, in a result of Voit [310, Corollary 2.6]. In the case \( \beta = 0 \), the second-order behaviour of \( X_n \) is somewhat different; see for instance [251, Theorem 7] and [310, Theorems 2.7–2.10].

Further remarks

In recent years, significant interest in processes with asymptotically zero drifts has come from a community of probabilists and statistical physicists from the point of view of modelling random polymers and interfaces, their structure, and their interactions with a medium or boundary. In the context of random polymers, the path of the process models the physical polymer chain; the asymptotically zero drift indicates the presence of long-range interaction with a boundary, which can be either attractive or repulsive. For a random interface, the walk models the behaviour of a liquid interface on a solid substrate (including wetting and pinning phenomena); in this context the drift may represent affinity for the boundary. We refer to [70, 116, 307] for recent surveys, and to [67, 140, 2] for recent works inspired by the random polymer motivation.

Lamperti-type ideas have been employed in the context of branching processes (cf. Example 3.5.9); in particular maximal branching processes were studied by Lamperti himself [193, 194], and much more recently in [12]. Another class of models studied by the methods of this chapter are growth processes, see e.g. [189, 159, 155, 173].

In the context of diffusion processes, Bessel processes can be viewed as a certain analogue of Lamperti-type processes; the analogy is rather narrow, as special methods are available for Bessel processes, comparable to the case of nearest-neighbour random walks. A Bessel process is a \( \mathbb{R}_+ \)-valued diffusion \( (X_t, t \in \mathbb{R}_+) \) determined by the stochastic differential equation

\[
dX_t = \frac{a}{X_t} dt + b dB_t,
\]
where $B_t$ is standard Brownian motion on $\mathbb{R}$. Similarly one can consider supercritical cases, where the drift is of order $X_t^{-\beta}$, $\beta \in (0,1)$, or cases where the drift is $o(1/X_t)$: see e.g. [68].
Chapter 3. Lamperti’s problem
Chapter 4

Many-dimensional random walks

4.1 Introduction

4.1.1 Chapter overview: Anomalous recurrence

Famously, a \(d\)-dimensional, spatially homogeneous random walk whose increments are non-degenerate, have finite second moments, and have zero mean is recurrent if \(d \in \{1, 2\}\) but transient if \(d \geq 3\): this is a consequence of the classical Chung–Fuchs theorem (see Theorem 1.5.2). Once spatial homogeneity is relaxed, this is no longer true, as discussed in Section 1.5: there are zero-drift spatially non-homogeneous random walks which, in any ambient dimension \(d \geq 2\), can be either transient or recurrent depending on the details of the jump distributions. Such potential anomalous recurrence behaviour is the first focus of this chapter. As described in Theorem 1.5.4, anomalous recurrence behaviour can be excluded under mild assumptions, provided we impose a fixed covariance structure on the increments of the walk. In that case, by analogy with the Lamperti problem described in Chapter 3, it is natural to subsequently relax the assumption of zero drift to asymptotically zero drift, to probe phase transitions in recurrence behaviour. The second focus of this chapter is the asymptotically zero drift case, and in particular the family of examples known as centrally biased random walks.

In this chapter we explore recurrence/transience phase transitions for spatially non-homogeneous random walks in \(\mathbb{R}^d\), and give proofs of the results discussed in Sections 1.5 and 1.7. Now we give an informal overview of the models and results that we will present, and an outline of the rest of
In the zero drift case, natural examples of anomalous recurrence behaviour are provided by random walks whose increments are supported on ellipsoids that are symmetric about the ray from the origin through the walk’s current position. These elliptic random walks generalize the classical homogeneous Pearson–Rayleigh random walk (PRRW): recall from Example 1.4.2 that a PRRW proceeds via a sequence of unit-length steps, each in an independent and uniformly random direction in $\mathbb{R}^d$. The PRRW can be represented via partial sums of sequences of i.i.d. random vectors that are uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Clearly the PRRW has zero drift. It is well known (cf. Theorem 1.5.2) that the PRRW is recurrent for $d \in \{1, 2\}$ and transient if $d \geq 3$.

Suppose that we replace the spherically symmetric increments of the PRRW by increments that instead have some elliptical structure, while retaining the zero drift. For example, one could take the increments to be uniformly distributed on the surface of an ellipsoid of fixed shape and orientation, as represented by the picture on the right of Figure 4.1. More generally, one should view the ellipses in Figure 4.1 as representing the covariance structure of the increments of the walk (we will give a concrete example later; the uniform distribution on the ellipse is actually not the most convenient for calculations).

A little thought shows that the walk represented by the picture on the right of Figure 4.1 is essentially no different to the PRRW: an affine transformation of $\mathbb{R}^d$ will map the walk back to a walk whose increments have the same covariance structure as the PRRW (cf. Remark 1.5.5). To obtain
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genuinely different behaviour, it is necessary to abandon spatial homogeneity.

In Section 4.2 we consider a family of spatially non-homogeneous elliptic random walks with zero drift. These include generalizations of the PRRW in which the increments are not i.i.d. but have a distribution supported on an ellipsoid of fixed size and shape but whose orientation depends upon the current position of the walk. Figure 4.2 gives representations of two important types of example, in which the ellipsoid is aligned so that its principal axes are parallel or perpendicular to the vector of the current position of the walk, which sits at the centre of the ellipse.

Figure 4.2: Pictorial representation of spatially non-homogeneous random walks with increments distributed on a radially-aligned ellipse with major axis aligned in the radial sense (left) and in the transverse sense (right).

The random walks represented by Figure 4.2 are no longer sums of i.i.d. variables. These modified walks can behave very differently to the PRRW. For instance, one of the two-dimensional random walks represented in Figure 4.2 is transient while the other (as in the classical case) is recurrent. The reader who has not seen this kind of example before may take a moment to identify which is which.

The elliptic random walk models represented by Figure 4.2 require uncountably many different increment distributions in order to generate their anomalous recurrence behaviour. A natural combinatorial question concerns the minimal number of different increment distributions required; this is the subject of Section 4.3. One can view this problem as one of control: the controller is able to implement a number of different increment distributions, and wishes to minimize this number while achieving the desired recurrence properties; thus we address this question in the context of controlled random walks.
Section 4.4 moves beyond the zero drift setting to discuss walks with *asymptotically zero drift*. Analogously to the one-dimensional Lamperti problem (Chapter 3), this is the critical regime in which to investigate various phase transitions in asymptotic behaviour for the process. In higher dimensions, one natural model is the *centrally biased random walk* in which the mean drift field has magnitude that tends to zero with the distance from the origin, and direction either away from or towards the origin: see Figure 4.3.

Figure 4.3: Pictorial representation of centrally biased random walks with asymptotically drift aligned in the radial sense, outward (left) and inward (right).

The final section of this chapter, Section 4.5, addresses the different (but related) topic of the *range* of a many-dimensional random walk, which means, roughly speaking, the volume of the state-space visited by the walk up to a given time.

### 4.1.2 Notation for many-dimensional random walks

Our notation in this chapter coincides, whenever possible, with that in Sections 1.4 and 1.5. We work in \( \mathbb{R}^d, \ d \geq 1 \). Our main interest is in \( d \geq 2 \), as we shall explain in the following sections. Write \( e_1, \ldots, e_d \) for the standard orthonormal basis vectors in \( \mathbb{R}^d \). Write \( \mathbf{0} \) for the origin and \( \mathbb{S}_{d-1} = \{ \mathbf{u} \in \mathbb{R}^d : ||\mathbf{u}|| = 1 \} \) for the unit sphere in \( \mathbb{R}^d \), and let \( || \cdot || \) denote the Euclidean norm and \( \langle \cdot, \cdot \rangle \) the Euclidean inner product on \( \mathbb{R}^d \). For \( \mathbf{x} \in \mathbb{R}^d \setminus \{ \mathbf{0} \} \), \( \hat{\mathbf{x}} = \mathbf{x}/||\mathbf{x}|| \) is the unit vector parallel to \( \mathbf{x} \); we use the convention \( \mathbf{0} := \mathbf{0} \). Vectors \( \mathbf{x} \in \mathbb{R}^d \) are viewed as column vectors throughout.
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In most (but not all) of this chapter, we will consider the following basic setting for our many-dimensional non-homogeneous random walk.

**MD0** Let \( d \geq 1 \). Suppose that \((\xi_n, n \geq 0)\) is a discrete-time, time-homogeneous Markov process on an unbounded subset \( \Sigma \) of \( \mathbb{R}^d \), with \( 0 \in \Sigma \).

For the increments of the random walk, we use the notation

\[
\theta_n := \xi_{n+1} - \xi_n, \quad n \in \mathbb{Z}_+.
\]

By assumption (MD0), given \( \xi_0, \ldots, \xi_n \), the distribution of \( \theta_n \) depends only on the position \( \xi_n \in \Sigma \) and not on \( n \).

We use the shorthand \( \mathbb{P}_x[\cdot] = \mathbb{P}[\cdot \mid \xi_0 = x] \) for probabilities when the walk is started from \( x \in \Sigma \); similarly we use \( \mathbb{E}_x \) for the corresponding expectations.

We will typically impose a moments assumption on \( \theta_n \) of the form,

\[
\sup_{x \in \Sigma} \mathbb{E}_x[\|\theta_0\|^p] < \infty, \tag{4.1}
\]

for appropriate \( p > 0 \). If (4.1) holds for \( p \geq 1 \), then the one-step mean drift vector is well defined, and will be denoted

\[
\mu(x) := \mathbb{E}_x[\theta_0], \quad x \in \Sigma. \tag{4.2}
\]

If (4.1) holds for \( p \geq 2 \), then the increment covariance matrix is also well defined, and will be denoted

\[
M(x) := \mathbb{E}_x[\theta_0\theta_0^\top], \quad x \in \Sigma. \tag{4.3}
\]

A little linear algebra shows that \( M(x) \) encodes various information about second moments, including, for \( y \in \mathbb{R}^d \),

\[
\mathbb{E}_x[(y, \theta_0)^2] = \mathbb{E}_x[y^\top \theta_0 \theta_0^\top y] = y^\top M(x) y, \tag{4.4}
\]

and \( \mathbb{E}_x[\|\theta_0\|^2] = \text{tr} M(x) \). \tag{4.5}

4.1.3 Beyond the Chung–Fuchs theorem

The starting point for much of the discussion in this chapter is the question raised in Section 1.5: to what extent does the recurrence classification for spatially homogeneous random walks extend to the spatially non-homogeneous case? Our main interest is in the case where the increments
of the walk have uniformly bounded second moments, and then it is a consequence of the Chung–Fuchs theorem that the conclusion of Pólya’s theorem (Theorem 1.2.1) holds for any zero drift, spatially homogeneous random walk: see Theorem 1.5.2. We now start to consider in detail the non-homogeneous case.

We use the notation of Section 4.1.2 and consider a non-homogeneous random walk $\xi_n$ on $\Sigma \subseteq \mathbb{R}^d$, $d \geq 1$. We suppose for this section that the increment moments condition (4.1) holds for some $p > 2$, so that the mean drift vector $\mu(x)$ given by (4.2) is well defined, and that the random walk has zero drift:

(MD1) Suppose that $\mu(x) = 0$ for all $x \in \Sigma$.

Our moments assumption also ensures that the increment covariance matrix $M(x)$ given by (4.3) is well defined. To rule out pathological cases, we impose the following non-degeneracy condition on the increments.

(MD2) There exists $v > 0$ such that $\text{tr} M(x) \geq v$ for all $x \in \Sigma$.

Note that assumption (MD2) is weaker than uniform ellipticity such as (3.13).

First, we state the following basic non-confinement result.

Lemma 4.1.1. Suppose that (MD0), (MD1), and (MD2) hold, and that (4.1) holds for some $p > 2$. Then

$$\limsup_{n \to \infty} \|\xi_n\| = +\infty, \text{ a.s.} \quad (4.6)$$

Lemma 4.1.1, which we prove at the end of this section, ensures that questions of the escape of trajectories to infinity are non-trivial. We are then interested in classifying $\xi_n$ as either recurrent or transient, in the sense of Definition 1.5.1. If $\xi_n$ is an irreducible time-homogeneous Markov chain on a locally finite state-space, these definitions reduce to the usual notions of recurrence and transience.

In dimension $d = 1$, it is a consequence of the Chung–Fuchs theorem (see Chapter 9 of [150]) that a spatially homogeneous random walk with zero drift is necessarily recurrent. For a spatially non-homogeneous random walk this is not true in general, and there is a counterexample with increments of infinite variance that is transient [273]; see the bibliographical notes to Section 2.5. Our conditions exclude such heavy-tailed phenomena, so that in $d = 1$ recurrence is assured in our setting. The following result is a variant of Theorem 2.5.7, which was stated for a locally finite state space.
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Theorem 4.1.2. Suppose that (MD0), (MD1), and (MD2) hold, and that (4.1) holds for some \( p > 2 \). Suppose that \( d = 1 \). Then \( \xi_n \) is recurrent.

In dimensions \( d \geq 2 \), we first seek an analogue of Theorem 1.5.2. The next result shows that one recovers the classical recurrence classification for zero-drift random walk as soon as one imposes a fixed increment covariance structure across all of space.

Theorem 4.1.3. Suppose that (MD0) and (MD1) hold, and that (4.1) holds for some \( p > 2 \). Suppose that for all \( x \in \Sigma \), \( M(x) = M \) for some positive-definite matrix \( M \). Then \( \xi_n \) is recurrent if \( d \leq 2 \) and transient if \( d \geq 3 \).

A weaker version of Theorem 4.1.3, assuming a uniform bound on the increments, was stated as Theorem 1.5.4 and established in Example 3.5.4. Importantly, the zero drift assumption can be relaxed: for example, for the conclusion of Theorem 4.1.3 one may replace \( \mu(x) = 0 \) by (essentially) \( \|\mu(x)\| = o(\|x\|^{-1}) \), as we shall see later. This fact leads naturally to the question of the extent to which one can ‘perturb’ a zero-drift walk and preserve its recurrence or transience. Lamperti showed that the critical scale for perturbations is \( O(\|x\|^{-1}) \). Roughly speaking, for any walk satisfying (4.1) for \( p > 2 \), regardless of its covariance structure, one can arrange \( \mu \) with \( \|\mu(x)\| = O(\|x\|^{-1}) \) for which the walk is recurrent or transient, as desired: see Section 4.1.5 for details.

We prove Theorems 4.1.2 and 4.1.3 in Section 4.1.5, after presenting a more general result (Theorem 4.1.7). The rest of this section is devoted to the proof of Lemma 4.1.1. We first present a general result for martingales on \( \mathbb{R}^d \) satisfying a non-degeneracy condition; the result can be viewed as a \( d \)-dimensional martingale version of Kolmogorov’s other inequality (Theorem 2.4.12). Theorem 4.1.4 extends the one-dimensional martingale result Theorem 2.4.12 to higher dimensions, and also replaces the uniformly bounded jumps assumption by a moments condition. The proof here uses a different idea from that of Theorem 2.4.12.

Theorem 4.1.4. Let \( d \in \mathbb{N} \). Suppose that \( (Y_n, n \geq 0) \) is an \( \mathbb{R}^d \)-valued process adapted to a filtration \( (\mathcal{F}_n, n \geq 0) \), with \( \mathbb{P}[Y_0 = 0 | \mathcal{F}_0] = 1 \). Suppose that there exist a stopping time \( \eta \) and constants \( p > 2, v > 0, B < \infty \) such that for all \( n \geq 0 \), a.s.,

\[
\mathbb{E}[\|Y_{n+1} - Y_n\|^p | \mathcal{F}_n] \leq B; \tag{4.7}
\]

\[
\mathbb{E}[\|Y_{n+1} - Y_n\|^2 | \mathcal{F}_n] \geq v \mathbb{1}\{n < \eta\}; \tag{4.8}
\]

\[
\mathbb{E}[Y_{n+1} - Y_n | \mathcal{F}_n] = 0. \tag{4.9}
\]
Then there exists $D < \infty$, depending only on $B$, $p$, and $v$, such that for all $n \in \mathbb{Z}_+$ and all $x \in \mathbb{R}_+$,

$$
\mathbb{P}\left[\max_{0 \leq m \leq n} \|Y_m\| \geq x \mid \mathcal{F}_0\right] \geq 1 - \frac{D(1 + x)^2}{n} - \mathbb{P}[\eta \leq n \mid \mathcal{F}_0], \text{ a.s.}
$$

Proof. Let $x \in \mathbb{R}_+$ and set $\sigma = \min\{n \geq 0 : \|Y_n\| \geq x\}$. For the increments of the process write $D_n = Y_{n+1} - Y_n$, and let

$$
W_n = \begin{cases} 
Y_n & \text{if } \|Y_n\| \leq A(1 + x), \\
Y_{n-1} + \hat{D}_{n-1}(A - 1)(1 + x) & \text{if } \|Y_n\| > A(1 + x),
\end{cases}
$$

where $A > 1$ is a constant to be specified later. Note that $W_n$ is $\mathcal{F}_n$-measurable. Now, on $\{\|Y_n\| \leq x\}$, $W_n = Y_n$ and

$$
\mathbb{E}[W_{n+1} - W_n \mid \mathcal{F}_n] = \mathbb{E}[D_n \mid \mathcal{F}_n] + \mathbb{E}[\hat{D}_n ((A - 1)(1 + x) - \|D_n\|)] 1\{\|Y_{n+1}\| > A(1 + x)\} \mid \mathcal{F}_n].
$$

But $\{\|Y_{n+1}\| > A(1 + x)\} \cap \{\|Y_n\| \leq x\}$ implies that $\|D_n\| > (A - 1)(1 + x)$, and by (4.9), $\mathbb{E}[D_n \mid \mathcal{F}_n] = 0$. Hence, on $\{\|Y_n\| \leq x\}$,

$$
\mathbb{E}[W_{n+1} - W_n \mid \mathcal{F}_n] \leq \mathbb{E}[\|D_n\| 1\{\|D_n\| > (A - 1)(1 + x)\} \mid \mathcal{F}_n]
\leq (A - 1)^{-1}(1 + x)^{-1} \mathbb{E}[\|D_n\|^2 \mid \mathcal{F}_n]
\leq B'(A - 1)^{-1}(1 + x)^{-1}, \text{ a.s.,}
$$

where, by (4.7) and Lyapunov’s inequality, $B' < \infty$ depends only on $B$ and $p$. Hence we can choose $A \geq A_0$ for some $A_0 = A_0(B, p, v)$ large enough so

$$
\mathbb{E}[W_{n+1} - W_n \mid \mathcal{F}_n] \leq (v/8)(1 + x)^{-1}, \text{ on } \{\|Y_n\| \leq x\}. \quad (4.10)
$$

Also, on $\{\|Y_n\| \leq x\}$, by a similar argument,

$$
\mathbb{E}[\|D_n\|^2 \mid \mathcal{F}_n] 
+ \mathbb{E}\left[\{(A - 1)^2(1 + x)^2 - \|D_n\|^2\} 1\{\|Y_{n+1}\| > A(1 + x)\} \mid \mathcal{F}_n\right]
\geq \mathbb{E}[\|D_n\|^2 \mid \mathcal{F}_n] - \mathbb{E}[\|D_n\|^2 1\{\|D_n\| > (A - 1)(1 + x)\} \mid \mathcal{F}_n]
\geq v 1\{n < \eta\} - (A - 1)^{2-p}(1 + x)^{2-p} \mathbb{E}[\|D_n\|^p \mid \mathcal{F}_n]
\geq (v/2) 1\{n < \eta\}, \quad (4.11)
$$

for all $x \geq 0$ and $A \geq A_1$ for some sufficiently large $A_1 = A_1(B, p, v)$, using (4.7) and (4.8).
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Now, set $Z_n = \|W_n\|^2$. Then, on $\{n < \sigma \wedge \eta\}$, by (4.10) and (4.11),

$$E[Z_{n+1} - Z_n | \mathcal{F}_n] = E[\|W_{n+1}\|^2 - \|W_n\|^2 | \mathcal{F}_n]$$

$$\geq \frac{v}{2} - \frac{2\|W_n\|v}{8(1+x)} \geq \frac{v}{2} - \frac{vx}{4(1+x)} \geq \frac{v}{4}.$$  

Hence $Z_n - \sum_{k=0}^{n-1} v_k$ is an $\mathcal{F}_n$-adapted submartingale, where

$$v_k = \frac{v}{4}1\{k < \sigma \wedge \eta\} \geq \frac{v}{4}1\{n < \sigma \wedge \eta\}, \quad \text{for } 0 \leq k < n.$$  

By construction, $0 \leq Z_n \leq A^2(1 + x)^2$, so

$$0 = E[Z_0 | \mathcal{F}_0] \leq E[Z_n | \mathcal{F}_0] - \sum_{k=0}^{n-1} E[v_k | \mathcal{F}_0]$$

$$\leq A^2(1 + x)^2 - \sum_{k=0}^{n-1} \frac{v}{4} P[n < \sigma \wedge \eta | \mathcal{F}_0],$$

which implies that $n(v/4) P[n < \sigma \wedge \eta | \mathcal{F}_0] \leq A^2(1 + x)^2$. Hence, since

$$P\left[\max_{0 \leq m \leq n} \|Y_m\| < x \mid \mathcal{F}_0\right] = P[n < \sigma | \mathcal{F}_0]$$

$$= P[n < \sigma, n < \eta | \mathcal{F}_0] + P[n < \sigma, n \geq \eta | \mathcal{F}_0]$$

$$\leq P[n < \sigma \wedge \eta | \mathcal{F}_0] + P[\eta \leq n | \mathcal{F}_0],$$

the statement of the theorem follows. \hfill \Box

Now we can give the proof of Lemma 4.1.1.

Proof of Lemma 4.1.1. Let $X_n = \|\xi_n\|$ and $\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)$. Then the assumption (MD0) on $\xi_n$ shows that $X_n$ satisfies (L0) with $\mathcal{S} = \{\|x\| : x \in \Sigma\} \subseteq \mathbb{R}_+$; note that $\inf \mathcal{S} = 0$ since $0 \in \Sigma$ and $\sup \mathcal{S} = \infty$ since $\Sigma$ is unbounded.

To prove the non-confinement statement, we show that $X_n$ satisfies (L3) and then apply Proposition 3.3.4. Thus we must verify (3.10). By the Markov property and time-homogeneity, it suffices to check that for any $x \in \mathbb{R}_+$ there exist $r_x \in \mathbb{N}$ and $\delta_x > 0$ such that

$$P\left[\max_{0 \leq m \leq r_x} X_m \geq x \mid \mathcal{F}_0\right] \geq \delta_x, \quad \text{on } \{X_0 \leq x\}. \quad (4.12)$$
To verify (4.12) we apply Theorem 4.1.4 with \( Y_n = \xi_n - \xi_0 \); the assumptions (4.1), (MD2), and (MD1) imply that the conditions (4.7), (4.8), and (4.9) are satisfied (with \( \eta = \infty \)). Hence the triangle inequality and Theorem 4.1.4 show that, on \( \{ \| \xi_0 \| \leq x \} \),

\[
P \left[ \max_{0 \leq m \leq n} X_m \geq x \left| F_0 \right. \right] \geq P \left[ \max_{0 \leq m \leq n} \| \xi_m - \xi_0 \| \geq 2x \left| F_0 \right. \right] \\
\geq 1 - \frac{D(1 + 2x)^2}{n}, \text{ a.s.}
\]

Thus for any \( x \) we may choose \( n = r_x \) big enough so that this last probability is at least \( 1/2 \), say, which verifies (4.12) and completes the proof. \( \square \)

### 4.1.4 Relation to Lamperti's problem

The basic idea behind the proof of Theorem 4.1.3 is the same as that described in Section 1.5 and carried through in Example 3.5.4, namely to consider the one-dimensional process \( X_n = \| \xi_n \| \), which can be studied by the methods of Chapter 3. The additional technical difficulty introduced by assuming (4.1) rather than uniformly bounded increments is overcome using truncation ideas similar to those employed in Theorems 2.5.7 and 2.5.18, and repeatedly in Chapter 3.

Let \( F_n = \sigma(\xi_0, \dots, \xi_n) \). The assumption (MD0) on \( \xi_n \) shows that \( X_n \) satisfies (L0) with \( S = \{ \| x \| : x \in \Sigma \} \subseteq \mathbb{R}_+ \); note that \( \inf S = 0 \) since \( \emptyset \in \Sigma \) and \( \sup S = \infty \) since \( \Sigma \) is unbounded. As in Chapter 3, we write \( \Delta_n = X_{n+1} - X_n \). To apply the results of Chapter 3, we must study the increments \( \Delta_n \) given \( \xi_n = x \in \Sigma \); in general, \( X_n \) is not itself a Markov process.

We make an important comment on notation. When we write \( O(\| x \|^{-1-\delta}) \), and similar expressions, these are understood to be uniform in \( x \). That is, if \( f : \mathbb{R}^d \to \mathbb{R} \) and \( g : \mathbb{R}_+ \to \mathbb{R}_+ \), we write \( f(x) = O(g(\| x \|)) \) to mean that there exist \( C \in \mathbb{R}_+ \) and \( r \in \mathbb{R}_+ \) such that

\[
|f(x)| \leq Cg(\| x \|) \text{ for all } x \in \Sigma \text{ with } \| x \| \geq r.
\]

**Lemma 4.1.5.** Suppose that (MD0) holds, and that (4.1) holds for some \( p > 0 \). Then, for \( X_n := \| \xi_n \| \), we have

\[
\sup_{x \in \Sigma} \mathbb{E}[|\Delta_n|^p | \xi_n = x] < \infty. \tag{4.14}
\]

Moreover, the radial increment moment functions satisfy the following.
(i) If $p > 1$, then, for any $r > 0$ with $r < (p \land 2) - 1$, as $\|x\| \to \infty$,
\[
\mathbb{E}[\Delta_n \mid \xi_n = x] = \langle \dot{x}, \mu(x) \rangle + O(\|x\|^{-r}). \tag{4.15}
\]

(ii) If $p > 2$, then, for some $\delta = \delta(p) > 0$, as $\|x\| \to \infty$,
\[
\mathbb{E}[\Delta_n \mid \xi_n = x] = \langle \dot{x}, \mu(x) \rangle + \frac{\text{tr} M(x) - \dot{x}^\top M(x) \dot{x}}{2\|x\|} + O(\|x\|^{-1-\delta}); \tag{4.16}
\]
\[
\mathbb{E}[\Delta_n^2 \mid \xi_n = x] = \dot{x}^\top M(x) \dot{x} + O(\|x\|^{-\delta}). \tag{4.17}
\]

Lemma 4.1.5 is central to our analysis of the recurrence and transience of $\xi_n$, by applying the results of Chapter 3 to $X_n = \|\xi_n\|$. For example, if $\mu(x) = 0$ then (4.16) shows that the drift of $X_n$ is of order $1/X_n$, the critical regime for Lamperti processes; see Section 4.1.5.

The rest of this section is devoted to the proof of Lemma 4.1.5. The following notation will be useful. Given $x \neq 0$ and $y \in \mathbb{R}^d$, write
\[
y_x := \frac{\langle x, y \rangle}{\|x\|} = \langle \hat{x}, y \rangle, \tag{4.18}
\]
so that $y_x$ is the component of $y$ in the $\hat{x}$ direction, and $y - y_x \hat{x}$ is a vector orthogonal to $\hat{x}$. For convenience, we write simply $\theta$ for $\theta_0 = \xi_1 - \xi_0$, and let $\theta_x$ be the radial component of $\theta$ at $\xi_0 = x$ in accordance with the notation defined at (4.18); no confusion should arise with our notation $\theta_n$ defined previously.

Given $\xi_0 = x$, the increment $\Delta_0 = X_1 - X_0$ is given in terms of $x$ and the increment $\theta = \xi_1 - \xi_0$ by
\[
\|x + \theta\| - \|x\| = \sqrt{(x + \theta, x + \theta)} - \|x\|
= \|x\| \left[ \left(1 + \frac{2\theta_x}{\|x\|} + \frac{\|\theta\|^2}{\|x\|^2} \right)^{1/2} - 1 \right]. \tag{4.19}
\]

We would like to use Taylor’s theorem here, as in Example 3.5.4, but this is only legitimate provided $\theta$ is not too big compared to $x$. Thus let
\[
A_x := \{\|\theta\| \leq \|x\|^{\gamma}\}, \tag{4.20}
\]
for some $\gamma \in (0, 1)$ to be chosen appropriately. On the event $A_x$ we approximate (4.19) using Taylor’s formula for $(1 + y)^{1/2}$, and on the event $A_x^c$ we bound (4.19) using (4.1). For various truncation estimates, we need the following analogue of Lemma 3.4.2.
Lemma 4.1.6. Suppose that (4.1) holds for some $p > 0$, and $A_x$ is as defined at (4.20) for some $\gamma \in (0, 1)$. Then, for any $q \in [0, p]$,

$$
\mathbb{E}_x [\|\theta\|^q 1(A^c_x)] = O(\|x\|^{-\gamma(p-q)}).
$$

Proof. By definition of $A_x$, we have for $q \in [0, p]$,

$$
\|\theta\|^q 1(A^c_x) = \|\theta\|^p \|\theta\|^{q-p} 1(A^c_x) \leq \|\theta\|^p \|x\|^{\gamma(q-p)},
$$

and the result follows on taking expectations, by (4.1). \qed

Now we can complete the proof of Lemma 4.1.5 by carrying through the scheme outlined above.

Proof of Lemma 4.1.5. By time-homogeneity, it suffices to consider the case $n = 0$. By the triangle inequality, $|X_1 - X_0| = \|\xi_0 + \theta - \|\xi_0\| \leq \|\theta\|$, so that (4.14) follows from (4.1).

Define the event $A_x$ as at (4.20), for some $\gamma \in (0, 1)$ to be specified later. We prove part (ii); so suppose that (4.1) holds for $p > 2$. For all $y > -1$,

$$
(1 + y)^{1/2} = 1 + \frac{1}{2} y - \frac{1}{8} y^2 (1 + \varphi y)^{-3/2},
$$

for some $\varphi = \varphi(y) \in [0, 1]$, so on the event $A_x$, $\|x + \theta\| - \|x\|$ is equal to

$$
\|x\| \left( \frac{\theta_x}{\|x\|} + \frac{\|\theta\|^2}{2\|x\|^2} - \frac{1}{8} \left( \frac{2\theta_x}{\|x\|} + \frac{\|\theta\|^2}{\|x\|^2} \right)^2 (1 + O(\|x\|^{-1})) \right),
$$

where $\theta_x = \langle \dot{x}, \theta \rangle$ as defined at (4.18). Here, on the event $A_x$,

$$
\left( \frac{2\theta_x}{\|x\|} + \frac{\|\theta\|^2}{\|x\|^2} \right)^2 = \frac{4\theta_x^2}{\|x\|^2} + \|\theta\|^2 O(\|x\|^{-3}),
$$

using the fact that $|\theta_x| \leq \|\theta\| \leq \|x\|^{\gamma}$ for $\gamma < 1$. Hence, on the event $A_x$,

$$
\|x + \theta\| - \|x\| = \theta_x + \frac{\|\theta\|^2 - \theta_x^2}{2\|x\|} + \|\theta\|^2 O(\|x\|^{-3}). \quad (4.21)
$$

It follows from (4.21) and the triangle inequality that

$$
\left| (\|x + \theta\| - \|x\|) 1(A_x) - \left( \theta_x + \frac{\|\theta\|^2 - \theta_x^2}{2\|x\|} \right) \right|
$$
Here, by the $q = 1$ and $q = 2$ cases of Lemma 4.1.6, we have
\[
\mathbb{E}_x[\|\theta\|1(A^c_x)] = O(\|x\|^{-\gamma(p-1)}), \quad \text{and} \quad \mathbb{E}_x[\|\theta\|^21(A^c_x)] = O(\|x\|^{-\gamma(p-2)}),$
so that, taking expectations in (4.22), we obtain
\[
\mathbb{E}_x[(\|x + \theta\| - \|x\|)1(A_x) - \left(\theta_x + \frac{\|\theta\|^2 - \theta_x^2}{2\|x\|}\right)] = O(\|x\|^{-\gamma(p-1)}) + O(\|x\|^{-\gamma(p-2)-1}) + O(\|x\|^{-\gamma-3}).
\]
Let $\delta \in (0, 1)$ with $\delta < p - 2$. Choose $\gamma$ such that $\frac{1 + \delta}{p-1} < \gamma < 1$. Then $-\gamma(p-2) - 1 < -\gamma(p-1) < -1 - \delta$ and $\gamma - 3 < -1 - \delta$, so it follows that
\[
\mathbb{E}_x[(\|x + \theta\| - \|x\|)1(A_x)] = \mathbb{E}_x \theta_x + \frac{\mathbb{E}_x[\|\theta\|^2 - \theta_x^2]}{2\|x\|} + O(\|x\|^{-1-\delta})
= \langle \hat{x}, \mu(x) \rangle + \frac{\text{tr} M(x) - \hat{x}^\top M(x) \hat{x}}{2\|x\|} + O(\|x\|^{-1-\delta}),
\]
using (4.4), (4.5), and the observation that, by linearity,
\[
\mathbb{E}_x \theta_x = \mathbb{E}_x \langle \hat{x}, \theta \rangle = \langle \hat{x}, \mathbb{E}_x \theta \rangle = \langle \hat{x}, \mu(x) \rangle.
\]
On the other hand, by the triangle inequality,
\[
\mathbb{E}_x[(\|x + \theta\| - \|x\|)1(A^c_x)] \leq \mathbb{E}_x[\|\theta\|1(A^c_x)] = O(\|x\|^{-\gamma(p-1)}),
\]
by the $q = 1$ case of Lemma 4.1.6; again, by choice of $\gamma$, we have that the right-hand side of (4.24) is $O(\|x\|^{-1-\delta})$. Since
\[
\|x + \theta\| - \|x\| = (\|x + \theta\| - \|x\|)1(A_x) + (\|x + \theta\| - \|x\|)1(A^c_x),
\]
we can combine (4.23) and (4.24) to obtain (4.16). For the second moment, we use the identity
\[
(\|x + \theta\| - \|x\|)^2 = \|x + \theta\|^2 - \|x\|^2 - 2\|x\|((\|x + \theta\| - \|x\|))
= 2\|x\|\theta_x + \|\theta\|^2 - 2\|x\|((\|x + \theta\| - \|x\|)),$
so that
\[
\mathbb{E}_x[\Delta_0^2] = 2\|x\| \mathbb{E}_x \theta_x + \mathbb{E}_x[\|\theta\|^2] - 2\|x\| \mathbb{E}_x \Delta_0 = \mathbb{E}_x[\theta_x^2] + O(\|x\|^{-\delta}),
\]
by (4.16), which gives (4.17). This completes the proof of part (ii).

The proof of part (i) is similar, now supposing that (4.1) holds for \( p > 1 \) only. In this case, Taylor’s theorem shows that the analogue of (4.21) is
\[
(\|x + \theta\| - \|x\|) 1(A_x) = \langle \dot{x}, \theta \rangle 1(A_x) + \|\theta\|^2 1(A_x) O(\|x\|^{-1}).
\]
(4.25)

Here, by definition of \( A_x \),
\[
\mathbb{E}_x[\|\theta\|^2 1(A_x)] \leq \mathbb{E}_x[\|\theta\|^p\|\theta\|^{(2-p)^+} 1(A_x)] \leq \|x\|^{\gamma(2-p)^+} \mathbb{E}_x[\|\theta\|^p],
\]
which is \( O(\|x\|^{\gamma(2-p)^+}) \) by (4.1), and, as shown above, \( \mathbb{E}_x[\|\theta\| 1(A_x)] = O(\|x\|^{-\gamma(p-1)}) \), so
\[
\mathbb{E}_x[\langle \dot{x}, \theta \rangle 1(A_x)] = \langle \dot{x}, \mathbb{E}_x \theta \rangle + O(\|x\|^{-\gamma(p-1)}).
\]
Thus taking expectations in (4.25) we obtain
\[
\mathbb{E}_x[(\|x + \theta\| - \|x\|) 1(A_x)] = \langle \dot{x}, \mathbb{E}_x \theta \rangle + O(\|x\|^{-\gamma(p-1)}) + O(\|x\|^{\gamma(2-p)^+ - 1}).
\]

On the other hand, (4.24) still applies, which contributes another \( O(\|x\|^{-\gamma(p-1)}) \) term, so that
\[
\mathbb{E}_x[\|x + \theta\| - \|x\|] = \langle \dot{x}, \mu(x) \rangle + O(\|x\|^{\gamma(2-p)^+ - \gamma}),
\]
since \( \gamma(1-p) = \gamma(2-p) - \gamma \leq \gamma(2-p)^+ - \gamma \). Since \( \gamma \in (0, 1) \) was arbitrary, and \( (2-p)^+ - 1 = 1 - (p \wedge 2) < 0 \), we obtain (4.15).

4.1.5 Sufficient conditions for recurrence and transience

In this section we present sufficient conditions for recurrence and transience of the non-homogeneous random walk \( \xi_n \), assuming (MD0) and that (4.1) holds for some \( p > 2 \). In the zero drift case, the main result of this section (Theorem 4.1.7) will allow us to deduce Theorems 4.1.2 and 4.1.3, but the result is also valid for the non-zero drift case, and will later enable us to deduce results on elliptic random walks (Section 4.2) and centrally biased random walks (Section 4.4).

For this section, since we do not impose a zero drift condition, we cannot rely on Lemma 4.1.1; instead we must assume a non-confinement condition.
4.1. Introduction

**(MD3)** Suppose that \( \limsup_{n \to \infty} \| \xi_n \| = +\infty \), a.s.

In the case where \( \Sigma \) is locally finite and \( \xi_n \) is irreducible, (MD3) is satisfied, as shown in Corollary 2.1.10.

For some of our results, we also need to assume a non-degeneracy condition on the increment covariance matrix \( M(x) \) as defined at (4.3), to ensure that the walk is genuinely \( d \)-dimensional.

**(MD4)** Suppose that there exists \( v_0 > 0 \) such that,

\[
\inf_{e \in \mathbb{S}_d} (e^\top M(x)e) \geq v_0, \quad \text{for all } x \in \Sigma. \tag{4.26}
\]

The assumption (MD4) is stronger than (MD2), in that it imposes a uniform lower bound not just on the trace of \( M(x) \) but on all its eigenvalues, but weaker than uniform ellipticity, since (3.13) implies (4.26).

**Theorem 4.1.7.** Suppose that (MD0) and (MD3) hold, and that (4.1) holds for some \( p > 2 \).

(i) A sufficient condition for \( \xi_n \) to be transient is

\[
\lim_{r \to \infty} \inf_{x \in \Sigma : \|x\| \geq r} \left( 2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x)\hat{x} \right) > 0. \tag{4.27}
\]

(ii) If (MD4) also holds, then a sufficient condition for \( \xi_n \) to be recurrent is, as \( r \to \infty \),

\[
\sup_{x \in \Sigma : \|x\| \geq r} \left( 2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x)\hat{x} \right) \leq o(\log^{-1} r). \tag{4.28}
\]

(iii) Let \( \lambda_r = \min \{ n \geq 0 : \|\xi_n\| \leq r \} \), and suppose that

\[
\lim_{r \to \infty} \sup_{x \in \Sigma : \|x\| \geq r} \left( 2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) \right) < 0. \tag{4.29}
\]

Then there exists \( r_0 \in \mathbb{R}_+ \) such that, for any \( r > r_0 \), \( \mathbb{E} \lambda_r < \infty \).

One interpretation of Theorem 4.1.7 is as follows: for fixed dimension \( d \in \mathbb{N} \) and increment covariance matrix function \( M(x) \), there is a constant \( c_{\text{tra}} \) such that a sufficient condition to ensure that \( \xi_n \) is transient is

\[
\liminf_{n \to \infty} \left( \|x\|\langle \mu(x), \hat{x} \rangle \right) > c_{\text{tra}};
\]
Chapter 4. Many-dimensional random walks

specifically, one may take

\[ c_{\text{tra}} = \lim_{r \to \infty} \sup_{\|x\| \geq r} (\hat{x}^\top M(x)\hat{x} - \frac{1}{2} \text{tr} M(x)). \]

Analogously, given (MD4), a sufficient for recurrence is the condition

\[ \limsup_{n \to \infty} (\|x\| \langle \mu(x), \hat{x} \rangle) < c_{\text{rec}}, \]

where

\[ c_{\text{rec}} = \lim_{r \to \infty} \inf_{\|x\| \geq r} (\hat{x}^\top M(x)\hat{x} - \frac{1}{2} \text{tr} M(x)). \]

In other words, given \( M(x) \) one can always arrange a drift field \( \mu(x) \), satisfying \( \|\mu(x)\| = O(\|x\|^{-1}) \), such that the walk is transient; or recurrent; or positive-recurrent. We give some concrete examples in Section 4.4. In Section 4.2 we will apply Theorem 4.1.7 to show that in any dimension \( d \geq 2 \), even if \( \mu(x) = 0 \) everywhere, by suitably adjusting \( M(x) \) one can achieve transience or recurrence as desired (but not positive-recurrence).

One useful corollary to Theorem 4.1.7 is the following sufficient condition for transience in the zero-drift setting. Recall that \( \lambda_{\text{max}}(M) \) denotes the maximum eigenvalue of matrix \( M \).

**Corollary 4.1.8.** Suppose that (MD0), (MD1) and (MD3) hold, and that (4.1) holds for some \( p > 2 \). A sufficient condition for \( \xi_n \) to be transient is

\[ \lim_{r \to \infty} \inf_{x \in \Sigma: \|x\| \geq r} \left( \text{tr} M(x) - 2 \lambda_{\text{max}}(M(x)) \right) > 0. \]  

**Proof.** We apply Theorem 4.1.7(i). We have from (MD1) that

\[ 2\|x\| \langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x)\hat{x} \geq \text{tr} M(x) - 2 \sup_{u \in \mathbb{S}^{d-1}} (u^\top M(x)u) \]

\[ = \text{tr} M(x) - 2 \lambda_{\text{max}}(M(x)), \]

by the variational characterization of the maximal eigenvalue. Under condition (4.30), we thus have that (4.27) holds, and then Theorem 4.1.7(i) completes the proof. \( \Box \)

Here is a different example of the application of Theorem 4.1.7.

**Example 4.1.9.** Suppose that \( \xi_n \) satisfying (MD0) is an irreducible Markov chain on \( \mathbb{Z}^d \), \( d \geq 1 \), with transition probabilities given for \( x \neq 0 \) by

\[ \mathbb{P}_x[\theta_0 = e_i] = \frac{1}{2d} \left( 1 - \frac{\epsilon}{\langle x, e_i \rangle} \right) \quad \text{and} \quad \mathbb{P}_x[\theta_0 = -e_i] = \frac{1}{2d} \left( 1 + \frac{\epsilon}{\langle x, e_i \rangle} \right), \]
for all $i \in \{1, \ldots, d\}$, where $\varepsilon \in (-1, 1)$ is a fixed parameter, and

$$P_0[\theta_0 = e_i] = P_0[\theta_0 = -e_i] = \frac{1}{2d}.$$ 

Then we calculate $M(x) = \frac{1}{d}I_d$ and, for $x \neq 0$,

$$\mu(x) = -\varepsilon \frac{e}{d} \sum_{i=1}^d \frac{e_i}{(x, e_i)}.$$ 

Thus for $x \neq 0$,

$$2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^T M(x) \hat{x} = -2\varepsilon + 1 - (2/d),$$

which is strictly positive if and only if $\varepsilon < \frac{d-2}{2d}$. Thus Theorem 4.1.7 parts (i) and (ii) show that this random walk is transient if $\varepsilon < \frac{d-2}{2d}$ and recurrent if $\varepsilon \geq \frac{d-2}{2d}$. Similarly, part (iii) shows that the walk is positive-recurrent if $\varepsilon > \frac{1}{2}$. \(\triangle\)

**Proof of Theorem 4.1.7.** We consider the process of norms $X_n = \|\xi_n\|$ and use Lemma 4.1.5 to apply the results of Chapter 3. The condition (MD3) shows that $X_n = \|\xi_n\|$ satisfies (L2). Given that (4.1) holds for some $p > 2$, Lemma 4.1.5 shows that $X_n = \|\xi_n\|$ satisfies (L1) for the same $p > 2$. In the notation of Section 3.3, we have for $k \in \{1, 2\}$ that

$$\mu_k(x) \geq \inf_{x \in \Sigma : \|x\| \geq x} \mathbb{E}[\mathcal{D}_n^k \mid \xi_n = x],$$

and similarly for $\bar{\mu}_k(x)$. Lemma 4.1.5(ii) shows that

$$2\|x\| \mathbb{E}[\mathcal{D}_n \mid \xi_n = x] - \mathbb{E}[\mathcal{D}_n^2 \mid \xi_n = x] = 2\|x\| \langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^T M(x) \hat{x} + O(\|x\|^{-\delta}).$$

Suppose that (4.27) holds. Then,

$$\liminf_{x \to \infty} (2x\mu_1(x) - \bar{\mu}_2(x)) > 0,$$

and Theorem 3.5.1 yields transience, proving part (i). Now consider part (ii). It follows from (MD4) and (4.17) that

$$\liminf_{x \to \infty} \mu_2(x) \geq \liminf_{r \to \infty} \mathbb{E}[\mathcal{D}_n^2 \mid \xi_n = x] \geq v_0.$$ 

Moreover, it follows from (4.16) and (4.17) that

$$2x\bar{\mu}_1(x) - \mu_2(x)$$
\[ \leq \sup_{x \in \Sigma : \|x\| \geq x} \left( 2\|x\| \langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x) \hat{x} \right) + O(x^{-\delta}), \]

which is \(o(\log^{-1} x)\) by (4.28). Together with (4.31), this shows that

\[ 2x \bar{\mu}_1(x) - \left( 1 + \frac{1 - \psi}{\log x} \right) \bar{\mu}_2(x) \leq - \frac{1 - \psi}{\log x} \bar{\mu}_2(x) + o(\log^{-1} x) \]

\[ \leq - \left( (1 - \psi)v_0 + o(1) \right) \log^{-1} x, \]

which is negative for \(\psi \in (0, 1)\) and all \(x\) sufficiently large, so Theorem 3.5.2 yields recurrence. This proves part (ii).

Finally, a similar argument shows that (4.29) implies that

\[ \limsup_{x \to \infty} \left( 2x \bar{\mu}_1(x) + \bar{\mu}_2(x) \right) < 0, \]

from which the \(\alpha = 1\) case of Theorem 3.7.1 gives part (iii).

To conclude this section, we apply Theorem 4.1.7 to give the proofs of Theorems 4.1.2 and 4.1.3.

**Proof of Theorem 4.1.2.** Suppose that \(d = 1\). In one dimension, \(M(x)\) is the scalar \(\mathbb{E}_x[\theta_0^2]\), and the condition (MD2) is equivalent to (MD4). Lemma 4.1.1 shows that (MD3) holds. Moreover, the zero drift condition (MD1) then shows that

\[ 2\|x\| \langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x) \hat{x} = - \mathbb{E}_x[\theta_0^2] \leq -v, \]

by (MD2). Hence the conditions of Theorem 4.1.7(ii) are satisfied, and this establishes recurrence.

**Proof of Theorem 4.1.3.** First, the argument in Remark 1.5.5 shows that, by considering an affine transformation of \(\mathbb{R}^d\), it suffices to consider the case where \(M = I_d\). Again, Lemma 4.1.1 shows that (MD3) holds. Given that \(M(x) = I_d\), we have \(\text{tr} M(x) = d\) and, for any \(u \in \mathbb{S}^{d-1}\), \(u^\top M(x)u = 1\); thus (MD4) holds. Now, we have

\[ 2\|x\| \langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x) \hat{x} = d - 2, \]

which is strictly positive if \(d \geq 3\), in which case transience follows from Theorem 4.1.7(i), and non-positive if \(d \leq 2\), in which case recurrence follows from Theorem 4.1.7(ii).
4.2 Elliptic random walks

4.2.1 Recurrence classification

Theorem 4.1.2 shows that in $d = 1$, under mild conditions, the classical Chung–Fuchs recurrence classification for homogeneous zero-drift random walks extends to zero-drift non-homogeneous random walks. This extension fails for dimension $d \geq 2$; in this section we demonstrate this, and hence prove Theorem 1.5.3. To do so, we introduce a family of non-homogeneous random walks that we call elliptic random walks, that demonstrate the possible anomalous behaviour of non-homogeneous random walks when compared to classical homogeneous random walks.

The following assumption on the asymptotic stability of the covariance structure of the process along rays is central; recall that $\| \cdot \|_{\text{op}}$ denotes the matrix (operator) norm.

(E1) Suppose that there exists a positive-definite matrix function $\sigma^2$ with domain $S^{d-1}$ such that, as $r \to \infty$,

$$
\varepsilon(r) := \sup_{x \in \Sigma : \|x\| \geq r} \|M(x) - \sigma^2(\hat{x})\|_{\text{op}} \to 0.
$$

A little informally, (E1) says that $M(x) \to \sigma^2(\hat{x})$ as $\|x\| \to \infty$; in what follows, we will often make similar statements, formal versions of which may be cast as in (E1).

Note that (MD2) and (E1) together imply that $\text{tr} \sigma^2(u) \geq v > 0$; next we impose a key assumption on the form of $\sigma^2$ that is considerably stronger. To describe this, it is convenient to introduce the notation $\langle \cdot, \cdot \rangle_u$ that defines, for each $u \in S^{d-1}$, an inner product on $\mathbb{R}^d$ via

$$
\langle y, z \rangle_u := y^\top \sigma^2(u) z = \langle y, \sigma^2(u) z \rangle, \quad \text{for } y, z \in \mathbb{R}^d.
$$

(E2) Suppose that there exist constants $U$ and $V$ with $0 < U \leq V < \infty$ such that, for all $u \in S^{d-1}$,

$$
\langle u, u \rangle_u = U, \quad \text{and} \quad \text{tr} \sigma^2(u) = V.
$$

Informally, $V$ quantifies the total variance of the increments, while $U$ quantifies the variance in the radial direction; necessarily $U \leq V$. The assumption that $U > 0$ excludes some degenerate cases. As we will see, one possible way to satisfy condition (E2) is to suppose that the eigenvectors of $\sigma^2(u)$ are all parallel or perpendicular to the vector $u$, and that the corresponding
eigenvalues are all constant as \( u \) varies; the level sets of the corresponding quadratic forms \( q_u(x) := \langle x, x \rangle_u \) for \( u \in S^{d-1} \) are then ellipsoids like those depicted in Figure 4.2.

Our main result on elliptic random walks is the following, which shows that both transience and recurrence are possible for any \( d \geq 2 \), depending on parameter choices; as seen in Theorem 4.1.2, this possibility of anomalous recurrence behaviour is a genuinely multidimensional phenomenon under our regularity conditions.

**Theorem 4.2.1.** Suppose that (MD0), (MD1), and (MD2) hold, and that (4.1) holds for some \( p > 2 \). Suppose also that (E1) and (E2) hold, with constants \( 0 < U \leq V < \infty \) as defined in (E2). Then the following recurrence classification is valid.

(i) If \( 2U < V \), then \( \xi_n \) is transient.

(ii) If \( 2U > V \), then \( \xi_n \) is recurrent.

(iii) If \( 2U = V \) and (E1) holds with \( \varepsilon(r) = o(\log^{-1} r) \), then \( \xi_n \) is recurrent.

An explicit family of examples satisfying conditions (E1) and (E2) is presented in Section 4.2.2. We deduce Theorem 4.2.1 by another application of Theorem 4.1.7.

**Proof of Theorem 4.2.1.** Lemma 4.1.1 shows that (MD3) holds. By definition of \( \varepsilon(r) \) at (E1) we have \( \|M(x) - \sigma^2(\hat{x})\|_{op} = O(\varepsilon(\|x\|)) \) as \( |x| \to \infty \). Then (E2) implies that

\[
\begin{align*}
\text{tr } M(x) &= \text{tr } \sigma^2(\hat{x}) + O(\varepsilon(\|x\|)) = V + O(\varepsilon(\|x\|)); \\
\hat{x}^\top M(x) \hat{x} &= \langle \hat{x}, \sigma^2(\hat{x}) \hat{x} \rangle + O(\varepsilon(\|x\|)) = U + O(\varepsilon(\|x\|)).
\end{align*}
\]

These expressions, together with (MD1), show that

\[
\sup_{x \in \Sigma : \|x\| \geq r} \left| 2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr } M(x) - 2\hat{x}^\top M(x) \hat{x} - (V - 2U) \right| = O(\varepsilon(r)).
\]

Given that \( \varepsilon(r) \to 0 \), if \( V - 2U > 0 \), then Theorem 4.1.7(i) yields transience, while if \( V - 2U < 0 \), then Theorem 4.1.7(ii) yields recurrence. In the boundary case \( V - 2U = 0 \), Theorem 4.1.7(ii) again yields recurrence, provided that \( \varepsilon(r) = o(\log^{-1} r) \). \( \square \)
4.2. Elliptic random walks

4.2.2 Example: Increments supported on ellipsoids

Let \( d \geq 2 \). We describe a specific model on \( \Sigma = \mathbb{R}^d \) where the jump distribution at \( x \in \mathbb{R}^d \) is supported on an ellipsoid having one distinguished axis aligned with the vector \( x \). The model is specified by two constants \( a, b > 0 \).

Construct \( \theta \) as follows. Let \( Q_{\hat{x}} \) be an orthogonal matrix representing a transformation of \( \mathbb{R}^d \) mapping \( e_1 \) to \( \hat{x} \), and write \( D = \sqrt{d} \text{diag}(a, b, \ldots, b) \).

Given \( X_0 = x \), take \( \zeta \) uniform on \( S^{d-1} \) and set

\[
\theta = \begin{cases} 
Q_{\hat{x}} D \zeta & \text{if } x \neq 0; \\
\zeta & \text{if } x = 0.
\end{cases}
\tag{4.32}
\]

See Figure 4.4 for an illustration of this construction.

Figure 4.4: Definition of \( \theta = Q_{\hat{x}} D \zeta \).

Thus \( \theta \) is a random point on an ellipsoid that has one distinguished semi-axis, of length \( a\sqrt{d} \), aligned in the \( \hat{x} \) direction, and all other semi-axes of length \( b\sqrt{d} \). Note that the law of \( \theta \) is well defined, since the distribution of \( Q_{\hat{x}} D \zeta \) is the same for any orthogonal matrix \( Q_{\hat{x}} \) satisfying \( Q_{\hat{x}} e_1 = \hat{x} \).

Concretely, observe that for \( x \neq 0 \), \( \theta \) can be rewritten as

\[
\tilde{\theta} = Q_{\hat{x}} D \zeta,
\]

where \( \tilde{\theta} = Q_{\hat{x}} \zeta \) is also uniform on \( S^{d-1} \), by the spherical symmetry of the uniform distribution. Moreover, the symmetric matrix \( H_{\hat{x}} := Q_{\hat{x}} D Q_{\hat{x}}^{\top} \) is determined explicitly in terms of \( \hat{x} \) via

\[
H_{\hat{x}} = Q_{\hat{x}}(b\sqrt{d}I_d + (a - b)\sqrt{d}e_1 e_1^{\top}) Q_{\hat{x}}^{\top} = b\sqrt{d}I_d + (a - b)\sqrt{d}\hat{x}\hat{x}^{\top}.
\]
Thus a more concrete specification of \( \theta \) is
\[
\theta = H_x\hat{\zeta} = b\sqrt{d}\hat{\zeta} + (a - b)\sqrt{d}\langle \hat{x}, \hat{\zeta} \rangle,
\]
with \( \hat{\zeta} \) taken to be uniform on \( S^{d-1} \).

As a corollary to Theorem 4.2.1, we get the following recurrence classification for this example.

**Corollary 4.2.2.** Let \( d \geq 1 \) and \( a, b \in (0, \infty) \). Let \( \xi_n \) be a random walk on \( \mathbb{R}^d \) with distribution determined by (4.32) for parameters \( a, b \in (0, \infty) \). Then \( \xi_n \) is transient if \( a^2 < (d - 1)b^2 \) and recurrent if \( a^2 \geq (d - 1)b^2 \).

**Proof.** Since \( \|\theta\| \) is bounded above by \( \sqrt{d}\max\{a, b\} \), assumption (4.1) holds for all \( p > 2 \). Moreover, by (4.32), we compute
\[
M(x) = E_x[\theta^\top] = E[Q_xD\zeta^\top DQ_x^\top] = Q_xD E[\zeta^\top]DQ_x^\top = \frac{1}{d}Q_xD^2Q_x^\top,
\]
by linearity of expectation, and using the fact that \( E[\zeta^\top] = \frac{1}{d}I_d \) for \( \zeta \) uniformly distributed on \( S^{d-1} \). Hence (E1) holds with
\[
\sigma^2(u) = \frac{1}{d}Q_uD^2Q_u^\top, \quad \text{for } u \in S^{d-1},
\]
and with \( \varepsilon(r) \) identically zero. Also, by a similar calculation (or a symmetry argument) we have \( \mu(x) = 0 \) for all \( x \in \mathbb{R}^d \), since \( E\zeta = 0 \); thus (MD1) holds. Finally, to verify that (MD2) and (E2) hold, note that the matrix \( \sigma^2(u) \) represented in coordinates for the orthonormal basis \( \{Q_u e_1 = u, Q_u e_2, \ldots, Q_u e_d\} \) is diagonal with entries \( a^2, b^2, \ldots, b^2 \). Indeed,
\[
\sigma^2(u) = Q_u[b^2I_d + (a^2 - b^2)e_1 e_1^\top]Q_u^\top = a^2uu^\top + b^2(I_d - uu^\top),
\]
and therefore \( \langle u, u \rangle_u = \langle u, \sigma^2(u)u \rangle = a^2 > 0 \) for all \( u \in S^{d-1} \), and
\[
\text{tr } M(x) = \text{tr } \sigma^2(x) = a^2 + (d - 1)b^2 > 0 \quad \text{for all } x \in \mathbb{R}^d.
\]
Thus we verify all the assumptions of Theorem 4.2.1, with \( U = a^2 \) and \( V = a^2 + (d - 1)b^2 \), and the claimed recurrence classification follows.

In two dimensions we can explicitly describe the random walk as follows. For \( x \in \mathbb{R}^2 \), \( x \neq 0 \) with \( x = (x_1, x_2) \) in Cartesian components, set \( x^\perp := (-x_2, x_1) \). Fix \( a, b \in (0, \infty) \). Let \( E_x(a, b) \) denote the ellipse with centre \( x \) and principal axes aligned in the \( x, x^\perp \) directions, with lengths \( 2\sqrt{2}a \), \( 2\sqrt{2}b \) respectively, given in parametrized form by
\[
E_x(a, b) := \left\{ x + \sqrt{2}a \frac{x}{\|x\|} \cos \varphi + \sqrt{2}b \frac{x^\perp}{\|x\|} \sin \varphi : \varphi \in (-\pi, \pi) \right\}, \quad (4.33)
\]
and for $x = 0$ set

$$E_0(a, b) := \{ e_1 \cos \varphi + e_2 \sin \varphi : \varphi \in (-\pi, \pi) \},$$

the unit circle. The parameter $\varphi$ in the parametrization (4.33) should be interpreted with caution: it is not, in general, the central angle of the parametrized point on the ellipse.

Given $\xi_n = x \in \mathbb{R}^2$, $\xi_{n+1}$ is taken to be distributed on $E_x(a, b)$, ‘uniformly’ with respect to the parametrization (4.33). Precisely, let $\varphi_0, \varphi_1, \ldots$ be a sequence of independent random variables uniformly distributed on $(-\pi, \pi]$. Then, on $\{ \xi_n \neq 0 \}$,

$$\xi_{n+1} = \xi_n + \sqrt{2a} \frac{\xi_n}{\|\xi_n\|} \cos \varphi_n + \sqrt{2b} \frac{\xi_n}{\|\xi_n\|} \sin \varphi_n,$$

while, on $\{ \xi_n = 0 \}$, $\xi_{n+1} = (\cos \varphi_n, \sin \varphi_n)$.

Figure 4.5 shows two simulated sample paths of this two-dimensional elliptic random walk, for different choices of $a$ and $b$, one recurrent and the other transient, corresponding roughly to the two case pictures in Figure 4.2.

4.3 Controlled driftless random walks

Consider $k$ spatially homogeneous random walks in $\mathbb{R}^d$, $d \geq 3$, with zero drift and with increment covariance matrices $M_1, \ldots, M_k$. Assume also that
$M_1, \ldots, M_k$ are full-rank, so that these walks are truly $d$-dimensional, and therefore transient.

Now, assume that at each discrete moment of time, we are able to choose (according to some rule that does not depend on the future) among the $k$ available jump laws; in other words, we control the distribution of the next step. The main question of this section is: Is it possible to make the walk recurrent by choosing a suitable control policy? Below, we are going to show that having $k < d$ walks is not enough to achieve recurrence, but there are recurrent examples with $k = d$.

We now give the precise definition of the walks we will be considering.

**Definition 4.3.1.** Let $\pi_1, \ldots, \pi_k$ be $k$ probability measures in $\mathbb{R}^d$ and let $(\theta_j, j = 1, \ldots, k, n = 1, 2, 3, \ldots)$ be independent random variables with $\theta_j \sim \pi_j$ for all $n$ and all $j = 1, \ldots, k$. Define an adapted rule $\ell = (\ell(n), n \geq 0)$ with respect to a filtration $(\mathcal{F}_n, n \geq 0)$ to be a process such that, for all $n \in \mathbb{Z}_+$, $\ell(n) \in \{1, \ldots, k\}$ is $\mathcal{F}_n$-measurable. We say that the walk $(\xi_n, n \geq 0)$ with $\xi_0 = 0$ is generated by the measures $\pi_1, \ldots, \pi_k$ and the rule $\ell$, if $\xi_n$ is $\mathcal{F}_n$-measurable and

$$\xi_{n+1} = \xi_n + \theta_{\ell(n)}^{(n)}.$$ 

In particular, observe that any Markov chain with at most $k$ different jump distributions fits into this framework. As announced, we assume that the jump distributions have mean zero, the corresponding covariance matrices are $M_1, \ldots, M_k$, and also that they satisfy the moment condition (4.1) for some $p > 2$.

Our first result is as follows.

**Theorem 4.3.2.** Let $\pi_1, \ldots, \pi_k$ be $d$-dimensional measures in $\mathbb{R}^d$, $d \geq 3$, with zero mean and $p$ moments, for some $p > 2$ and $k < d$. If $\xi_n$ is a random walk generated by these measures and an arbitrary adapted rule $\ell$, then $\xi_n$ is transient.

**Proof.** Clearly, without loss of generality we may assume that $k = d - 1$. The key to the proof is the following fact, which is a consequence of Theorem 1 of [87]: there exists a matrix $A$ such that

$$\text{tr}(AM_iA^\top) > 2\lambda_{\text{max}}(AM_iA^\top), \text{ for all } i \in \{1, \ldots, d - 1\}. \quad (4.34)$$

With (4.34) to hand, the rest of the proof is quite straightforward. Define $\tilde{\xi}_n = A\xi_n$. Clearly, $\tilde{\xi}_n$ is also a controlled walk with zero drift and covariance matrices $AM_iA^\top, i = 1, \ldots, d - 1$. As we saw on the route to Corollary 4.1.8, the Lyapunov function $\|x\|^{-\alpha}$ with small enough $\alpha > 0$ “works” for each of
4.3. Controlled driftless random walks

the \(d-1\) measures belonging to \(\hat{\xi}\); so, this function also shows transience of \(\hat{\xi}\) itself. \(\square\)

We now construct an example that shows that \(k = d\) may be already enough for recurrence. Let us consider a nearest-neighbour random walk \((\xi_n, n \geq 0)\) on \(\mathbb{Z}^d, d \geq 3\), defined in the following way. Fix a parameter \(\gamma > 0\), and for \(x = (x_0, \ldots, x_{d-1}) \in \mathbb{Z}^d\) define \(\varrho(x) = \min\{k : |x_k| = \max_{j=0,\ldots,d-1} |x_j|\}\). Then

\[
\xi_{n+1} = \xi_n + \theta_{n+1},
\]

where \(\theta_{n+1} = \pm e_{\varrho(\xi_n)}\) with probabilities \(\frac{\gamma}{2(\gamma + d - 1)}\) and \(\delta_{n+1} = \pm e_k\) for \(k \neq \varrho(\xi_n)\) with probabilities \(\frac{1}{2(\gamma + d - 1)}\). In words, we choose the maximal (in absolute value) coordinate of \(\xi_n\) with weight \(\gamma\) and all the other coordinates with weight 1, and then add 1 or \(-1\) to the chosen coordinate with equal probabilities.

**Theorem 4.3.3.** For any dimension \(d \geq 3\) there exists large enough \(\gamma = \gamma(d)\) such that the above random walk is recurrent.

The proof of this result is somewhat different from the arguments used in the rest of the book. Instead of taking a particular Lyapunov function (maybe depending on a few parameters), we work with an “unknown” function, write down the conditions that it must satisfy, and prove that such a function exists.

**Proof of Theorem 4.3.3.** By Theorem 2.5.2, to prove recurrence it is enough to find a nonnegative function \(f\) such that \(f(x) \rightarrow \infty\) as \(\|x\| \rightarrow \infty\), and

\[
\mathbb{E}[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = x] \leq 0, \text{ for all } x \text{ with } \|x\| \text{ large enough. } \quad (4.35)
\]

Before presenting the explicit construction of such a function, let us informally explain the intuition behind this construction. First of all, recall from (1.4) and (1.5) that if \(S_n\) is a simple random walk in \(\mathbb{Z}^d\), then

\[
\mathbb{E}[\|S_{n+1}\| - \|S_n\| \mid S_n = x] = \frac{d-1}{2d} \frac{1}{\|x\|} + O(\|x\|^{-2});
\]

\[
\mathbb{E}[(\|S_{n+1}\| - \|S_n\|)^2 \mid S_n = x] = \frac{1}{d} + O(\|x\|^{-1}).
\]

One can observe that the ratio of the drift to the second moment behaves as \(\frac{d-1}{2\|x\|}\); combined with the well-known fact that the SRW is recurrent for \(d = 2\) and transient for \(d \geq 3\), this suggests that, to obtain recurrence, the
constant in this ratio should not be too large (in fact, at most $\frac{1}{2}$). Then, the second moment depends essentially on the dimension, and thus it is crucial to look at the drift. So, consider a (smooth in $\mathbb{R}^d \setminus \{0\}$) function $g(x) = \Theta(||x||)$; we shall try to figure out how the level sets of $g$ should be so that the “drift outside” with respect to $g$ “behaves well” (i.e., the drift multiplied by $||x||$ is uniformly bounded above by a not-so-large constant).

For that, let us look at Figure 4.6: level sets of $g$ are indicated by solid lines, vectors’ sizes correspond to transition probabilities. Then, it is intuitively clear that the case of “moderate” drift corresponds to the following:

- the “preferred” direction is radial, the curvature of level lines is large, or
- the “preferred” direction is transversal and the curvature of level lines

Figure 4.6: Looking at the level sets: how large is the drift? We have very small drift in case 1, very large drift in case 2, and moderate drifts in cases 3 and 4.
is small;
also, it is clear that “very flat” level lines always generate small drift. However, one cannot hope to make the level lines very flat everywhere, as they should go around the origin. So, the idea is to find in which places one can afford “more curved” level lines.

Observe that, for the random walk we are considering now, the preferred direction near the axes is the radial one, while in the “diagonal” regions it is in some intermediate position between transversal and radial. This indicates that the level sets of the Lyapunov function should look as depicted on Figure 4.7: more curved near the axes, and more flat off the axes.

We are going to use the Lyapunov function

\[ f(x) = \|x\|^\alpha \varphi(\hat{x}), \]

where \( \alpha \) is a positive constant and \( \varphi : \mathbb{S}^{d-1} \rightarrow \mathbb{R} \) is a positive continuous function, symmetric in the sense that for any \((u_0, \ldots, u_{d-1}) \in \mathbb{S}^{d-1}\) we have \( \varphi(u_0, \ldots, u_{d-1}) = \varphi(\tau_0 u_{\sigma(0)}, \ldots, \tau_{d-1} u_{\sigma(d-1)}) \) for any permutation \( \sigma \) and any \( \tau \in \{-1, 1\}^d \). By the previous discussion, to have the level sets as on
Figure 4.7, we are aiming at constructing \( \varphi \) with values close to 1 near the “diagonals” and less than 1 near the axes.

By symmetry, it is enough to define the function \( \varphi \) for \( u \in S^{d-1} \) such that \( u_0 \geq u_1, \ldots, u_{d-1} \geq 0 \) (clearly, it then holds that \( u_0 > 0 \)), and, again by symmetry, it is enough to prove (4.35) for all large enough \( x \in \mathbb{Z}^d \) of the same kind. For such \( u \) abbreviate \( s_j = u_j/u_0, j = 1, \ldots, d-1 \); observe that, if \( u = \tilde{x} \), then \( s_j = x_j/x_0 \). We are going to look for the function (for \( u \) as above) \( \varphi(u) = 1 - \alpha \psi(s_1, \ldots, s_{d-1}) \), where \( \psi \) is a function with continuous third partial derivatives on \([0,1]^{d-1}\) (in fact, it will become clear that the function \( \psi \) extended by means of symmetry on \([-1,1]^d\) has continuous third derivatives on \([-1,1]^d\).

Next, we proceed in the following way: we do calculations in order to figure out, which conditions the function \( \psi \) should satisfy in order to guarantee that (4.35) holds, and then try to construct a concrete example of \( \psi \) that satisfies these conditions.

In the computations below, we will use the abbreviations
\[
\psi'_{j} := \frac{\partial \psi(s_1, \ldots, s_{d-1})}{\partial s_j}, \quad j = 1, \ldots, d-1,
\]
\[
\psi''_{ij} := \frac{\partial^2 \psi(s_1, \ldots, s_{d-1})}{\partial s_i \partial s_j}, \quad i, j = 1, \ldots, d-1.
\]

Let us now consider \( x \in \mathbb{Z}^d \). From now on we will refer to the situation when \( x_0 > x_1, \ldots, x_{d-1} \geq 0 \) as the “non-boundary case” and \( x_0 = x_1 = \cdots = x_m > x_{m+1} \geq \cdots \geq x_{d-1} \geq 0 \) for some \( m \geq 1 \) as the “boundary case”. Observe for the boundary case the corresponding \( s \) will be of the form \( s = (1, \ldots, (1)_m, s_{m+1}, \ldots, s_{d-1}) \); here and in the sequel we indicate the position of the symbol in a row by placing parentheses and putting a subscript.

First we deal with the non-boundary case.

Let us consider \( x \in \mathbb{Z}^d \) such that \( x_0 > x_1, \ldots, x_{d-1} \geq 0 \). After some extensive but straightforward computations (see (3.6) in [257] for details) it can be obtained that
\[
\mathbb{E}[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = x] = -\alpha \|x\|^{\alpha-2} \Phi(x, \psi) + \alpha \|x\|^{\alpha-2} (\gamma^{-1} \Phi_1(x, \psi, \gamma, \alpha) + \alpha \Phi_2(x, \psi, \gamma, \alpha)),
\]
(4.36)
where \( \Phi_1 \) and \( \Phi_2 \) are uniformly bounded for large enough \( x \), and
\[
\Phi(x, \psi) = \frac{x_0^2}{\|x\|^2} - \frac{1}{2} \sqrt{\frac{2}{x_0^2}} \left( \sum_{j=1}^{d-1} s_j \psi'_{j} + \frac{1}{2} \sum_{i,j=1}^{d-1} s_i s_j \psi''_{ij} \right).
\]
(4.37)
The idea is then to prove that, with a suitable choice for \( \psi \), the quantity \( \Phi(x, \psi) \) will be uniformly positive for all large enough \( x \), and then the second term in the right-hand side of (4.36) can be controlled by choosing large \( \gamma \) and small \( \alpha \). This will make (4.36) negative for all large \( x \).

Now, in order to obtain a simplified form for (4.37), we pass to the (hyper)spherical coordinates:

\[
s_1 = r \cos \beta_1,
\]
\[
s_2 = r \sin \beta_1 \cos \beta_2,
\]
\[
\ldots
\]
\[
s_{d-2} = r \sin \beta_1 \ldots \sin \beta_{d-3} \cos \beta_{d-2},
\]
\[
s_{d-1} = r \sin \beta_1 \ldots \sin \beta_{d-3} \sin \beta_{d-2}.
\]

Since \( \|x\|^2 = 1 + r^2 \), and (abbreviating \( \psi'_r = \frac{\partial \psi}{\partial r} \) and \( \psi''_{rr} = \frac{\partial^2 \psi}{\partial r^2} \))

\[
\psi'_r = \frac{1}{r} \sum_{j=1}^{d-1} s_j \psi'_j, \quad \psi''_{rr} = \frac{1}{r^2} \sum_{i,j=1}^{d-1} s_i s_j \psi''_{ij},
\]

we have

\[
\Phi(x, \psi) = \frac{1}{1 + r^2} - \frac{1}{2} + (1 + r^2) \left( r \psi'_r + \frac{r^2}{2} \psi''_{rr} \right)
\]
\[
= \frac{1 + r^2}{2} \left( \frac{1 - r^2}{(1 + r^2)^2} + \left( r^2 \psi'_r \right)' \right). \tag{4.38}
\]

Now, we define the function \( \psi \) (it will depend on \( r \) only, not on \( \beta_1, \ldots, \beta_{d-2} \)) in the following way. First, clearly, we need to define \( \psi(r) \) for \( r \in [0, \sqrt{d - 1}] \).

Then, observe that

\[
\int_0^{\sqrt{d-1}} \frac{1 - r^2}{(1 + r^2)^2} \, dr = \frac{\sqrt{d-1}}{d} > 0, \tag{4.39}
\]

so, for a suitable (small enough) \( \varepsilon_0 \) we can construct a smooth function \( h \) with the following properties (on the Cartesian plane with coordinates \( (r, y) \), think of going from the origin along \( y = \frac{r^2}{4\varepsilon_0} \) until it intersects with \( y = \frac{1 - r^2}{(1 + r^2)^2} \) and then modify a little bit the curve around the intersection point to make it smooth, see Figure 4.8):

(i) \( 0 \leq h(r) \leq \frac{1 - r^2}{(1 + r^2)^2} \) for all \( r < 2\varepsilon_0 \) and \( h(r) = \frac{1 - r^2}{(1 + r^2)^2} \) for \( r \geq 2\varepsilon_0 \);
Figure 4.8: On the construction of \( h \)

(ii) \( h(0) = 0 \) and \( h(r) \sim \frac{r^2}{4\varepsilon_0^2} \) as \( r \to 0 \);

(ii) \( \frac{1-r^2}{(1+r^2)^2} - h(r) > \frac{1}{2} \) for \( r \leq \varepsilon_0 \);

(iv) \( b := \int_0^{\sqrt{d-1}} h(r) \, dr > 0 \) (by (4.39) it holds in fact that \( b \in (0, 1) \));

(v) \( \int_0^r h(u) \, du > \frac{br^3}{3(d-1)^{3/2}} \) for all \( r \in (0, \sqrt{d-1}] \).

Denote \( H(r) = \int_0^r h(u) \, du \), so that we have \( H(\sqrt{d-1}) = b \). Then, define for \( r \in [0, \sqrt{d-1}] \)

\[
\psi(r) = \int_r^{\sqrt{d-1}} \left( \frac{H(v)}{v^2} - \frac{bv}{3(d-1)^{3/2}} \right) \, dv. \tag{4.40}
\]

For the function \( \psi \) defined in this way, we have \( r^2 \psi'(r) = \frac{br^3}{3(d-1)^{3/2}} - H(r) \), so \( h(r) + (r^2 \psi'(r))' = b(d-1)^{-3/2}r^2 \). By construction, it then holds that

\[
\inf_{r \in [0, \sqrt{d-1}]} \left( \frac{1-r^2}{(1+r^2)^2} + (r^2 \psi'(r))' \right) \geq b(d-1)^{-3/2} \varepsilon_0^2 \wedge \frac{1}{2}, \tag{4.41}
\]

and this (recall (4.37) and (4.38)) shows that, if \( \gamma \) is large enough and \( \alpha \) is small enough then the right-hand side of (4.36) is negative for all large enough \( x \in \mathbb{Z}^d \).
4.4. Centrally biased random walks

To complete the proof of the theorem, it remains to deal with the boundary case. Let $x_0 = x_1 = \cdots = x_m > x_{m+1} \geq \cdots \geq x_{d-1} \geq 0$ for some $m \geq 1$. We have (again, we omit some long computations; see (3.12) in [257] for more details)

$$
\mathbb{E}[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = x] = \alpha \|x\|^n \frac{\gamma + m}{2(\gamma + d - 1)} \left[ \frac{1}{x_0} \left( \sum_{k=1}^{m-1} \psi'_k + 2\psi'_m + \sum_{k=m+1}^{d-1} s_k \psi'_k \right) + o(\|x\|^{-1}) \right].
$$

(4.42)

Now simply note that by the property (v), we have $\psi'(r) < 0$ for all $r \in (0, \sqrt{d-1}]$. Observe also that for some positive constant $\delta_0$ it holds that $\psi'(r) \leq -\delta_0$ for all $r \in [1, \sqrt{d-1}]$. Then (recall that in the boundary case $s_1 = 1$ and $s_j \geq 0$ for all $j = 2, \ldots, d-1$) we have

$$
\psi'_j = \frac{s_j}{r} \psi'_r \leq 0 \quad \text{for all } j = 1, \ldots, d-1 \quad \text{and} \quad \psi'_1(s) \leq -\frac{\delta_0}{\sqrt{d-1}}.
$$

This implies that the right-hand side of (4.42) is negative for all large enough $x \in \mathbb{Z}^d$ and thus concludes the proof of Theorem 4.3.3.

4.4 Centrally biased random walks

4.4.1 Model and notation

The results of Sections 4.2 and 4.3 demonstrate that in the spatially non-homogeneous setting, even assuming zero drift is no guarantee of predictable behaviour for a many-dimensional random walk, at least in terms of recurrence versus transience. However, we saw in Theorem 4.1.3 that one regains some control given suitable assumptions on the covariance structure of the increments of the walk: then the classical recurrence theory for homogeneous random walk has a natural analogue for non-homogeneous random walk. In this setting, it turns out that the zero drift assumption can be relaxed to an assumption of asymptotically zero drift, which also turns out to be the correct regime in which to probe the recurrence/transience phase transition, in much the same way as in the one-dimensional setting of Chapter 3. The most natural framework in which to investigate these questions, and more, is that of the centrally biased random walk, which is the focus for this part of the present chapter.
Chapter 4. Many-dimensional random walks

Once more we use the notation of Section 4.1.2 and consider a non-homogeneous random walk \( \xi \) on \( \Sigma \subseteq \mathbb{R}^d \), \( d \geq 1 \). In this section, our basic assumption will again be (MD0).

As mentioned in Section 4.1.3, under mild conditions, a non-homogeneous random walk with zero drift and fixed increment covariance matrix is recurrent if and only if \( d \leq 2 \). We will see in Section 4.4.2 that the assumption of \( \mu(x) = 0 \) may be relaxed to (essentially) \( \|\mu(x)\| = o(\|x\|^{-1}) \). On the other hand, it is possible to achieve either transience or recurrence (even positive-recurrence) in any dimension under the condition \( \|\mu(x)\| = O(\|x\|^{-1}) \), as described in Section 4.1.5. In view of these phenomena, and others that we will see later, we call the case where \( \|\mu(x)\| = O(\|x\|^{-1}) \) the critical case. Before considering in detail that case, we examine in turn the subcritical case in which \( \|x\|\|\mu(x)\| \to 0 \) and the supercritical case in which \( \|x\|\|\mu(x)\| \to \infty \).

Let us comment briefly on our use of the terminology ‘centrally biased’ random walk (see the bibliographical notes at the end of this chapter for historical remarks): in its broadest use we mean any random walk whose drift field \( \mu \) is perturbed around the origin, so, typically, \( \mu(x) \) is asymptotically zero as \( \|x\| \to \infty \). In some, but certainly not all, of the results in this section, the ‘central bias’ takes on the additional sense that the dominant component of the drift field is in the radial direction, towards or away from the origin.

### 4.4.2 Subcritical case

The following result extends Theorem 4.1.3 to admit non-zero drifts that decay sufficiently fast.

**Theorem 4.4.1.** Suppose that (MD0), (MD3), and (MD4) hold, and that (4.1) holds for some \( p > 2 \). Suppose that for some positive-definite \( M \),

\[
\|\mu(x)\| = o(\|x\|^{-1}) \quad \text{and} \quad \|M(x) - M\|_{op} \to 0, \quad \text{as} \quad \|x\| \to \infty, \quad (4.43)
\]

Then \( \xi_n \) is recurrent if \( d = 1 \) and transient if \( d \geq 3 \).

If \( d = 2 \) and, as \( \|x\| \to \infty \),

\[
\|\mu(x)\| = o(\|x\|^{-1} \log^{-1} \|x\|) \quad \text{and} \quad \|M(x) - M\|_{op} = o(\log^{-1} \|x\|), \quad (4.44)
\]

then \( \xi_n \) is recurrent.

**Proof.** The idea of Remark 1.5.5 shows that it suffices to suppose that \( M = I_d \). We again apply Theorem 4.1.7. First suppose that (4.43) holds. Then
4.4. Centrally biased random walks

tr $M(x) = d + o(1)$ and, for any $u \in \mathbb{S}^{d-1}$, $u^\top M(x)u = 1 + o(1)$; thus

$$2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x)\hat{x} = d - 2 + o(1),$$

which, for $\|x\|$ sufficiently large, is strictly positive for if $d \geq 3$, in which case transience follows from Theorem 4.1.7(i), and negative if $d \leq 1$, in which case recurrence follows from Theorem 4.1.7(ii).

Finally, suppose that $d = 2$ and (4.44) holds. Then, similarly,

$$2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x)\hat{x} = o(\log^{-1} \|x\|),$$

which with another application of Theorem 4.1.7(ii) yields recurrence.

4.4.3 Supercritical case

In this section we suppose that the moments condition (4.1) holds for $p > 1$ (at least), so that $\mu$ as given by (4.2) is well defined, and we will study models in which $\|x\|\|\mu(x)\|$ is unbounded. The model that we focus on in this section has dominant drift in the (outwards) radial direction. In this case, to conclude about asymptotic behaviour one typically needs no assumptions on the covariance structure of the walk; indeed, without further assumptions the covariance matrix (4.3) may not exist.

Specifically, we suppose that for some $\rho \in \mathbb{R}^+$ and $\beta \geq 0$,

$$\mu(x) = \rho\|x\|^{-\beta} \hat{x} + O(\|x\|^{-\beta} \log^{-2} \|x\|), \quad (4.45)$$

as $\|x\| \to \infty$. In equation (4.45) the notational convention described at (4.13) is in force, i.e., (4.45) is to be understood as

$$\sup_{x \in \Sigma: \|x\| \geq r} \|\mu(x) - \rho\|x\|^{-\beta} \hat{x}\| = O(\|r\|^{-\beta} \log^{-2} \|r\|), \quad \text{as } r \to \infty,$$

in the usual scalar sense of $O(\cdot)$. We will assume the following.

(CB1) Suppose that for some $\beta \in [0, 1)$, $\rho \in \mathbb{R}^+$, and $p > 2 + 2\beta$, the conditions (4.1) and (4.45) hold.

Our main result on the supercritical case is the following, which shows that $\xi_n$ satisfies a strong law of large numbers with a super-diffusive rate of escape and has a limiting direction; this result implies Theorem 1.7.3.

**Theorem 4.4.2.** Suppose that (MD0), (MD3), and (CB1) hold. Then there exists a random $u \in \mathbb{S}^{d-1}$ such that, as $n \to \infty$,

$$n^{-\frac{1}{1+p}} \xi_n \to (\rho(1 + \beta))^{\frac{1}{1+p}} u, \quad \text{a.s.}$$

In particular, $\lim_{n \to \infty} \hat{\xi}_n = u,$ a.s.
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Figure 4.9: Simulation of $10^5$ steps of a centrally biased random walk satisfying (CB1) with $\beta = 3/4$, showing the trajectory (left) and a histogram of the corresponding angles (right).

The first step in the proof of Theorem 4.4.2 is to appeal to Lemma 4.1.5(i), which shows that the process $X_n = \|\xi_n\|$ is of supercritical Lamperti type, so that we may apply the results of Section 3.12. In particular, we establish the following law of large numbers for $X_n$.

**Lemma 4.4.3.** Suppose that (MD0), (MD3), and (CB1) hold. Then,

$$n^{-\frac{1}{1+\beta}}\|\xi_n\| \to (\rho(1+\beta))^{\frac{1}{1+\beta}} \text{ a.s., as } n \to \infty.$$ 

**Proof.** We take $X_n = \|\xi_n\|$. Since $\beta < 1$, (4.15) shows that, for some $\delta > 0$,

$$E[\Delta_n \mid \xi_n = x] = \langle \hat{x}, \mu(x) \rangle + O(\|x\|^{-\beta-\delta})$$

$$= \rho\|x\|^{-\beta} + O(\|\xi\|^{-\beta} \log^{-2} \|x\|),$$

by (4.45). Then Theorem 3.12.4 applied to $X_n$ gives the result. \qed

The second step in the proof of Theorem 4.4.2 is to show that the process $\xi_n$ has a limiting direction. To this end, the next result concerns the angular process $\hat{\xi}_n$ on $\mathbb{S}^{d-1} \cup \{0\}$.

**Lemma 4.4.4.** Suppose that (MD0) holds. Let $\beta \in [0,1)$. Suppose that (4.1) holds for some $p > 1 + \beta$, and that, as $\|x\| \to \infty$,

$$\sup_{y \in \mathbb{S}^{d-1} : \langle y, x \rangle = 0} |\langle y, \mu(x) \rangle| = O(\|x\|^{-\beta} \log^{-2} \|x\|).$$

(4.46)
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Then, as \( \|x\| \to \infty \),

\[
\mathbb{E}[\xi_{n+1} - \hat{\xi}_n | \xi_n = x] = O(\|x\|^{-1-\beta} \log^{-2} \|x\|); \quad (4.47)
\]

\[
\mathbb{E}[\|\xi_{n+1} - \hat{\xi}_n\|^2 | \xi_n = x] = O(\|x\|^{-1-\beta} \log^{-2} \|x\|). \quad (4.48)
\]

Proof. Without loss of generality, suppose that \( p \in (1 + \beta, 2] \). For the duration of this proof, condition on \( \xi_n = x \) for \( x \in \Sigma \setminus \{0\} \). Recall the definition of \( A_x \) from (4.20), and let \( \gamma \in (0, 1) \) with \( \gamma > \frac{1+\beta}{p} \), which is possible since \( p > 1 + \beta \). Also, write

\[
\theta_n^\perp := \theta_n - \langle \theta_n, \hat{x} \rangle \hat{x};
\]

note that \( \langle \theta_n^\perp, \hat{x} \rangle = 0 \) and \( \|\theta_n^\perp\|^2 = \|\theta_n\|^2 - \langle \theta_n, \hat{x} \rangle^2 \).

Now, a computation shows, given \( \xi_n = x \) with \( \|x\| > 0 \), for \( \|\theta_n\| < \|x\| \),

\[
\hat{\xi}_{n+1} - \hat{\xi}_n = \frac{\theta_n}{\|x + \theta_n\|} - \hat{x} \left( \frac{\|x + \theta_n\| - \|x\|}{\|x + \theta_n\|} \right)
= \frac{\theta_n^\perp}{\|x + \theta_n\|} - \hat{x} \left( \frac{2\langle \theta_n, x \rangle + \|\theta_n\|^2}{\|x\|^2 + \|x + \theta_n\|^2} - \frac{\langle \theta_n, x \rangle}{\|x\|^2} \right).
\]

Note that, here,

\[
\frac{2\langle \theta_n, x \rangle}{\|x\|^2 + \|x + \theta_n\|^2} - \frac{\langle \theta_n, x \rangle}{\|x\|^2} = \langle \theta_n, x \rangle \left( \frac{\|x\|^2 - \|x + \theta_n\|^2}{\|x\|^2(\|x\|^2 + \|x + \theta_n\|^2)} \right),
\]

and, by the triangle inequality,

\[
\left| \frac{\|x\|^2 - \|x + \theta_n\|^2}{\|x\|^2(\|x\|^2 + \|x + \theta_n\|^2)} \right| \leq \frac{\|\theta_n\|}{\|x\|^2}.
\]

Hence

\[
\left| \frac{2\langle \theta_n, x \rangle + \|\theta_n\|^2}{\|x\|^2 + \|x + \theta_n\|^2} - \frac{\langle \theta_n, x \rangle}{\|x\|^2} \right| \leq \frac{2\|\theta_n\|^2}{\|x\|^2}.
\]

By definition of \( A_x \), we may choose \( \|x\| \) large enough so that \( \|\theta_n\| < \|x\|/2 \), and then

\[
\left| \frac{\hat{x}}{\|x + \theta_n\|} \left( \frac{2\langle \theta_n, x \rangle + \|\theta_n\|^2}{\|x\|^2 + \|x + \theta_n\|^2} - \frac{\langle \theta_n, x \rangle}{\|x\|^2} \right) \right| \|A_x\| \leq 4\|\theta_n\|^2\|x\|^{-2}1(A_x)
\leq 4\|\theta_n\|^p\|x\|^{(2-p)-2},
\]

which has an expectation of \( O(\|x\|^{(2-p)-2}) \), by (4.1). In particular,

\[
\mathbb{E}[\hat{\xi}_{n+1} - \hat{\xi}_n | \xi_n = x] = \mathbb{E}[\langle \theta_n^\perp, 1(A_x) \rangle | \xi_n = x] = \|x\|^{-1}(1 + o(1)) \mathbb{E}[\theta_n^\perp 1(A_x) | \xi_n = x]
\]
Expressed in an orthonormal basis containing \( \hat{x} \), the (magnitudes of the) non-zero components of \( \theta_n^+ \) are of the form \( \langle y, \theta_n^+ \rangle \) for \( y \) with \( \langle y, \hat{x} \rangle = 0 \), where
\[
\mathbb{E}[\langle y, \theta_n^+ \rangle \mid \xi_n = \mathbf{x}] = \mathbb{E}[\langle y, \theta_n \rangle \mid \xi_n = \mathbf{x}] = \langle y, \mu(\mathbf{x}) \rangle = O(||\mathbf{x}||^{-\beta} \log^{-2} ||\mathbf{x}||),
\]
by assumption (4.46). Hence \( ||\mathbb{E}[\theta_n^+ \mid \xi_n = \mathbf{x}]|| = O(||\mathbf{x}||^{-\beta} \log^{-2} ||\mathbf{x}||) \). Also, by the \( q = 1 \) case of Lemma 4.1.6,
\[
||\mathbb{E}[\theta_n^+ 1(A^c_x) \mid \xi_n = \mathbf{x}]|| \leq \mathbb{E}||\theta_n|| 1(A_x) \mid \xi_n = \mathbf{x}] = O(||\mathbf{x}||^{-\gamma(p-1)}).
\]
Hence, using the fact that \( \gamma > \frac{1+\beta}{p} > \frac{\beta}{p-1} \),
\[
\mathbb{E}[\theta_n^+ 1(A_x) \mid \xi_n = \mathbf{x}] = O(||\mathbf{x}||^{-\beta} \log^{-2} ||\mathbf{x}||),
\]
(4.50)
We also have that \( 2(\gamma - 1) - p\gamma < -p\gamma < -1 - \beta \) since \( \gamma > \frac{1+\beta}{p} \), so we obtain from (4.49) and (4.50) that
\[
\mathbb{E}[\langle \hat{\xi}_{n+1} - \hat{\xi}_n \rangle 1(A_x) \mid \xi_n = \mathbf{x}] = O(||\mathbf{x}||^{-1-\beta} \log^{-2} ||\mathbf{x}||).
\]
On the other hand,
\[
\mathbb{E}[||\hat{\xi}_{n+1} - \hat{\xi}_n|| 1(A^c_x) \mid \xi_n = \mathbf{x}] \leq 2 \mathbb{P}[||\theta_n||^p > ||\mathbf{x}||^p \mid \xi_n = \mathbf{x}] = O(||\mathbf{x}||^{-p\gamma}),
\]
by Markov’s inequality and (4.1). Since \( \gamma > \frac{\beta}{p-1} \), we obtain (4.47).
Now note that, given \( \xi_n = \mathbf{x} \) with \( ||\mathbf{x}|| > 0 \), for \( ||\theta_n|| < ||\mathbf{x}|| \),
\[
||\hat{\xi}_{n+1} - \hat{\xi}_n|| = \left\| \frac{\mathbf{x}(||\mathbf{x}|| - ||\mathbf{x} + \theta_n||) + \theta_n ||\mathbf{x}||}{||\mathbf{x} + \theta_n||} \right\| \leq \frac{2||\theta_n||}{||\mathbf{x} + \theta_n||},
\]
by the triangle inequality. Hence, for all \( ||\mathbf{x}|| \) sufficiently large,
\[
\mathbb{E}[||\hat{\xi}_{n+1} - \hat{\xi}_n||^2 1(A_x) \mid \xi_n = \mathbf{x}] \leq 8||\mathbf{x}||^{-2} \mathbb{E}[||\theta_n||^2 1(A_x) \mid \xi_n = \mathbf{x}]
\]
\[
\leq 8||\mathbf{x}||^{-\gamma(2-p)-2} \mathbb{E}[||\theta_n||^p \mid \xi_n = \mathbf{x}],
\]
which is \( O(||\mathbf{x}||^{-1-\beta} \log^{-2} ||\mathbf{x}||) \), by (4.1). On the other hand,
\[
\mathbb{E}[||\hat{\xi}_{n+1} - \hat{\xi}_n||^2 1(A^c_x) \mid \xi_n = \mathbf{x}] \leq 4 \mathbb{P}[A^c_x \mid \xi_n = \mathbf{x}] = O(||\mathbf{x}||^{-p\gamma}),
\]
by Markov’s inequality and (4.1). Since \( \gamma > \frac{1+\beta}{p} \), we obtain (4.48).

Now we can complete the proof of Theorem 4.4.2. \( \square \)
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Proof of Theorem 4.4.2. Set $F_n := \sigma(\xi_0, \ldots, \xi_n)$. Then $(\xi_n, n \geq 0)$ is an $(F_n, n \geq 0)$-adapted process taking values in $S^{d-1} \cup \{0\}$, and using the vector-valued version of Doob’s decomposition we may write

$$\hat{\xi}_n = A_n + M_n,$$

(4.51)

where $M_0 = \hat{\xi}_0$, $M_n$ is a $d$-dimensional martingale, and $A_n$ is the previsible sequence defined by $A_0 = 0$ and $A_n = \sum_{m=0}^{n-1} E[\hat{\xi}_{m+1} - \hat{\xi}_m \mid F_m]$ for $n \geq 1$. To show that $\hat{\xi}_n$ converges, we show that both $A_n$ and $M_n$ converge.

We have from (4.47) that, for finite positive constants $C_1$ and $C_2$,

$$\|A_{n+1} - A_n\| \leq C_1 \|\xi_n\|^{-1-\beta} \log^{-2} \|\xi_n\|, \text{ on } \|\xi_n\| \geq C_2.$$

By Lemma 4.4.3, there is a positive constant $c$ such that $\|\xi_n\| \sim cn^{1/(1+\beta)}$, a.s. It follows that

$$\limsup_{n \to \infty} \frac{\|A_{n+1} - A_n\|}{n^{-1}(\log n)^{-2}} < \infty, \text{ a.s.},$$

so $\sum_{n \geq 0} \|A_{n+1} - A_n\| < \infty$, a.s., implying that $A_n$ converges almost surely to some limit $A_\infty \in \mathbb{R}^d$.

Taking expectations in the vector identity $\|M_{n+1} - M_n\|^2 = \|M_{n+1}\|^2 - 2M_n \cdot (M_{n+1} - M_n)$ and using the martingale property, we have

$$E[\|M_{n+1}\|^2 - \|M_n\|^2 \mid F_n] = E[\|M_{n+1} - M_n\|^2 \mid F_n]$$

$$= E[\|\hat{\xi}_{n+1} - \hat{\xi}_n - \hat{\xi}_n - \hat{\xi}_n \mid F_n\|^2 \mid F_n].$$

Expanding out the expression in the latter expectation, we obtain

$$E[\|M_{n+1}\|^2 - \|M_n\|^2 \mid F_n] = E[\|\hat{\xi}_{n+1} - \hat{\xi}_n\|^2 \mid F_n] - (E[\hat{\xi}_{n+1} - \hat{\xi}_n \mid F_n])^2$$

$$\leq E[\|\hat{\xi}_{n+1} - \hat{\xi}_n\|^2 \mid F_n].$$

Then by (4.48) and the fact that $\|\xi_n\| \sim cn^{1/(1+\beta)}$, by Lemma 4.4.3, it follows that

$$\limsup_{n \to \infty} \frac{E[\|M_{n+1}\|^2 - \|M_n\|^2 \mid F_n]}{n^{-1}(\log n)^{-2}} < \infty, \text{ a.s.}$$

Hence $\sum_{n \geq 0} E[\|M_{n+1}\|^2 - \|M_n\|^2 \mid F_n] < \infty$, a.s., which implies that $M_n$ has an almost-sure limit $M_\infty \in \mathbb{R}^d$, by e.g. the $d$-dimensional version of [83, Theorem 5.4.9, p. 254].

Taking $n \to \infty$ in (4.51), we conclude that, a.s., $\hat{\xi}_n \to A_\infty + M_\infty =: u$, for some random vector $u \in S^{d-1} \cup \{0\}$. Finally, note that (MD3) implies
\(\mathbb{P}[\mathbf{u} = \mathbf{0}] = 0\), since, any sequence \(S^{d-1} \cup \{0\}\) with \(\mathbf{0}\) as a limit must be eventually equal to \(\mathbf{0}\), so \(\mathbf{u} = \mathbf{0}\) if and only if \(\xi_n = \mathbf{0}\) for all but finitely many \(n\), which contradicts (MD3); hence we may assume that \(\mathbf{u} \in S^{d-1}\).

Combining the convergence of \(\hat{\xi}_n\) with the asymptotics for \(\|\xi_n\|\) given in Lemma 4.4.3 completes the proof of the theorem.

\subsection*{4.4.4 Critical case: Angular asymptotics}

Throughout this section, we assume \(d \geq 2\). The main result of the previous section, Theorem 4.4.2, showed that the angular process \(\hat{\xi}_n\) has a limit in the supercritical case, where \(\|\mathbf{x}\|\|\mu(\mathbf{x})\| \to \infty\). We will see that the case in which \(\|\mathbf{x}\|\|\mu(\mathbf{x})\|\) remains bounded is fundamentally different: under mild assumptions, \(\hat{\xi}_n\) does not converge. This angular wandering is a classical property of zero-drift homogeneous random walk in dimensions \(d \geq 2\) (see Theorem 1.7.2), and in this section we show the extent to which this property carries across to more general models. We emphasize that the phenomena exhibited in this section are genuinely many dimensional (\(d \geq 2\) is essential), as we explain below.

In addition to taking \(d \geq 2\), for this section we will make the following assumption on the moments and drift of the increments.

\textbf{(CB2)} Suppose that the moments condition (4.1) holds for some \(p > 2\), and that there exists \(C < \infty\) such that

\(\|\mu(\mathbf{x})\| \leq C(1 + \|\mathbf{x}\|)^{-1}\), for all \(\mathbf{x} \in \Sigma\). \quad (4.52)

The main result of this section is the following.

\textbf{Theorem 4.4.5.} Suppose that, for \(d \geq 2\), (MD0), (MD3), (CB2), and (MD4) hold. Then \(\mathbb{P}[\lim_{n \to \infty} \hat{\xi}_n \text{ exists}] = 0\).

The remainder of this section is devoted to the proof of Theorem 4.4.5. First we state an important consequence of Theorem 2.4.8.

\textbf{Lemma 4.4.6.} Suppose that (4.1) holds for some \(p \geq 2\) and that (4.52) holds. Then for any \(\delta > 0\) there exists \(\varepsilon > 0\) such that

\[\mathbb{P}\left[\max_{0 \leq n \leq \lfloor \varepsilon \|\xi_0\|^2 \rfloor} \|\xi_n - \xi_0\| \leq \frac{1}{2}\|\xi_0\|\right] \geq 1 - \delta.\]

\textbf{Proof.} Again let \(\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)\). Set \(X_n = \|\xi_n - \xi_0\|^2\), so \(X_0 = 0\) and \(X_n \geq 0\). Let

\[\eta = \min\{n \geq 0 : \|\xi_n - \xi_0\| \geq \frac{1}{2}\|\xi_0\|\} = \min\{n \geq 0 : X_n \geq \frac{1}{4}\|\xi_0\|^2\}.\]
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Then $X_{n+1} - X_n = \|\theta_n\|^2 + 2\langle \theta_n, \xi_n - \xi_0 \rangle$, so that

$$
|\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]| \leq \mathbb{E}[\|\theta_n\|^2 \mid \mathcal{F}_n] + 2\|\xi_n - \xi_0\||\mu(\xi_n)||

\leq B + \frac{2C\|\xi_n - \xi_0\|}{1 + \|\xi_n\|},
$$

for some constants $B < \infty$ and $C < \infty$, by the $p = 2$ case of (4.1) and (4.52). On $\{n < \eta\}$, we have $\|\xi_n\| \geq \|\xi_0\| - \|\xi_n - \xi_0\| \geq \frac{1}{2}\|\xi_0\|$, so there is some constant $B' < \infty$ for which

$$
\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \leq B + \frac{C\|\xi_0\|}{1 + \frac{1}{2}\|\xi_0\|} \leq B', \text{ on } \{n < \eta\}.
$$

Hence we may apply Theorem 2.4.8 with the stopping time $\tau = \eta = \sigma_x$, where $x = \frac{1}{4}\|\xi_0\|^2$ and, as usual, $\sigma_x = \min\{n \geq 0 : X_n \geq x\}$, to obtain

$$
x \mathbb{P}[\sigma_x \leq n] \leq \mathbb{E}[X_n \wedge \sigma_x] \leq B' n.
$$

In other words, $\mathbb{P}[\sigma_x > n]$ is equal to

$$
\mathbb{P}\left[\max_{0 \leq m \leq n} \|\xi_m - \xi_0\| \leq \frac{1}{2}\|\xi_0\|\right] = \mathbb{P}\left[\max_{0 \leq m \leq n} X_m \leq \frac{1}{4}\|\xi_0\|^2\right] \geq 1 - \frac{4B' n}{\|\xi_0\|^2}.
$$

Taking $n = \lfloor \varepsilon\|\xi_0\|^2 \rfloor$ gives the result, on choosing $\varepsilon > 0$ small enough so that $4B'\varepsilon < \delta$. \qed
The next ingredient is the following estimate for the probability that the walk makes a significant deviation in an orthogonal direction.

Lemma 4.4.7. Suppose that (CB2) and (MD4) hold. There exists \( y_0 < \infty \) such that, for any \( \delta > 0 \), we can choose \( \varepsilon > 0 \) and \( h > 0 \) such that, for all \( x \in \Sigma \) with \( \|x\| \geq y_0 \),

\[
\inf_{e \in S^{d-1} : \langle e, x \rangle = 0} \mathbb{P}\left[ \max_{0 \leq n \leq \lceil \varepsilon \|x\|^2 \rceil} \left| \langle \xi_n, e \rangle \right| \geq h \|x\| \left| \xi_0 = x \right| \geq 1 - \delta. \right.
\]

Proof. Fix \( \delta > 0 \). Suppose that \( \xi_0 = x \in \Sigma \) with \( \|x\| = y > 0 \). Fix \( e \in S^{d-1} \) such that \( \langle e, x \rangle = 0 \). Denote the Doob decomposition of \( \langle \xi_n, e \rangle \) as

\[
\langle \xi_n, e \rangle = \langle \xi_0, e \rangle + M_n + A_n = M_n + A_n,
\]

where \( M_n \) is a martingale with \( M_0 = 0 \), and

\[
A_n = \sum_{m=0}^{n-1} \mathbb{E}[\langle \xi_{m+1} - \xi_m, e \rangle | \mathcal{F}_m] = \sum_{m=0}^{n-1} \langle e, \mu(\xi_m) \rangle.
\]

Again let \( \eta = \min\{n \geq 0 : \|\xi_n - \xi_0\| \geq \frac{1}{2} \|\xi_0\|\} \). By Lemma 4.4.6, there exists \( \varepsilon_1 = \varepsilon_1(\delta) > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_1) \), for any \( y > 0 \),

\[
\mathbb{P}[\eta \geq \varepsilon y^2 | \xi_0 = x] \geq 1 - \delta.
\]

Moreover, for \( \xi_0 = x \) and \( n < \eta \) we have \( \|\xi_n\| \geq y/2 \) so that, on \( \{n < \eta\} \),

\[
|\langle e, \mu(\xi_m) \rangle| \leq \|\mu(\xi_m)\| \leq \frac{C}{1 + (y/2)} \leq \frac{2C}{y}, \text{ a.s.,}
\]

by (4.52). Hence \( |A_{n \wedge \eta}| \leq 2Cn/y \), a.s. In particular,

\[
\mathbb{P}\left[ \max_{0 \leq n \leq \lceil \varepsilon y^2 \rceil} |A_n| \geq 4C\varepsilon y \left| \xi_0 = x \right] \leq \mathbb{P}[\eta \leq \varepsilon y^2 | \xi_0 = x] \leq \delta. \tag{4.53}
\]

Next we show that the martingale \( M_n \) satisfies the hypotheses of the \( d = 1 \) case of Theorem 4.1.4. Indeed,

\[
M_{n+1} - M_n = \langle \theta_n - \mu(\xi_n), e \rangle, \tag{4.54}
\]

so that, a.s.,

\[
\mathbb{E}[\langle M_{n+1} - M_n \rangle^2 | \mathcal{F}_n] = \mathbb{E}[\langle \theta_n \rangle^2 | \mathcal{F}_n] - \langle \theta_n \rangle^2 \geq v_0 - \|\mu(\xi_n)\|^2,
\]

Because \( \mathbb{E}[\langle \theta_n \rangle^2 | \mathcal{F}_n] \) is indepen}
for \( v_0 > 0 \) as given by (MD4). Hence, on \( \{ n < \eta \} \), a.s., by (4.52),

\[
\mathbb{E}[(M_{n+1} - M_n)^2 \mid F_n] \geq v_0 - (2C/y)^2 \geq v_0/2,
\]

for all \( y \geq y_0 \) sufficiently large; note that \( y_0 \) depends only on \( C \) and \( v_0 \), and not on \( \delta \). Hence (4.8) is satisfied for \( Y_n = M_n \) for this choice of \( \eta \) and \( v = v_0/2 \). Also, (4.54) shows that

\[
\mathbb{E}[\|\theta_n - \mu(\xi_n)\|^p \mid F_n] \leq \mathbb{E}[\|\theta_n\|^p \mid \xi_n = x],
\]

by Jensen’s inequality. Hence, by Minkowski’s inequality, (4.7) is also satisfied for \( Y_n = M_n \). So applying Theorem 4.1.4 we obtain, for some constant \( D < \infty \) depending on \( v_0 \) and \( C \), but not on \( \varepsilon \) or \( \delta \),

\[
P\left[ \max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} |M_n| \geq 2hy \mid \xi_0 = x \right] \geq 1 - \frac{D(2hy)^2}{\varepsilon y^2} - P[\xi_0 \leq \varepsilon y^2 \mid \xi_0 = x] = 1 - \frac{4Dh^2}{\varepsilon} - \delta,
\]

for \( \varepsilon \in (0, \varepsilon_1) \). Now take \( h = h(\varepsilon) = \frac{1}{2} \sqrt{\delta \varepsilon / D} \), so that

\[
P\left[ \max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} |M_n| \geq 2hy \mid \xi_0 = x \right] \geq 1 - 2\delta, \tag{4.55}
\]

for all \( y \geq y_0 \). Thus, from (4.53) and (4.55), we see that with probability at least \( 1 - 3\delta \), the event

\[
\left\{ \max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} |M_n| \geq 2hy \right\} \cap \left\{ \max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} |A_n| \leq 4C\varepsilon y \right\} \tag{4.56}
\]

occurs. Since \( h/\varepsilon \to \infty \) as \( \varepsilon \downarrow 0 \), there exists a constant \( \varepsilon_0 = \varepsilon_0(\delta) \in (0, \varepsilon_1) \) such that \( h \geq 4C\varepsilon \) for all \( \varepsilon \in (0, \varepsilon_0) \). Then, for such \( \varepsilon \), on the event (4.56),

\[
\max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} \|\xi_n, e\| \geq \max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} |M_n| - \max_{0 \leq n \leq \lfloor \varepsilon y^2 \rfloor} |A_n| \geq 2hy - 4C\varepsilon y \geq hy,
\]

for all \( y \geq y_0 \). This completes the proof, after a relabelling of the arbitrary positive constant \( \delta \). \( \square \)
Now we can complete the proof of Theorem 4.4.5.

**Proof of Theorem 4.4.5.** Again let $\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)$. Suppose that $\xi_0 = x$ with $\|x\| \geq y_0$, the constant in the statement of Lemma 4.4.7, and take $e \in S^{d-1}$ with $\langle x, e \rangle = 0$. Note

$$
\|\hat{\xi}_n - \hat{\xi}_0\| \geq |\langle \hat{\xi}_n - \hat{\xi}_0, e \rangle| = \frac{|\langle \xi_n, e \rangle|}{\|\xi_n\|}, \quad (4.57)
$$

since $\langle \xi_0, e \rangle = 0$. By the $\delta = 1/4$ cases of Lemmas 4.4.6 and 4.4.7, we can choose $\varepsilon > 0$ and $h > 0$ such that, with probability at least $1 - 2\delta = 1/2$, the event

$$
\left\{ \max_{0 \leq n \leq \lfloor \varepsilon \|x\|^2 \rfloor} \|\xi_n - \xi_0\| \leq \frac{1}{2} \|x\| \right\} \cap \left\{ \max_{0 \leq n \leq \lfloor \varepsilon \|x\|^2 \rfloor} |\langle \xi_n, e \rangle| \geq h \|x\| \right\}
$$

occurs. On this event, we have from (4.57) that

$$
\max_{0 \leq n \leq \lfloor \varepsilon \|x\|^2 \rfloor} \|\hat{\xi}_n - \hat{\xi}_0\| \geq \frac{h\|x\|}{3\|x\|/2} = \frac{2h}{3}.
$$

With the strong Markov property, we have thus shown that, for any finite stopping time $\tau$, $\mathbb{P}[E(\tau) \mid \mathcal{F}_\tau] \geq \frac{1}{2} \mathbb{1}\{\|\xi_\tau\| \geq y_0\}$, where

$$
E(\tau) := \left\{ \max_{0 \leq n \leq \lfloor \varepsilon \|\xi_\tau\|^2 \rfloor} \|\hat{\xi}_{\tau+n} - \hat{\xi}_\tau\| \geq \frac{2h}{3} \right\}.
$$

Define $\tau_0 = \min\{n \geq 0 : \|\xi_n\| \geq y_0\}$, and recursively, for $k \in \mathbb{Z}_+$,

$$
\tau_{k+1} = \min\{n > \tau_k + \varepsilon\|\xi_{\tau_k}\|^2 : \|\xi_n\| \geq y_0\}.
$$

Then, by (MD3), the stopping times $(\tau_k, k \geq 0)$ are a.s. finite, and, by construction,

$$
E(\tau_k) \in \mathcal{F}_{\tau_{k+1}}, \quad \text{and} \quad \mathbb{P}[E(\tau_k) \mid \mathcal{F}_{\tau_k}] \geq \frac{1}{2}.
$$

Hence, by Lévy’s extension of the Borel–Cantelli lemma (Theorem 2.3.19), the events $E(\tau_k)$ occur for infinitely many $k$, a.s. But this implies that

$$
\liminf_{m \to \infty} \sup_{n \geq 0} \|\hat{\xi}_{m+n} - \hat{\xi}_m\| > 0, \; \text{a.s.,}
$$

and so $\hat{\xi}_n$ does not converge to a limit. $\square$
4.4.5 Critical case: Recurrence classification and excursions

Having established some angular properties in Section 4.4.4, we now return to the question of (compact set) recurrence. Again, we use the same basic notation as in Section 4.4.1; in particular, recall the definitions of the increment mean drift \( \mu(x) \) and increment covariance matrix \( M(x) \) from (4.2) and (4.3) respectively.

The sufficient conditions in Theorem 4.1.7 are not, in general, sharp. For the rest of this section we impose additional regularity in order to obtain a spectrum of fine results on the behaviour of the walk. We will impose the following many-dimensional analogue of the assumption (M2) for the one-dimensional Lamperti problem.

\[(CB3) \text{ Suppose that there exist } \rho \in \mathbb{R} \text{ and } \sigma^2 \in (0, \infty) \text{ for which, as } \|x\| \to \infty, \]
\[
\mu(x) = \rho \hat{x} \|x\|^{-1} + o(\|x\|^{-1} \log^{-1} \|x\|); \text{ and } \\
\|M(x) - \sigma^2 I_d\|_{\text{op}} = o(\log^{-1} \|x\|). \]

For convenience, the following results also assume that the state space \( \Sigma \) is locally finite and that \( \xi_n \) is irreducible; versions of all of the following results still hold in the case of more general state spaces, under appropriate conditions. First we have the following recurrence classification.

**Theorem 4.4.8.** Suppose that \((MD0)\) holds for \( \Sigma \subseteq \mathbb{R}^d \) locally finite with \( 0 \in \Sigma \), and that \( \xi_n \) is irreducible. Suppose also that \((4.1)\) holds for some \( p > 2 \), and that \((CB3)\) holds. Then \( \xi_n \) is

(i) transient if \( 2\rho/\sigma^2 > 2 - d \);

(ii) null-recurrent if \( -d \leq 2\rho/\sigma^2 \leq 2 - d \);

(iii) positive-recurrent if \( 2\rho/\sigma^2 < -d \).

Next we give a result on existence of moments of return times.

**Theorem 4.4.9.** Suppose that \((MD0)\) holds for \( \Sigma \subseteq \mathbb{R}^d \) locally finite with \( 0 \in \Sigma \), and that \( \xi_n \) is irreducible. Suppose also that \((CB3)\) holds, and that \((4.1)\) holds with \( p > 2 \). Let \( s \in (0, p/2) \) and write \( s_0 := 1 - (d/2) - (\rho/\sigma^2) \). Then \( \mathbb{E}[\tau^{+}_{s_0}^s] < \infty \) if \( s < s_0 \) and \( \mathbb{E}[\tau^{+}_{s_0}^s] = \infty \) if \( s > s_0 \).

The next result gives almost-sure scaling behaviour for \( \|\xi_n\| \).
Theorem 4.4.10. Suppose that (MD0) holds for $\Sigma \subseteq \mathbb{R}^d$ locally finite with $0 \in \Sigma$, and that $\xi_n$ is irreducible. Suppose also that (CB3) holds.

(i) Suppose that $2\rho/\sigma^2 > 2 - d$ and (4.1) holds with $p > 2$. Then

$$\lim_{n \to \infty} \frac{\log \|\xi_n\|}{\log n} = \frac{1}{2}, \text{ a.s.}$$

(ii) Suppose that $-d \leq 2\rho/\sigma^2 \leq 2 - d$ and (4.1) holds with $p > 2$. Then

$$\limsup_{n \to \infty} \frac{\log \|\xi_n\|}{\log n} = \frac{1}{2}, \text{ a.s.}$$

(iii) Suppose that $2\rho/\sigma^2 < -d$ and (4.1) holds with $p > 2 - d - (2\rho/\sigma^2)$. Then

$$\limsup_{n \to \infty} \frac{\log \|\xi_n\|}{\log n} = \frac{1}{2 - d - (2\rho/\sigma^2)}, \text{ a.s.}$$

The rest of this section is devoted to the proofs of the preceding theorems. The following consequence of Lemma 4.1.5 shows the relation to Lamperti’s problem in this case.

Lemma 4.4.11. Suppose that (MD0) holds for $\Sigma \subseteq \mathbb{R}^d$ locally finite with $0 \in \Sigma$, and that $\xi_n$ is irreducible. Suppose also that (4.1) holds for some $p > 2$, and that (CB3) holds. Then, for $X_n := \|\xi_n\|$, we have that (4.14) holds for the given $p > 2$, and

$$2x\mu_1(x) = 2\rho + \sigma^2(d - 1) + o(\log^{-1} x) = 2x\bar{\mu}_1(x);$$

$$\mu_2(x) = \sigma^2 + o(\log^{-1} x) = \bar{\mu}_2(x).$$

Proof. It follows from (CB3) that

$$\langle \hat{x}, \mu(x) \rangle = \rho\|x\|^{-1} + o(\|x\|^{-1} \log^{-1} \|x\|);$$

$$\operatorname{tr} M(x) = d\sigma^2 + o(\log^{-1} \|x\|);$$

$$\sup_{e \in \mathbb{S}^{d-1}} |e^T M(x)e - \sigma^2| = o(\log^{-1} \|x\|).$$

Lemma 4.1.5(ii) then yields the result.

Now we can complete the proofs of the theorems in this section.
4.5. Range and local time

Proof of Theorem 4.4.8. As in the proof of Lemma 4.4.11, (CB3) shows that
\[ 2\|x\|\langle \hat{x}, \mu(x) \rangle + \text{tr} M(x) - 2\hat{x}^\top M(x)\hat{x} = 2\rho + (d-2)\sigma^2 + o(\log^{-1}\|x\|). \]
It follows from Theorem 4.1.7(i) that \( \xi_n \) is transient if \( 2\rho/\sigma^2 > 2 - d \), and it follows from Theorem 4.1.7(ii) that \( \xi_n \) is recurrent if \( 2\rho/\sigma^2 \leq 2 - d \).

It remains to distinguish between positive- and null-recurrence; we apply the results of Section 3.8 to the process \( X_n = \|\xi_n\| \) and use Lemma 4.4.11. Then, for the process \( X_n \),
\[
\limsup_{x \to \infty} \left( 2x \bar{\mu}_1(x) + (2\alpha - 1)\bar{\mu}_2(x) \right) \leq 2\rho + \sigma^2(d + 2\alpha - 2). \tag{4.58}
\]
If \( \alpha = 1 \) and \( 2\rho/\sigma^2 < -d \), then (4.58) is strictly negative, and so Theorem 3.8.1(ii) yields positive-recurrence. On the other hand, for any \( \theta \in (0,1) \),
\[
2x\mu_1(x) + \left( 1 + \frac{1 - \theta}{\log x} \right) \mu_2(x) \geq 2\rho + \sigma^2d + \left( \frac{1 - \theta}{\log x} \right) (\sigma^2 + o(1)),
\]
which is positive for all \( x \) sufficiently large if \( \sigma^2 > 0 \) and \( 2\rho/\sigma^2 \geq -d \). In this case Theorem 3.8.3 shows that \( \mathbb{E}[\tau_0^+] = \infty \), and hence the random walk, if recurrent, is null-recurrent.

Proof of Theorem 4.4.9. Consider \( X_n = \|\xi_n\| \). Lemma 4.4.11 shows that we may apply Theorems 3.8.1 and 3.8.2 to obtain, respectively, the existence and non-existence of moments results.

Proof of Theorem 4.4.10. Consider \( X_n = \|\xi_n\| \). Lemma 4.4.11 shows that we may apply Theorem 3.10.1 to obtain part (i) of the theorem and Theorem 3.9.5 to obtain parts (ii) and (iii) of the theorem.

4.5 Range and local time of many-dimensional martingales

Throughout this section we assume that \( d \geq 2 \) and consider a discrete-time process \((\xi_n, n \geq 0)\) with values in \( \Sigma \subseteq \mathbb{R}^d \), adapted to a filtration \((\mathcal{F}_n, n \geq 0)\). We suppose that \( \Sigma \) is a set of infinite cardinality; the elements of \( \Sigma \) will be called sites. Without restriction of generality, we assume that \( 0 \in \Sigma \). For the process \( \xi_n \), we suppose that it is uniformly elliptic (can advance in any given direction with uniformly positive probability), has uniformly bounded jumps, and is a martingale in each coordinate (see Definition 4.5.1 below for
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In principle, we do not assume homogeneity in space or time, or even that the process is Markovian.

In this section we study two related questions:

- How many different sites can be visited by the process \( \xi_n \) by time \( n \)?
- How big can be the number of visits to a given site?

Of course, in the absence of space/time homogeneity one cannot hope to be able to characterize the precise behavior of the quantities of interest; in this section we content ourselves in proving that with probability \( 1 - \exp(-n^\varepsilon) \) the number of visits to any fixed site by time \( n \) is less than \( n^{\frac{1}{2} - \delta} \) for some \( \delta > 0 \). This in its turn implies that the number is different sites visited by the process by time \( n \) with very high probability will be at least \( n^{\frac{1}{2} + \delta} \).

Although it is not important for the formulation of our results, while reading this section one may always assume that \( \Sigma \) is the vertex set of the integer lattice \( \mathbb{Z}^d \), the vertex set of some other mosaic, or just any locally finite (in particular, countable) set. This makes sense in light of the discrete nature of our questions of interest raised above; however, it is not hard to formulate ‘continuous’ versions of these questions: see the bibliographical notes at the end of this chapter.

We give formal definitions and state the results of this section. Let

\[
L_n(x) := \sum_{k=0}^{n} 1\{\xi_k = x\}
\]

be the local time of the process in \( x \in \Sigma \) up to time \( n \), and denote by

\[
\Xi_n := \{\xi_0, \xi_1, \ldots, \xi_n\}
\]

the set of visited sites up to time \( n \). The range is \( \#\Xi_n \), the number of visited sites up to time \( n \). Define the random variable \( D_n \in \mathbb{R}^d \) to be the (conditional) drift of the process at time \( n \):

\[
D_n := \mathbb{E}[\xi_{n+1} - \xi_n \mid F_n] = \mathbb{E}[\xi_{n+1} \mid F_n] - \xi_n.
\]

**Definition 4.5.1.** We say that the \( F_n \)-adapted process \( \xi_n \) in \( \Sigma \subseteq \mathbb{R}^d, d \geq 2 \),

(a) has uniformly bounded jumps if there exists \( K > 0 \) such that \( \|\xi_{n+1} - \xi_n\| \leq K \) a.s. for all \( n \) (we assume without restriction of generality that \( K \geq 1 \));
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(b) is uniformly elliptic if there exist \( h, r > 0 \) such that for all \( e \in \mathbb{S}^{d-1} \) we have \( \mathbb{P}[(\xi_{n+1} - \xi_n) \cdot e > r \mid \mathcal{F}_n] > h \) a.s. (we assume without restriction of generality that \( r \leq 1 \));

(c) is a \( d \)-dimensional martingale, if \( D_n = 0 \) a.s. for all \( n \).

The main result of this section is the following.

**Theorem 4.5.2.** Let \( d \geq 2 \). Suppose that \( \xi_n \) is a uniformly elliptic, \( d \)-dimensional martingale with uniformly bounded jumps. Then, there exist \( \gamma \in (0, \frac{1}{2}) \), \( C_1, C_2, \delta > 0 \) such that, for any \( x \in \Sigma \), for all \( n \geq 0 \),

\[
\mathbb{P}[L_n(x) > n^\gamma] \leq C_1 \exp\{-C_2n^\delta\};
\]

and \( \mathbb{P}[\#\Xi_n < n^{1-\gamma}] \leq C_1 n \exp\{-C_2n^\delta\} \).

**Proof.** So, suppose that \( \xi_n \) is a martingale in dimension \( d \geq 2 \), with bounded jumps and uniform ellipticity. To begin, we show that there exist \( b \in (0, 1) \) close enough to \( 1 \) and \( \gamma' > 0 \) (depending only on the constants \( K, h, r \) in Definition 4.5.1(a) and (b)) such that

\[
\mathbb{E}[\|\xi_{n+1}\|^b \mid \mathcal{F}_n] \geq \|\xi_n\|^b 1\{\|\xi_n\| > \gamma'\}.
\]

To see that this inequality holds, first observe that for a fixed \( y \in \mathbb{R}^d \),

\[
\|x + y\|^b = (\|x\|^2 + 2x \cdot y + \|y\|^2)^{b/2}
= \|x\|^b \left(1 + b \frac{x \cdot y}{\|x\|^2} + \frac{b\|y\|^2}{2\|x\|^2} - \frac{1}{2} b(2 - b) \frac{(x \cdot y)^2}{\|x\|^4} + o(\|x\|^{-2})\right),
\]

as \( \|x\| \to \infty \). Taking \( x = \xi_n \) and \( y = \theta_n := \xi_{n+1} - \xi_n \), denoting by \( \varphi_n \) the angle between \( \xi_n \) and \( \theta_n \), and using the martingale property,

\[
\mathbb{E}[\|\xi_{n+1}\|^b - \|\xi_n\|^b \mid \mathcal{F}_n] = \frac{b}{2\|\xi_n\|^{2-b}} \left( \mathbb{E}[\|\theta_n\|^2 \mid \mathcal{F}_n] - (2 - b) \mathbb{E}[\|\theta_n\|^2 \cos^2 \varphi_n \mid \mathcal{F}_n] + o_{\|\xi_n\|^b}(\|\xi_n\|^{-b})\right).
\]

Using the uniform ellipticity and the boundedness of jumps, one can obtain that

\[
\mathbb{E}[\|\theta_n\|^2 \cos^2 \varphi_n \mid \mathcal{F}_n] < (1 - \epsilon') \mathbb{E}[\|\theta_n\|^2 \mid \mathcal{F}_n]
\]

for some \( \epsilon' > 0 \). It follows that if \( b < 1 \) is close enough to \( 1 \), the right-hand side of (4.59) is positive for all large enough \( \|\xi_n\| \), since \( \mathbb{E}[\|\theta_n\|^2 \mid \mathcal{F}_n] \geq \frac{b}{2} \mathbb{E}[\|\theta_n\|^2 \mid \mathcal{F}_n] \).
Theorem 4.5.2 can be extended to a more general class of processes, namely, the so-called strongly directed submartingales. To give a formal definition, let us denote by $P_L$ the operator of projection on the linear subspace $L \subset \mathbb{R}^d$. Assuming that $L$ is a two-dimensional subspace of $\mathbb{R}^d$, $\ell \in \mathbb{S}^{d-1} \cap L$ and $u \in \mathbb{R}$, define

$$H^n_{\ell,L} = \left\{ x \in \mathbb{R}^d : P_L x = 0 \text{ or } \frac{P_L x \cdot \ell}{\|P_L x\|} \geq u \right\},$$

see Figure 4.11.

**Definition 4.5.3.** We say that the $\mathcal{F}_n$-adapted process $\xi_n$ is a $(u, \ell, \mathcal{L})$-strongly directed submartingale, if $u > 0$ and $\mathbb{P}[D_n \in H^n_{\ell,L} \mid \mathcal{F}_n] = 1$ a.s.

Informally, the drift of a strongly directed submartingale should always belong to the “interior of the open book” as in Figure 4.11. Observe that
Figure 4.11: On the definition of $H_{\ell,\mathcal{L}}^u$; observe that $H_{\ell,\mathcal{L}}^u$ and $H_{-\ell,\mathcal{L}}^{-u}$ “complement” each other (i.e., their union is $\mathbb{R}^d$ and they intersect on a set of measure 0).
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for \((u, \ell, \mathcal{L})\)-strongly directed submartingale it holds that the (conditional on the history) expected projection of the drift to \(\ell\) is always nonnegative (so that it is what one would naturally call a “submartingale in direction \(\ell\)”).

Then, we have the following generalization of Theorem 4.5.2:

**Theorem 4.5.4.** Let \(d \geq 2\). Suppose that \(\xi_n\) is a \((u, \ell, \mathcal{L})\)-strongly directed submartingale, which is uniformly elliptic and has uniformly bounded jumps.

is a uniformly elliptic, \(d\)-dimensional martingale with uniformly bounded jumps. Then, there exist constants \(\gamma \in (0, \frac{1}{2})\), \(C_1, C_2, \delta > 0\) (apart from the dimension, depending only on \(u\) and on \(K, r, h\) from Definition 4.5.1 (a)–(b)) such that for any \(x \in \Sigma\), for all \(n \geq 0\),

\[
P[L_n(x) > n^\gamma] \leq C_1 \exp\{-C_2 n^\delta\};
\]

and

\[
P[\#\Xi_n < n^{1-\gamma}] \leq C_1 n \exp\{-C_2 n^\delta\}.
\]

We do not give the proof of this theorem here, since it is quite technical, does not rely on Lyapunov functions, and is not as instructive as the proof of Theorem 4.5.2.

**Bibliographical notes**

**Section 4.1**

The non-confinement result for zero-drift random walks, Lemma 4.1.1, can be found as Proposition 2.1 in [114]; the \(d\)-dimensional martingale version of Kolmogorov’s other inequality, Theorem 4.1.4, on which the non-confinement result rests, is a variation of Lemma 4.1 of [114].

As mentioned in the text, Theorem 4.1.2, which states that in a one-dimensional zero-drift random walk is recurrent under mild second moment conditions, is a relative of Theorem 2.5.7: see the bibliographical notes to Section 2.5 for further discussion.

The recurrence classification for a zero-drift random walk with fixed increment covariance matrix, Theorem 4.1.3, is contained in Theorem 4.1 of Lamperti [190], under a uniform bound on the increments of the walk. The version stated here is contained, for example, in Theorem 2.2 of [114].

The analysis of many-dimensional random walks by considering the process of norms, as in Section 4.1.4, was an important contribution of Lamperti’s seminal work [190, 192]. Early versions of the computations for Lemma 4.1.5, assuming uniformly bounded increments, can be found for example in [190,
§4]. The sufficient conditions for recurrence and transience given in Theorem 4.1.7 extend Theorem 4.1 of Lamperti [190], which, as well as assuming uniformly bounded increments, applies only to the case where the increment covariance matrix $M(x)$ is (asymptotically) constant. The approach here unifies the proofs not only of Theorems 4.1.2 and 4.1.3, but also of Theorems 4.2.1, 4.4.2, and 4.4.8.

The model in Example 4.1.9 is due to Gillis [117], who calls it a ‘centrally biased discrete random walk’. Gillis established recurrence for (only) $\varepsilon > \frac{d-2}{2d}$ and transience for $\varepsilon < \frac{d-2}{2d}$; his method used generating functions. Building on Gillis’s argument, Flatto [104] established recurrence for $\varepsilon = \frac{d-2}{2d}$.

Section 4.2

The material on elliptic random walks is based on [114], although the germ of the model goes back to earlier work of two of the authors of the present book (see the Acknowledgements in [114]).

Consider the example with increments supported on ellipsoids discussed in Section 4.2.2. It follows from (4.32) that

$$
\|\xi_1\|^2 = \|\xi_0\|^2 + 2\|\xi_0\|\langle \hat{\xi}_0, \theta \rangle + \|\theta\|^2 \\
= \|\xi_0\|^2 + 2\|\xi_0\|\langle e_1, D\zeta \rangle + \langle \zeta, D^2\zeta \rangle \\
= \|\xi_0\|^2 + 2a\sqrt{d}\|\xi_0\|\langle e_1, \zeta \rangle + (a^2 - b^2)d\langle e_1, \zeta \rangle^2 + b^2d.
$$

(4.60)

In particular, for this family of models $(\|\xi_n\|, n \geq 0)$ is itself a Markov process, since the distribution of $\|\xi_{n+1}\|$ depends only on $\|\xi_n\|$ and not $\xi_n$; however, in the general setting of Section 4.2.1, this need not be the case.

One-dimensional processes with evolutions reminiscent to that given by (4.60) have been studied previously by Kingman [172] and Bingham [26]. Those processes can be viewed, respectively, as the distance from its start point of a random walk in Euclidean space, and the geodesic distance from its start point of a random walk on the surface of a sphere, but in both cases the increments of the random walk have the property that the distribution of the jump vector is a product of the independent marginal distributions of the length and direction of the jump vector. In contrast, for the elliptic random walk the laws of $\|\theta_n\|$ and $\langle \hat{\xi}_n, \hat{\theta}_n \rangle$ are not independent (except when $a = b$).
Section 4.3

As mentioned before, the results in this section are from [257] and [87]. In fact, the main result of [87] is a generalization of (4.34), and there are also some bounds on hitting probabilities for the controlled walks.

The work [257] originated from the following question, posed in [20]. Let \( \pi_1 \) and \( \pi_2 \) be two zero mean measures in \( \mathbb{R}^4 \) with finite supports that span the whole space. On the first visit to a site the jump of the process has law \( \pi_1 \) and at further visits it has law \( \pi_2 \). Is the resulting walk transient?

More generally, one can consider any adapted rule (a rule depending on the history of the process) for choosing between \( \pi_1 \) and \( \pi_2 \), and ask the same question. It turns out that the answer to this question is positive, even in \( \mathbb{R}^3 \), as proved in Theorem 4.3.2.

This naturally fits into the wider context of random walks that are not Markovian, namely where the next step the walk takes also depends on the past. Recently there has been a lot of interest in random walks of this kind. A large class of such walks are the so-called vertex (or edge) reinforced random walks, where the walker chooses the next vertex to jump to with weight proportional to the number of visits to that vertex up to that time; see e.g. [18, 237, 256, 262, 301, 312]. Another class of such walks is the so-called excited random walks, when the transition probabilities depend on whether it is the first visit to a site or not, see e.g. [21, 22, 181, 222, 306, 321].

Section 4.4

Many-dimensional Markov processes with asymptotically zero drift, such as those in this section, were studied by Lamperti [190, 192] under the name centrally biased random walks, due to the nature of the drift field; the name had been used earlier by Gillis [117] for the model in Example 4.1.9.

The results on the supercritical case given in Section 4.4.3 are adapted from [236, 211, 53]. In (4.45) the exponent \(-2\) on the logarithm is chosen for simplicity; it could be replaced with any exponent strictly less than \(-1\). The law of large numbers for the process of norms, Lemma 4.4.3, is a variation on Theorem 3.2 of [236]. The limiting direction part of Theorem 4.4.2 is a variation on Theorem 2.2 of [211]. The method of proof of the limiting direction, via Lemma 4.4.4, is different from (and more transparent than) the method in [211], and extends the approach of [53, §6.2], which (in a slightly different context) was restricted to two dimensions and assumed a uniform bound on the increments. A particular case of the model with \( \beta = 0 \) is studied in [109], who also obtain a central limit theorem to accompany
the law of large numbers.

The conclusion of Theorem 4.4.5 that $\hat{\xi}_n$ does not have a limit can be strengthened in many circumstances. For example, a stronger property is angular recurrence, which says that $\hat{\xi}_n$ visits every neighbourhood on $S^{d-1}$ infinitely often; see [211, 212] for some approaches to studying angular recurrence for non-homogeneous random walks. Stronger still is angular ergodicity, such as

$$\liminf_{n \to \infty} n^{-1} \sum_{m=0}^{n-1} 1\{\|u - \hat{\xi}_m\| < \varepsilon\} > 0.$$ 

The proof of Theorem 4.4.5 given here uses some ideas similar to those in [211, 212], but is considerably more transparent.

The recurrence classification in Section 4.4.5 is similar to that given in [135], and builds on work going back to Lamperti. Specifically, Theorem 4.4.8 extends results of Lamperti [190, 192], who assumed uniformly bounded increments and a stronger version of (CB3) with the error term $\log^{-1} \|x\|$ replaced by $\|x\|^{-\delta}$ for $\delta > 0$: see Theorem 4.1 of [190, p. 324] and Theorem 5.1 of [192, p. 142]; the latter result is not sharp enough to decide between null- or positive-recurrence at the boundary case $2\rho/\sigma^2 = -d$. The result on moments of return times, Theorem 4.4.9, also extends Theorem 5.1 of [192]; Lamperti’s result only covers integer $s$.

The almost-sure bounds in Theorem 4.4.10 are a variation on similar results from [135]. Upper bounds similar to those in Theorem 4.4.10 can be derived from [232, §3]; see Theorem 2.4 of [53] for a similar application of such results, albeit under more restrictive assumptions.

Section 4.5

The range (i.e., the cardinality of the set visited sites, or sometimes this set itself) and the local time (i.e., the number of visits to a given site) for space-homogeneous discrete-time random walks have been extensively studied in the literature. It is a classical result that the expected range of the simple random walk is $O\left( \frac{n}{\log n} \right)$ for $d = 2$ and $O(n)$ for $d \geq 3$, see e.g. Section 6.1 of [139]. It is not difficult to obtain from this fact (using an independence argument as e.g. in Lemma 3.1 of [3]) that with very high probability the walk visits at least $n^{1-\delta}$ distinct sites by time $n$. Finer results for the range of homogeneous random walks can be found in a number of papers; see e.g. in [19, 77, 124] and references therein. For nonhomogeneous random walks these questions, of course, are more difficult; we mention [263] that contains
results on the range of simple random walk on a supercritical percolation cluster.

The behaviour of the local time (i.e., the number of visits) in a fixed site, or the field of local times in all sites, was much studied in the literature as well. It is quite elementary to obtain that the expected number of visits to the origin by time \( n \) for the simple random walk is \( O(\log n) \) for \( d = 2 \) and \( O(1) \) for \( d \geq 3 \). Also, one can easily obtain for the simple random walk in dimension 2 (using e.g. E1 of Section III.16 of [293]) that, with stretched-exponentially small probability, the number of visits to the origin is less than \( n^d \) for any fixed \( \delta > 0 \). Of course, finer results (for more general random walks as well) are available; see e.g. [36, 62, 63, 66, 217].

Theorem 4.5.2 is taken from [222], although the exposition basically follows [221]; the last paper also contains the proof of more general Theorem 4.5.4.

Theorem 4.5.2 shows that the range of a many-dimensional martingale is of order at least \( n^{(1/2) + \varepsilon} \) with high probability. As discussed above, for simple random walk the range is in fact of order at least \( n^{1-\varepsilon} \). A natural question is thus:

**Open problem 4.5.5.** Under the conditions of Theorem 4.5.2, either prove that \( \#\Xi_n \) is at least \( n^{1-\varepsilon} \) with high probability for any \( \varepsilon > 0 \), or produce a counterexample.

In general state-spaces \( \Sigma \subseteq \mathbb{R}^d \), one natural definition of the range is

\[
R_n := \left| \bigcup_{\mathbf{x} \in \Xi_n} B(\mathbf{x}; r) \right|, \quad \text{for fixed } r > 0, \tag{4.61}
\]

the volume of the \( r \)-radius ‘sausage’ around the path. If \( \Sigma \) is locally finite, then, as \( r \downarrow 0 \),

\[
r^{-d} R_n \rightarrow v_d \#\Xi_n,
\]

where \( v_d = |B(0; 1)| \) is the volume of the unit-radius \( d \)-ball. So for locally finite \( \Sigma \), it is usual to work simply with \( \#\Xi_n \), the number of sites visited, and also call this the range of the walk. This is the approach taken in Section 4.5.

**Open problem 4.5.6.** Study the behaviour of \( R_n \) as defined by (4.61) for the elliptic random walks of Section 4.2.
Chapter 5

Heavy tails

5.1 Chapter overview

So far, the processes that we have considered in this book have typically had increments whose first (and second) moments have been uniformly bounded. The focus of this chapter is on the case where first (or second) moments do not exist, i.e., the increments are *heavy-tailed*. We consider Markov processes on $\mathbb{R}$, and study questions of asymptotic behaviour, such as recurrence or transience.

This chapter deals with two different types of process. First, in Section 5.2, we study processes whose increments have, in some sense, heavier tails in one direction than the other. We investigate sufficient conditions for transience in the direction of the heavier tail, and quantify the transience by deriving results on the rate of escape and on moments of first passage and last exit times.

Second, in Section 5.3, we study processes whose increments are of two different types, depending on whether the current position is to the left or to the right of the origin. Such processes are known as *oscillating random walks*, and their study (via classical methods) goes back to Kemperman [157]. We consider a version of the model in which the increment distribution is signed, i.e., only jumps towards the origin are allowed. For these models we study recurrence and transience.

In keeping with the theme of this book, the methods of this chapter are based on the semimartingale ideas of Chapter 2: for appropriate choices of Lyapunov function, we verify Foster–Lyapunov style drift conditions. Verification of drift conditions usually entails some Taylor’s formula expansions (as in Chapter 3) as well as some careful truncation ideas to deal with the
heavy tails. The resulting proofs are relatively short, and based on some intuitively appealing ideas.

5.2 Directional transience

5.2.1 Introduction

In this section we assume the following.

(H0) Let $X_n$ be a time-homogeneous Markov chain on $\Sigma \subseteq \mathbb{R}$ with $0 \in \Sigma$, $\inf \Sigma = -\infty$ and $\sup \Sigma = +\infty$. Suppose that $X_0 = 0$.

For $n \in \mathbb{Z}_+$, we write $\Delta_n := X_{n+1} - X_n$ for the increments of $X_n$. In most of our results, we impose ‘heavy tail’ conditions on either $\Delta_n^+$ (positive jumps) or $\Delta_n^-$ (negative jumps) or both; typically these conditions are one-sided (i.e., inequalities). Two of the most common assumptions are as follows.

(H1) There exist $\alpha > 0, c > 0, y_0 < \infty$ for which, for all $x \in \Sigma$ and all $y \geq y_0$, $\mathbb{P}[\Delta_n^+ > y \ | \ X_n = x] \geq cy^{-\alpha}$.

(H2) There exist $\alpha \in (0, 1), c > 0, y_0 < \infty$ for which, for all $x \in \Sigma$ and all $y \geq y_0$, $\mathbb{E}[\Delta_n^+ \mathbb{1}\{\Delta_n^+ \leq y\} \ | \ X_n = x] \geq cy^{1-\alpha}$.

The following non-confinement result shows that, under the conditions of most of the results in this section, the process $X_n$ has non-trivial asymptotic behaviour.

**Proposition 5.2.1.** Suppose that (H0) holds. Suppose that either (H1) or (H2) holds, or either holds with $\Delta_n^-$ instead of $\Delta_n^+$. Then

$$\limsup_{n \to \infty} |X_n| = \infty, \ \text{a.s.}$$

(5.1)

**Proof.** We claim that under any of the conditions in the proposition, it is the case that for any $y \geq 0$ sufficiently large, there exists $\delta(y) > 0$ for which,

$$\mathbb{P}[|\Delta_n| > y \ | \ X_n = x] \geq \delta(y), \ \text{for all } x \in \Sigma.$$  

(5.2)

Consider the events $A_n := \{|X_n| > B\}$ and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Given (5.2), for any $B < \infty$ large enough,

$$\mathbb{P}[A_{n+1} \cup A_n \ | \ \mathcal{F}_n] = \mathbb{P}[A_{n+1} \ | \ \mathcal{F}_n] \mathbb{1}(A_n^c) + \mathbb{1}(A_n),$$
and on $A_n^c$ we have $\mathbb{P}[A_{n+1} \mid \mathcal{F}_n] \geq \delta(2B) > 0$. Thus $\sum_{n \geq 0} \mathbb{P}[A_{n+1} \cup A_n \mid \mathcal{F}_n] = \infty$ a.s., and $A_{n+1} \cup A_n \in \mathcal{F}_{n+1}$, so Lévy’s extension of the Borel–Cantelli lemma (Theorem 2.3.19) shows that $\limsup_{n \to \infty} |X_n| \geq B$, a.s. Since $B$ was arbitrarily large, (5.1) follows.

It remains to verify (5.2). Since $|\Delta_n| = \Delta^+_n + \Delta^-_n$, it suffices to verify (5.2) with one of $\Delta^+_n$ or $\Delta^-_n$ in place of $|\Delta_n|$. In the case where (H1) holds, the claim is immediate. If (H2) holds, then, for any $y \geq y_0$, for $z > y$,

$$E[\Delta^+_n 1\{y \leq \Delta^+_n \leq z\} \mid X_n = x] \geq E[\Delta^+_n 1\{\Delta^+_n \leq z\} \mid X_n = x] - y > 1,$$

provided $z > y_0 + ((1 + y)/c)^{1/(1-\alpha)}$, say. Then,

$$1 < E[\Delta^+_n 1\{y \leq \Delta^+_n \leq z\} \mid X_n = x] \leq z \mathbb{P}[y \leq \Delta^+_n \leq z \mid X_n = x] \leq z \mathbb{P}[\Delta^+_n \geq y \mid X_n = x],$$

which implies (5.2) in this case also.

### 5.2.2 A condition for directional transience

Our first main result, which gives conditions under which $\lim_{n \to \infty} X_n = +\infty$, imposes a moments condition on the negative jumps of the process:

(H3) There exist $\beta > 0$ and $C < \infty$ for which $E[(\Delta^+_n)^\beta \mid X_n = x] \leq C$ for all $x \in \Sigma$.

**Theorem 5.2.2.** Suppose that (H0) holds, and that (H2) and (H3) hold for $\alpha \in (0, 1)$ and $\beta > \alpha$. Then $X_n \to +\infty$ a.s. as $n \to \infty$.

To prove Theorem 5.2.2, we use the Lyapunov function $f_{z,\delta} : \mathbb{R} \to [0, 1]$ defined for $z \in \mathbb{R}$ and $\delta > 0$ by

$$f_{z,\delta}(y) := \begin{cases} 1 & \text{if } y \leq z; \\ (1 + y - z)^{-\delta} & \text{if } y > z. \end{cases}$$

(5.3)

Note that $f_{z,\delta}$ is non-increasing on $\mathbb{R}$.

**Lemma 5.2.3.** Suppose that (H0) holds, and that (H2) and (H3) hold for $\alpha \in (0, 1)$ and $\beta > \alpha$. Then for any $\delta \in (0, \beta - \alpha)$ and some $A > 0$ sufficiently large, for any $z \in \mathbb{R}$,

$$E[f_{z,\delta}(X_{n+1}) - f_{z,\delta}(X_n) \mid X_n = x] \leq 0, \text{ for all } x > z + A.$$
Proof. It suffices to suppose that \( n = 0 \) and \( z = 1 \). Let \( \delta > 0 \), and let \( f_\delta := f_{1, \delta} \) be as defined at (5.3). Let \( \gamma \in (0, 1) \); we will specify \( \delta \) and \( \gamma \) later. Since \( f_\delta \) is non-increasing and \([0, 1]-\)valued, we have for \( x > 1 \) sufficiently large that

\[
f_\delta(x + \Delta_0) - f_\delta(x) \leq \left[ (x + \Delta_0^+) - x \right] 1\{\Delta_0^+ \leq x^\gamma\} + \left[ (x - \Delta_0^-)^- - x \right] 1\{\Delta_0^- \leq x^\gamma\} + 1\{\Delta_0^- > x^\gamma\}. \tag{5.4}
\]

We will take expectations on both sides of (5.4), conditioning on \( X_0 = x \). The final term in (5.4) then becomes, by Markov's inequality and (H3),

\[
P_x[\Delta_0^- > x^\gamma] = P_x[(\Delta_0^-)^\beta > x^{\gamma \beta}] \leq C x^{-\gamma \beta}. \tag{5.5}
\]

For the second term on the right-hand side of (5.4), since \( \gamma < 1 \), Taylor's formula implies that, as \( x \to \infty \),

\[
\left[ (x - \Delta_0^-)^- - x \right] 1\{\Delta_0^- \leq x^\gamma\} = \delta(1 + o(1))x^{-1-\delta} \Delta_0^- 1\{\Delta_0^- \leq x^\gamma\}, \tag{5.6}
\]

where the \( o(1) \) term is non-random. Here we have for the product of the final two terms in (5.6) that

\[
\Delta_0^- 1\{\Delta_0^- \leq x^\gamma\} = (\Delta_0^-)^\beta 1\{\Delta_0^- \leq x^\gamma\} \leq (\Delta_0^-)^{\beta(1 - \beta)} 1\{\Delta_0^- \leq x^\gamma\}. \tag{5.7}
\]

Combining (5.6) and (5.7), and using (H3), we obtain

\[
E_x\left[ \left( (x - \Delta_0^-)^- - x \right) 1\{\Delta_0^- \leq x^\gamma\} \right] = O(x^{-1-\delta + \gamma(1-\beta)^+}). \tag{5.8}
\]

For the first term on the right-hand side of (5.4), another application of Taylor's formula implies that, as \( x \to \infty \),

\[
\left[ (x + \Delta_0^+) - x \right] 1\{\Delta_0^+ \leq x^\gamma\} = -\delta(1 + o(1))x^{-1-\delta} \Delta_0^+ 1\{\Delta_0^+ \leq x^\gamma\}. \tag{5.9}
\]

Taking expectations and using (H2) we obtain, for \( x \) sufficiently large,

\[
E_x\left[ \left( (x + \Delta_0^+) - x \right) 1\{\Delta_0^+ \leq x^\gamma\} \right] \leq -(c\delta/2)x^{-1-\delta + \gamma(1-\alpha)}. \tag{5.9}
\]

Thus from (5.4), using the estimates (5.5), (5.8) and (5.9), we verify that

\[
E_x[f_\delta(X_1) - f_\delta(X_0)] \leq 0, \text{ for } x > A \text{ with } A \text{ sufficiently large, provided that the negative term arising from (5.9) dominates, i.e.,}
\]

\[-1 - \delta + \gamma(1 - \alpha) > -\gamma \beta, \quad \text{and} \quad -1 - \delta + \gamma(1 - \alpha) > -1 - \delta + \gamma(1 - \beta)^+.
\]
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The second inequality holds since $\alpha < \beta \wedge 1$. The first inequality holds provided we choose $\delta \in (0, \beta - \alpha)$, which we may do since $\alpha < \beta$, and then choose $\gamma \in (\frac{1+\delta}{1+\beta-\alpha}, 1)$.

Now we can give the proof of Theorem 5.2.2.

Proof of Theorem 5.2.2. The argument is similar to Theorem 3.5.6. First we show that, under the conditions of the theorem,

$$\mathbb{P}\left[\liminf_{n \to \infty} X_n = -\infty\right] = 0. \quad (5.10)$$

Let $a > 0$, to be chosen later. For $x \in \mathbb{R}$, set

$$\nu_x := \min\{n \geq 0 : X_n > x + a\}; \quad \eta_x := \min\{n \geq \nu_x : X_n \leq x\}.$$

In particular, since $X_0 = 0$, we have that $\nu_x = 0$ for all $x < -a$.

Let $\delta \in (0, \beta - \alpha)$ and write $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Then Lemma 5.2.3 shows that, on $\{\nu_x < \infty\}$, $(f_{x-A,\delta}(X_n \wedge \eta_x), n \geq \nu_x)$ is a non-negative supermartingale, and so converges a.s. to a finite limit, $L_x$, say. On $\{\nu_x < \infty\}$, we have the supermartingale property that

$$\mathbb{E}[f_{x-A,\delta}(X_n \wedge \eta_x) \mid \mathcal{F}_{\nu_x}] \leq f_{x-A,\delta}(X_{\nu_x}) \leq (1 + A + a)^{-\delta}, \text{ a.s.,}$$

while by Fatou’s lemma, also on $\{\nu_x < \infty\},$

$$\lim_{n \to \infty} \mathbb{E}[f_{x-A,\delta}(X_n \wedge \eta_x) \mid \mathcal{F}_{\nu_x}] \geq \mathbb{E}[L_x \mid \mathcal{F}_{\nu_x}] \geq \mathbb{E}[L_x 1\{\eta_x < \infty\} \mid \mathcal{F}_{\nu_x}] \geq (1 + A)^{-\delta} \mathbb{P}[\eta_x < \infty \mid \mathcal{F}_{\nu_x}],$$

since, on $\{\eta_x < \infty\}$, $X_n \wedge \eta_x \leq x$ for all $n$ sufficiently large. So on $\{\nu_x < \infty\}$ we have, a.s.,

$$\mathbb{P}[\eta_x < \infty \mid \mathcal{F}_{\nu_x}] \leq \left(\frac{1 + A + a}{1 + A}\right)^{-\delta}.$$

Let $\varepsilon > 0$. Then we can take $a$ sufficiently large so that $\mathbb{P}[\eta_x = \infty \mid \mathcal{F}_{\nu_x}] \geq 1 - \varepsilon$, a.s., on $\{\nu_x < \infty\}$. For such a choice of $a$, suppose that $x < -a$; then $\nu_x < \infty$ a.s. (indeed, since $X_0 = 0$, $\nu_x = 0$ a.s.). Hence for such an $x$,

$$\mathbb{P}\left[\liminf_{n \to \infty} X_n > x\right] = \mathbb{P}[\eta_x = \infty] \geq \mathbb{E}[\mathbb{P}[\eta_x = \infty \mid \mathcal{F}_{\nu_x}] 1\{\nu_x < \infty\}] \geq 1 - \varepsilon. \quad (5.11)$$
It follows from (5.11) that
\[ P \left[ \lim \inf_{n \to \infty} X_n = -\infty \right] \leq P \left[ \lim \inf_{n \to \infty} X_n \leq -a - 1 \right] \leq \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, (5.10) follows.

Proposition 5.2.1 applies under condition (H2). Hence, together with (5.1), (5.10) implies that, a.s., \( \limsup_{n \to \infty} X_n = \infty \); in other words, for any \( a > 0 \) and any \( x \in \mathbb{R}, \nu_x < \infty \) a.s. Hence the argument for (5.11) extends to any \( x \in \mathbb{R} \), which implies that for any \( x \in \mathbb{R} \), a.s., \( \liminf_{n \to \infty} X_n > x \), so \( X_n \to \infty \) a.s. \( \Box \)

5.2.3 Almost-sure bounds

Our next two results deal with the growth rate of \( X_n \). First we have the following upper bounds.

**Theorem 5.2.4.** Suppose that there exist \( \theta \in (0, 1), \varphi \in \mathbb{R}, y_0 < \infty \) and \( C < \infty \) such that, for all \( y \geq y_0 \),
\[ P[\Delta_n^+ \geq y \mid X_n = x] \leq Cy^{-\theta}(\log y)^\varphi, \quad \text{for all } x \in \Sigma. \] (5.12)

(i) If \( \theta \in (0, 1) \), then, for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \geq 0 \),
\[ X_n \leq n^{1/\theta}(\log n)^{x^{+2}/\theta^2 + \varepsilon}. \]

(ii) If \( \theta = 1 \), then, for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \geq 0 \),
\[ X_n \leq n(\log n)^{(1+\varphi)^+ + 1 + \varepsilon}. \]

The next result shows that if we impose condition (H1) instead of the condition (H2) in Theorem 5.2.2, not only does \( X_n \to +\infty \), a.s., but it does so at a particular rate of escape.

**Theorem 5.2.5.** Suppose that (H0) holds, and that (H1) and (H3) hold for \( \alpha \in (0, 1) \) and \( \beta > \alpha \). Then for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \geq 0 \),
\[ X_n \geq n^{1/\alpha}(\log n)^{-(1/\alpha) - \varepsilon}. \]

Theorems 5.2.4 and 5.2.5 have the following corollary.
Corollary 5.2.6. Suppose that (H0) holds, and that (H3) holds. Suppose also that for \( \alpha \in (0, 1) \) with \( \alpha < \beta \),

\[
\lim_{y \to \infty} \sup_{x \in \Sigma} \left| \frac{\log \mathbb{P}[\Delta_n^+ > y \mid X_n = x]}{\log y} + \alpha \right| = 0.
\]

Then

\[
\lim_{n \to \infty} \frac{\log X_n}{\log n} = \frac{1}{\alpha}, \text{ a.s.}
\]

Proof. The condition in the corollary ensures that for any \( \varepsilon > 0 \) there exists \( y_0 < \infty \) such that, for all \( y \geq y_0 \) and all \( x \in \Sigma \),

\[
y^{\alpha - \varepsilon} \leq \mathbb{P}[\Delta_n^+ > y \mid X_n = x] \leq y^{\alpha + \varepsilon}, \text{ a.s.}
\]

Theorem 5.2.4 with the upper bound in the last display then shows that for any \( \varepsilon > 0 \), a.s., \( X_n \leq n^{(1/\alpha) + \varepsilon} \) for all but finitely many \( n \). On the other hand, Theorem 5.2.5 with the lower bound in the last display and (H3) shows that for any \( \varepsilon > 0 \), a.s., \( X_n \geq n^{(1/\alpha) - \varepsilon} \) for all but finitely many \( n \). Since \( \varepsilon > 0 \) was arbitrary, the result follows.

The rest of this section is devoted to the proofs of Theorems 5.2.4 and 5.2.5. First we give a general condition for obtaining almost-sure upper bounds.

Lemma 5.2.7. Let \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) be increasing and concave. Suppose that there exists \( C < \infty \) such that \( \mathbb{E}[h(\Delta_n^+)] \mid X_n = x \leq C \), for all \( x \in \Sigma \). Then for any \( \varepsilon > 0 \), a.s., for all but finitely many \( n \geq 0 \),

\[
X_n \leq \sum_{m=0}^{n-1} \Delta_n^+ \leq h^{-1}(n(\log n)^{1+\varepsilon}).
\]

Proof. Set \( Y_0 := 0 \) and for \( n \in \mathbb{N} \) let \( Y_n := \sum_{m=0}^{n-1} \Delta_n^+ \). Then \( Y_n \geq 0 \) is non-decreasing and \( X_n \leq X_0 + Y_n = Y_n \), since \( X_0 = 0 \). Since \( h \) is non-negative and concave, it is subadditive, i.e., \( h(a + b) \leq h(a) + h(b) \) for \( a, b \in \mathbb{R}_+ \). Hence

\[
\mathbb{E}[h(Y_n + \Delta_n^+) - h(Y_n) \mid X_n = x] \leq \mathbb{E}[h(\Delta_n^+) \mid X_n = x] \leq C,
\]

by hypothesis. The almost-sure upper bound in Theorem 2.8.1 applied to \( h(Y_n) \) implies that, for any \( \varepsilon > 0 \), a.s., \( h(Y_n) \leq n(\log n)^{1+\varepsilon} \), for all but finitely many \( n \). Since \( h \) is increasing and \( X_n \leq Y_n \), it follows that for any \( \varepsilon > 0 \), a.s., \( h(X_n) \leq n(\log n)^{1+\varepsilon} \).
Next we need a result on the maxima of the increments of $X_n$.

**Lemma 5.2.8.** Suppose that (H1) holds for $\alpha \in (0, \infty)$. Then for any $\varepsilon > 0$, a.s., for all but finitely many $n \geq 0$,

$$\max_{0 \leq m \leq n} \Delta_m^+ \geq n^{1/\alpha} (\log n)^{-(1/\alpha) - \varepsilon}.$$  

**Proof.** By repeated applications of the Markov property and (H1),

$$\mathbb{P} \left[ \max_{0 \leq m \leq n} \Delta_m^+ \leq y \right] \leq \prod_{m=0}^{n} (1 - cy^{-\alpha}) \leq (1 - cy^{-\alpha})^n,$$

for $y \geq y_0$. Taking $y = n^{1/\alpha} (\log n)^q$ in (5.14) we obtain, for $n$ sufficiently large,

$$\mathbb{P} \left[ \max_{0 \leq m \leq n} \Delta_m^+ < n^{1/\alpha} (\log n)^q \right] \leq (1 - cn^{-1} (\log n)^{-\alpha q})^n$$

$$= O \left( \exp \left( -c (\log n)^{-\alpha q} \right) \right),$$

which is summable over sufficiently large $n$ provided $q < -1/\alpha$. Hence the Borel–Cantelli lemma completes the proof. \(\square\)

Now we can complete the proofs of Theorems 5.2.4 and 5.2.5.

**Proof of Theorem 5.2.4.** First we prove part (i), so let $\theta \in (0, 1)$. For $\varepsilon > 0$, take $h(x) = (K+x)^\theta (\log (K+x))^{-\varphi - 1 - \varepsilon}$. For a large enough choice of $K \geq 1$, $h$ is non-negative, increasing, and concave. Moreover, $\mathbb{E}[h(\Delta_n^+) \mid X_n = x]$ is uniformly bounded provided $\sum_{k=1}^{\infty} h'(k) \mathbb{P}[\Delta_n^+ > k \mid X_n = x]$ is uniformly bounded; see e.g. [123, p. 76]. This is indeed the case under the hypothesis of the theorem, by (5.12), since $h'(x) = O(x^{\theta-1} (\log x)^{-\varphi-1-\varepsilon})$. Now (i) follows from Lemma 5.2.7, noting that $h^{-1}(x) = O(x^{1/\theta} (\log x)^{\varphi+1+\varepsilon})$. The proof of (ii) is similar, this time taking $h(x) = (K+x)(\log (K+x))^{-(\varphi+1-\varepsilon)}$. \(\square\)

**Proof of Theorem 5.2.5.** Let $\varepsilon > 0$. Lemma 5.2.7 applied to $-X_n$ with $h(x) = x^{\beta \wedge 1}$, using (H3), shows that for all but finitely many $n$,

$$\sum_{m=0}^{n-1} \Delta_m^- \leq n^{1/(\beta \wedge 1)} (\log n)^{(1/(\beta \wedge 1)) + \varepsilon}, \text{ a.s.}$$

On the other hand, Lemma 5.2.8 implies that for all but finitely many $n$,

$$\sum_{m=0}^{n-1} \Delta_m^+ \geq \max_{0 \leq m \leq n-1} \Delta_m^+ \geq n^{1/\alpha} (\log n)^{-(1/\alpha) - \varepsilon}, \text{ a.s.}$$

Combining these bounds and using the fact that $\alpha < \beta \wedge 1$ completes the proof. \(\square\)
5.2.4 Moments of passage times

Recall that $\rho_x = \min\{n \geq 0 : X_n \geq x\}$ denotes the first passage time into the half-line $[x, \infty)$. Under the conditions of Theorem 5.2.2, $X_n \to +\infty$, a.s., so that $\rho_x < \infty$ a.s., for all $x \in \mathbb{R}$. It is natural to study the tails or moments of the random variable $\rho_x$ in order to quantify the transience.

**Theorem 5.2.9.** Suppose that (H0) holds, and that (H2) and (H3) hold for $\alpha \in (0, 1)$ and $\beta > \alpha$. Then for any $x \in \mathbb{R}$ and any $s \in [0, \beta/\alpha)$, $\mathbb{E}[\rho_{sx}^\alpha] < \infty$.

**Theorem 5.2.10.** Let $\alpha \in (0, 1]$ and $\beta > 0$. Suppose that, for some $C < \infty$, $\mathbb{E}[(\Delta_n^+)^\alpha | X_n = x] \leq C$ for all $x \in \Sigma$, and $\mathbb{E}[(\Delta_n^-)^\beta | X_n = x] = \infty$ for all $x \in \Sigma$. Then, for any $x > 0$, $\mathbb{E}[\rho_{x}^{\beta/\alpha}] = \infty$.

Note that in Theorem 5.2.9, $\beta/\alpha > 1$, so in particular $\mathbb{E}[\rho_x] < \infty$ for any $x \in \mathbb{R}$. The proof of Theorem 5.2.9 will use another Lyapunov function, described in the next result.

**Lemma 5.2.11.** Suppose that (H0) holds, and that (H2) and (H3) hold for $\alpha \in (0, 1)$ and $\beta > \alpha$. For $\gamma \in (\alpha, \beta)$ and $y \in \mathbb{R}$ let $W_n := (y - X_n)^\gamma \mathbb{1}\{X_n < y\}$. Then the following hold.

(i) Take $\gamma = \beta - \varepsilon$. Then for any $\varepsilon \in (0, \beta(\beta - \alpha)/(1 + \beta - \alpha))$ there exists a finite constant $K$ such that, for all $n$,

$$
\mathbb{E}[W_{n+1} - W_n | X_n = x] \leq K, \text{ for all } x \in \Sigma.
$$

(ii) For any $\eta \in (0, 1 - (\alpha/\beta))$, we can choose $z < y$ and $\gamma \in (\alpha, \beta)$ such that, for some $c > 0$, for all $x < z$,

$$
\mathbb{E}[W_{n+1} - W_n | X_n = x] \leq -cW_n^{\eta}.
$$

**Proof.** Fix $y \in \mathbb{R}$. Also take $\gamma \in (\alpha, \beta)$ and $\theta \in (0, 1)$; we will make more restrictive specifications for these parameters later. It suffices to suppose that $n = 0$. If $X_0 = x < y - 1$, we have $(y - X_0)^\theta < y - X_0$ and so

$$
W_1 - W_0 = (y - x - \Delta_0)^\gamma \mathbb{1}\{X_1 < y\} - (y - x)^\gamma
\leq [(y - x - \Delta_0^0)^\gamma - (y - x)^\gamma] \mathbb{1}\{\Delta_0^+ \leq (y - x)^\theta\}
+ [(y - x + \Delta_0^-)^\gamma - (y - x)^\gamma] \mathbb{1}\{\Delta_0^- \leq (y - x)^\theta\}
+ (y - x + \Delta_0^-)^\gamma \mathbb{1}\{\Delta_0^- \geq (y - x)^\theta\}.
$$

(5.15)
Chapter 5. Heavy tails

We bound the three terms on the right-hand side of (5.15) in turn. For the first term, we have from Taylor’s formula that
\[
(y - x - \Delta_0^-) \gamma - (y - x)^\gamma \mathbf{1}\{\Delta_0^- \leq (y - x)^\theta\} = -\gamma \Delta_0^- (y - x)^{\gamma - 1}(1 + o(1)) \mathbf{1}\{\Delta_0^- \leq (y - x)^\theta\},
\]
where the \(o(1)\) is non-random and as \(y - x \to \infty\). Hence, taking expectations and using (H2), it follows that for a fixed \(y\) and any \(x\) for which \(y - x\) is large enough,
\[
\mathbb{E}_x \left[ (y - x - \Delta_0^+) \gamma - (y - x)^\gamma \mathbf{1}\{\Delta_0^+ \leq (y - x)^\theta\} \right] \leq -\frac{c\gamma}{2}(y - x)^{\gamma - 1}(1 - \alpha) + \mathbb{E}_x \left[ (y - x + \Delta_0^-) \gamma - (y - x)^\gamma \mathbf{1}\{\Delta_0^- \leq (y - x)^\theta\} \right] \leq 2\gamma C.
\]
For the second term on the right-hand side of (5.15), a similar application of Taylor’s formula yields, for \(y - x\) sufficiently large,
\[
(y - x + \Delta_0^-) \gamma - (y - x)^\gamma \mathbf{1}\{\Delta_0^- \leq (y - x)^\theta\} \leq 2\gamma (y - x)^{\gamma - 1}(\Delta_0^-)^{\beta\Lambda 1}(\Delta_0^-)^{(1 - \beta)^+} \mathbf{1}\{\Delta_0^- \leq (y - x)^\theta\} \leq 2\gamma (y - x)^{\gamma - 1}(\Delta_0^-)^{\beta\Lambda 1}.
\]
Taking expectations and using (H3), we obtain
\[
\mathbb{E}_x \left[ (y - x + \Delta_0^-) \gamma - (y - x)^\gamma \mathbf{1}\{\Delta_0^- \geq (y - x)^\theta\} \right] \leq K(y - x)^{\gamma - 1}(\Delta_0^-)^{(1 - \beta)^+},
\]
for a constant \(K < \infty\). For the final term in (5.15),
\[
(y - x + \Delta_0^-) \gamma \mathbf{1}\{\Delta_0^- \geq (y - x)^\theta\} \leq ((\Delta_0^-)^{1/\theta} + \Delta_0^-)^\gamma \leq 2^\gamma (\Delta_0^-)^{\gamma/\theta}.
\]
Taking \(\gamma = \theta\beta\), which requires \(\theta \in (\alpha/\beta, 1)\), and using (H3), we see that
\[
\mathbb{E}_x \left[ (y - x + \Delta_0^-) \gamma \mathbf{1}\{\Delta_0^- \geq (y - x)^\theta\} \right] \leq 2^\gamma C.
\]
Combining these estimates and taking expectations in (5.15) we see that the negative term dominates asymptotically provided
\[
\gamma - 1 + \theta(1 - \alpha) > 0 \quad \text{and} \quad \gamma - 1 + \theta(1 - \alpha) > \gamma - 1 + \theta(1 - \beta)^+.
\]
The first inequality requires \(\theta > 1/(1 + \beta - \alpha)\), which is a stronger condition than \(\theta > \alpha/\beta\) that we had already imposed, but which can be achieved with \(\theta \in (\alpha/\beta, 1)\) since \(\alpha < \beta\). The second inequality reduces to \(1 - \alpha > (1 - \beta)^+\) which is satisfied since \(\alpha < \beta \wedge 1\). Part (i) follows. Moreover, for \(\gamma = \theta\beta\),
1/(1 + β − α) < θ < 1, we can take y − x large enough so that, for some ε > 0,

\[ \mathbb{E}_x[W_1 - W_0] \leq -\varepsilon(y - x)^{\theta \beta - 1 + \theta(1 - \alpha)} = -\varepsilon W_0^\eta, \]

where \( \eta = (\theta \beta - 1 + \theta(1 - \alpha))/(\theta \beta) \) can be anywhere in \((0, 1 - (\alpha/\beta))\), by appropriate choice of \( \theta \), which proves part (ii).

Lemma 5.2.11 has as a consequence the following tail bound.

**Lemma 5.2.12.** Suppose that (H0) holds, and that (H2) and (H3) hold for \( \alpha \in (0, 1) \) and \( \beta > \alpha \). Then for any \( \varphi > 0 \) and any \( \varepsilon > 0 \), as \( n \to \infty \),

\[ \mathbb{P} \left[ \min_{0 \leq m \leq n} X_m \leq -n^{\varphi} \right] = O(n^{1-\beta \varphi + \varepsilon}). \]

**Proof.** As in Lemma 5.2.11, choosing \( y = 0 \) there, let \( W_n = (-X_n)^\gamma \mathbf{1}\{X_n < 0\} \). Take \( \gamma = \beta - \varepsilon \) for \( \varepsilon \in (0, \beta(1 - \alpha)/(1 + \beta - \alpha)) \). For \( n > 0 \),

\[ \mathbb{P} \left[ \min_{0 \leq m \leq n} X_m \leq -n^{\varphi} \right] \leq \mathbb{P} \left[ \max_{0 \leq m \leq n} W_m \geq n^{\varphi \gamma} \right] = O(n^{1-\varphi \gamma}), \]

by Lemma 5.2.11(i) and Theorem 2.4.8. The result follows.

Now we can give the proof of our result on existence of moments for \( \rho_x \).

**Proof of Theorem 5.2.9.** Define \( W_n = (y - X_n)^\gamma \mathbf{1}\{X_n < y\} \) as in Lemma 5.2.11. For \( z > 0 \), let \( \tau_z = \min\{n \geq 0 : W_n \leq z\} \). Since \( \{W_n \leq z\} = \{X_n \geq y - z^{1/\gamma}\} \), we have that \( \rho_x = \tau_{(y-x)^\gamma} \) for \( x \leq y \). Now fix \( x \in \mathbb{R} \). Under the conditions of the theorem, Lemma 5.2.11(ii) implies that, for any \( \eta \in (0, 1 - (\alpha/\beta)) \), for \( y > x \) sufficiently large, on \( \{n < \sigma_{(y-x)^\gamma}\} \), a.s.,

\[ \mathbb{E}[W_{n+1} - W_n \mid F_n] \leq -\varepsilon W_n^\eta. \]

Then Corollary 2.7.3 shows that for any \( x \in \mathbb{R} \), \( \mathbb{E}[\rho_x^s] = \mathbb{E}[\tau_{(y-x)^\gamma}^s] < \infty \), for any \( s < \beta/\alpha \).

Next we prove our non-existence of moments result for \( \rho_x \). We use an idea based on Theorem 2.4.8: roughly speaking, we show that with good probability \( X_n \) travels a long way in the negative direction with a single heavy-tailed jump, and then must take a long time to come back.

For processes with heavy-tailed increments, it is often a single big jump that dominates asymptotics; technically, this idea often gives estimates in one direction that are close to sharp.
Proof of Theorem 5.2.10. Fix $x > 0$ and let $y < x$. Let $W'_n = (X_n - y)^\alpha 1\{X_n > y\}$. Then on \{$X_n \leq y$\}, $W'_{n+1} - W'_n \leq (\Delta^+_n)^\alpha$. On the other hand, on \{$X_n > y$\},

$$W'_{n+1} - W'_n \leq (X_n + \Delta^+_n - y)^\alpha - (X_n - y)^\alpha \leq (\Delta^+_n)^\alpha,$$

by concavity since $\alpha \in (0, 1]$. Hence for $C < \infty$ (not depending on $y$), $E[W'_{n+1} - W'_n \mid F_n] \leq C$, a.s., so the maximal inequality Theorem 2.4.8 implies that, for any $y < x$,

$$P\left[\max_{0 \leq k \leq m} W'_{n+k} \geq (x - y)^\alpha \mid F_n\right] \leq \frac{Cm + W'_n}{(x - y)^\alpha}, \text{ a.s.}$$

In particular, on \{$X_n \leq y$\}, $W'_n = 0$ and so

$$P\left[\max_{0 \leq k \leq m} X_{n+k} \geq x \mid F_n\right] \leq P\left[\max_{0 \leq k \leq m} W'_{n+k} \geq (x - y)^\alpha \mid F_n\right] \leq \frac{Cm}{(x - y)^\alpha}, \text{ a.s.}$$

Setting $m = (x - y)^\alpha/(2C)$ in the last display, we obtain that for some $\varepsilon > 0$ (not depending on $x$ or $y$), on \{${n < \rho_x} \cap \{X_n \leq y\}$\}, for any $y < x$,

$$P[\rho_x \geq \varepsilon(x - y)^\alpha \mid F_n] \geq 1/2, \text{ a.s.} \quad (5.16)$$

Since $X_0 = 0$ and $x > 0$, we have that \{$\Delta^-_0 > y^-$\} implies $\{\rho_x > 1\}$ and \{$X_1 \leq y$\}. So applying (5.16) at $n = 1$ we have that

$$P[\rho_x \geq \varepsilon(x - y)^\alpha] \geq E\left[1\{\Delta^-_0 > y^-\} \mid \rho_x \geq \varepsilon(x - y)^\alpha \mid F_1\right] \geq \frac{1}{2} P[\Delta^-_0 > y^-].$$

Taking $y = -\varepsilon^{-1/\alpha} z^{1/\alpha} < 0$, we have that for any $z > 0$,

$$P[\rho_x \geq z] \geq P[\rho_x \geq \varepsilon(x - y)^\alpha] \geq \frac{1}{2} P[\Delta^-_0 > \varepsilon^{-1/\alpha} z^{1/\alpha}].$$

Hence for any $\gamma > 0$,

$$E[\rho_x^\gamma] = \int_0^\infty P[\rho_x > z^{1/\gamma}] dz \geq \frac{1}{2} \int_0^\infty P[\Delta^-_0 > \varepsilon^{-1/\alpha} z^{1/\alpha}] dz.$$

Using the substitution $w = \varepsilon^{-\gamma} z$ we obtain

$$E[\rho_x^\gamma] \geq \frac{1}{2} \varepsilon^{-\gamma} \int_0^\infty P[\Delta^-_0 > w^{1/(\alpha \gamma)}] dw = \frac{1}{2} \varepsilon^{-\gamma} E[(\Delta^-_0)^{\alpha \gamma}],$$

which is infinite provided $\alpha \gamma \geq \beta$, i.e., $\gamma \geq \beta/\alpha$. \qed
5.2.5 Moments of last exit times

Similarly to (3.51), for \( x \in \mathbb{R} \) we denote the last exit time from \((-\infty, x]\) by
\[
\eta_x := \max\{n \geq 0 : X_n \leq x\}.
\]

If \( X_n \to +\infty \) a.s. (such as under the conditions of Theorem 5.2.2) then \( \eta_x < \infty \) a.s. for all \( x \in \mathbb{R} \), and the moments of the random variables \( \eta_x \) provide a quantitative characterization of the transience.

**Theorem 5.2.13.** Suppose that (H0) holds, and that (H2) and (H3) hold for \( \alpha \in (0, 1) \) and \( \beta > \alpha \). Then for any \( x \in \mathbb{R} \) and any \( s \in [0, (\beta/\alpha) - 1) \), \( \mathbb{E}[\eta_x^s] < \infty \).

**Theorem 5.2.14.** Suppose that (H0) holds. Let \( \alpha \in (0, 1] \) and \( \beta > \alpha \). Suppose that there exist \( c > 0, C < \infty \), and \( y_0 < \infty \) such that \( \mathbb{E}[(\Delta_n^\pm)^\alpha | X_n = x] \leq C \) for all \( x \in \Sigma \), and \( \mathbb{P}[\Delta_n^- \geq y | X_n = x] \geq cy^{-\beta} \) for all \( y \geq y_0 \) and all \( x \in \Sigma \). Then for any \( x \in \mathbb{R} \) and any \( s > (\beta/\alpha) - 1 \), \( \mathbb{E}[\eta_x^s] = \infty \).

We will again use the Lyapunov function \( f_{z, \delta} \) defined at (5.3).

**Lemma 5.2.15.** Let \( \alpha \in (0, 1) \) and \( \beta > \alpha \). Suppose that there exist \( C < \infty \), \( c > 0 \), and \( x_0 < \infty \) for which \( \mathbb{E}[(\Delta_n^\pm)^\alpha | X_n = x] \leq C \) for all \( x \in \Sigma \) and \( \mathbb{P}[(\Delta_n^-)^\alpha | X_n = x] \geq cy^{-\beta} \) for all \( y \geq y_0 \) and all \( x \in \Sigma \). Then for any \( \delta > \beta - \alpha \) and some \( A > 0 \) sufficiently large, for any \( z \in \mathbb{R} \),
\[
\mathbb{E}[f_{z, \delta}(X_{n+1}) - f_{z, \delta}(X_n) | X_n = x] \geq 0, \text{ for all } x > z + A.
\]

**Proof.** As in the proof of Lemma 5.2.3, it suffices to take \( n = 0 \) and \( z = 1 \). Let \( \delta > 0 \), and let \( f_\delta := f_{1, \delta} \) be as defined at (5.3). Let \( \gamma \in (0, 1) \); we will specify \( \delta \) and \( \gamma \) later. For \( x > 1 \) we have
\[
f_\delta(x + \Delta_0) - f_\delta(x) \geq [(x + \Delta_0^\pm)^{-\delta} - x^{-\delta}] \mathbb{1}\{\Delta_0^\pm \leq x^\gamma\} + (1 - x^{-\delta})\mathbb{1}\{\Delta_0^- \geq x\} - x^{-\delta} \mathbb{1}\{\Delta_0^+ > x^\gamma\}. \quad (5.17)
\]

We bound the three terms on the right-hand side of (5.17). For the first term, we have that by Taylor’s formula, as \( x \to \infty \), since \( \gamma < 1 \),
\[
[(x + \Delta_0^\pm)^{-\delta} - x^{-\delta}] \mathbb{1}\{\Delta_0^\pm \leq x^\gamma\} = -\delta(1 + o(1))x^{-1-\delta} \Delta_0^\pm \mathbb{1}\{\Delta_0^\pm \leq x^\gamma\}, \text{ a.s.,}
\]
where, as usual, the \( o(1) \) term is non-random. Similarly to (5.7), we have that \( \Delta_n^\pm \mathbb{1}\{\Delta_0^\pm \leq x^\gamma\} \leq (\Delta_0^\pm)^\alpha x^{(1-\alpha)\gamma} \), so that
\[
[(x + \Delta_0^\pm)^{-\delta} - x^{-\delta}] \mathbb{1}\{\Delta_0^\pm \leq x^\gamma\} = O(\Delta_0^\pm \mathbb{1}\{\Delta_0^\pm \leq x^\gamma\}).
\]
It follows that
\[
\mathbb{E}_x [(x + \Delta_0^+)^{-\delta} - x^{-\delta} \mathbb{1}\{\Delta_0^+ \leq x^\gamma\}] = O(x^{(1-\alpha)\gamma - 1-\delta} \mathbb{E}_x [(\Delta_0^+)\alpha])
\]
\[
= O(x^{(1-\alpha)\gamma - 1-\delta}).
\]
(5.18)

For the second term on the right-hand side of (5.17), we have that for some
\(A > 1\) sufficiently large,
\[
\mathbb{E}_x [(1 - x^{-\delta}) \mathbb{1}\{\Delta_0^- \geq x\}] \geq (1/2) \mathbb{P}_x[\Delta_0^- \geq x] \geq (c/2)x^{-\beta}.
\]
(5.19)

For the third term on the right-hand side of (5.17), we have that, by
Markov’s inequality,
\[
\mathbb{E}_x [x^{-\delta} \mathbb{1}\{\Delta_0^- > x^\gamma\}] \leq x^{-\delta} x^{-\alpha\gamma} \mathbb{E}_x [(\Delta_0^+)\alpha] = O(x^{-\beta - \alpha\gamma}).
\]
(5.20)

Combining (5.17) with (5.18), (5.19) and (5.20) we have that
\[
\mathbb{E}_x [f_{\delta}(X_1) - f_{\delta}(X_0)] \geq (c/2)x^{-\beta} + O(x^{-\beta - \alpha\gamma}) + O((1-\alpha)\gamma - 1-\delta).
\]

The positive \(x^{-\beta}\) term here dominates for \(A\) large enough provided that
\[-\beta > -\delta - \alpha\gamma\quad \text{and} \quad -\beta > (1-\alpha)\gamma - 1-\delta.
\]

For any \(\delta > \beta - \alpha\), the second inequality holds since \(\alpha \in (0,1)\) and \(\gamma < 1\). Given any such \(\delta\), the first inequality holds provided we choose \(\gamma \in \left(\frac{\beta - \delta}{\alpha}, 1\right)\).

Finally, we can complete the proofs of our results on last exit times.

**Proof of Theorem 5.2.13.** Fix \(x \in \mathbb{R}\) and let \(y > x\), to be specified later. For this proof, define the stopping time \(\lambda_{y,x} := \min\{n \geq \rho_y : X_n \leq x\}\), the time of reaching \((-\infty, x]\) after having first reached \([y, \infty)\). To prove our result on finiteness of moments for \(\lambda_x\), we prove an upper tail bound for \(\eta_x\). For \(y > x\), \(\{\rho_y \leq n\} \cap \{\lambda_{y,x} = \infty\}\) implies \(\eta_x \leq n\), so
\[
\mathbb{P}[\eta_x > n] \leq \mathbb{P}[\lambda_{y,x} < \infty] + \mathbb{P}[\rho_y > n].
\]
(5.21)

We obtain an upper bound for \(\mathbb{P}[\lambda_{y,x} < \infty]\) using our usual supermartingale argument. Under the conditions of the theorem, Lemma 5.2.3 applies. It follows that for \(\delta \in (0, \beta - \alpha)\), on \(\{\rho_y < \infty\}\), \((f_{x-A,\delta}(X_{n\wedge \eta_{y,x}}), n \geq \rho_y)\) is a non-negative supermartingale adapted to \((\mathcal{F}_n, n \geq \rho_y)\), and hence converges a.s. as \(n \to \infty\) to a limit, \(L_{y,x}\), say. Then, on \(\{\rho_y < \infty\}\), by Fatou’s lemma,
\[
f_{x-A,\delta}(X_{\rho_y}) \geq \mathbb{E}[L_{y,x} | \mathcal{F}_{\rho_y}] \geq \mathbb{E}[f_{x-A,\delta}(X_{\lambda_{y,x}}) \mathbb{1}\{\lambda_{y,x} < \infty\} | \mathcal{F}_{\rho_y}]
\]
By definition, on \( \{ \rho_y < \infty \} \), \( X_{\rho_y} \geq y \), so \( f_{x-A,\delta}(X_{\rho_y}) \leq (1 + A + y - x)^{-\delta} \).

Hence,

\[
P[\lambda_{y,x} < \infty] = E[P[\lambda_{y,x} < \infty \mid \mathcal{F}_{\rho_y}]1\{\rho_y < \infty\}] = O(y^{-\delta}).
\] (5.22)

For the final term in (5.21), for \( y > 0 \), \( P[\rho_y > n] = P[\max_{0 \leq m \leq n} X_m < y] \), where

\[
P[\max_{0 \leq m \leq n} X_m < y] \leq P[\max_{0 \leq m \leq n} X_m \leq y, \min_{0 \leq m \leq n} X_m \geq -y] + P[\min_{0 \leq m \leq n} X_m \leq -y] \leq P[\max_{0 \leq m \leq n-1} \Delta_m^+ \leq 2y] + P[\min_{0 \leq m \leq n} X_m \leq -y].
\] (5.23)

We choose \( y = n^{\frac{1}{\alpha} - \varepsilon} \), for \( \varepsilon \in (0, 1/\alpha) \). Then we have from (5.14) that for \( c' > 0 \),

\[
P[\max_{0 \leq m \leq n-1} \Delta_m^+ \leq 2n^{\frac{1}{\alpha} - \varepsilon}] = O(\exp\{-c'n^\alpha\}).
\] (5.24)

On the other hand, the \( \varphi = (1/\alpha) - \varepsilon \) case of Lemma 5.2.12 implies that

\[
P[\min_{0 \leq m \leq n} X_m \leq -n^{\frac{1}{\alpha} - \varepsilon}] = O(n^{1-(\beta/\alpha)+\varepsilon}).
\] (5.25)

Using the bounds (5.24) and (5.25) in the \( y = n^{\frac{1}{\alpha} - \varepsilon} \) case of (5.23), we obtain

\[
P[\max_{0 \leq m \leq n} X_m < n^{\frac{1}{\alpha} - \varepsilon}] = O(n^{1-(\beta/\alpha)+(\beta+1)\varepsilon}).
\] (5.26)

Thus taking \( y = n^{\frac{1}{\alpha} - \varepsilon} \) in (5.21) and \( \delta \) as close as we wish to \( \beta - \alpha \), and combining (5.22) with (5.26), we conclude that, for any \( \varepsilon > 0 \), \( P[\eta_x > n] = O(n^{1-(\beta/\alpha)+\varepsilon}) \), which yields the claimed moment bounds. \( \square \)

**Proof of Theorem 5.2.14.** Fix \( x \in \mathbb{R} \) and let \( y > x \). For this proof, define \( \lambda_{n,x} := \min\{m \geq n : X_m \leq x\} \), the first time of reaching \( (-\infty, x] \) after time \( n \). Similarly, set \( \rho_{n,y} := \min\{m \geq n : X_m \geq y\} \). We have that, for \( r > 0 \),

\[
P[\lambda_x > n] \geq E[1\{X_n \leq r\} P[\lambda_{n,x} < \infty \mid \mathcal{F}_n]].
\] (5.27)

Under the conditions of the theorem, Lemma 5.2.15 applies. It follows that for \( \delta > \beta - \alpha \), \( (f_{x-A,\delta}(X_{m\wedge \lambda_{n,x} \wedge \rho_{n,y}}), m \geq n) \) is a non-negative submartingale adapted to \( (\mathcal{F}_m, m \geq n) \); moreover, it is uniformly bounded and
so converges a.s. and in $L^1$, as $m \to \infty$, to the limit $f_{x - A, \delta}(X_{\lambda_n, x \wedge \rho_n, y})$, since $\nu_{n, x} \wedge \rho_{n, y} < \infty$ a.s., by (5.1), which is available since Proposition 5.2.1 applies under the conditions of the theorem. Hence, a.s.,

$$f_{x - A, \delta}(X_n) \leq \mathbb{E}[f_{x - A, \delta}(X_{\lambda_n, x \wedge \rho_n, y}) \mid \mathcal{F}_n] \leq \mathbb{P}[\lambda_{n, x} < \infty \mid \mathcal{F}_n] + f_{x - A, \delta}(y).$$

Since $y$ was arbitrary, and $f_{x - A, \delta}(y) \to 0$ as $y \to \infty$, it follows that, a.s.,

$$\mathbb{P}[\nu_{n, x} < \infty \mid \mathcal{F}_n] \geq f_{x - A, \delta}(X_n) \geq f_{x - A, \delta}(r),$$

on \{$X_n \leq r$\}. Hence from (5.27) we obtain for $r \geq x$,

$$\mathbb{P}[\lambda_x > n] \geq f_{x - A, \delta}(r) \mathbb{P}[X_n \leq r] \geq (1 + A + r - x)^{-\delta} \mathbb{P}[X_n \leq r]. \quad (5.28)$$

It remains to obtain a lower bound for $\mathbb{P}[X_n \leq r]$, for a suitable choice of $r$. Let $Y_n = \sum_{m=0}^{n-1} \Delta_m^+$. Following the argument for (5.13), with $h(y) = y^\alpha$, $\alpha \in (0, 1]$, we may apply Theorem 2.4.8 to $Z_n = Y_n^\alpha$ to obtain

$$\mathbb{P}\left[\max_{0 \leq m \leq n} Y_m^\alpha \geq x\right] = \mathbb{P}[Y_n \geq x^{1/\alpha}] \leq C_n x^{-1},$$

for some $C < \infty$ and all $n \geq 0$, $x > 0$, which implies that

$$\mathbb{P}\left[X_n \leq (2Cn)^{1/\alpha}\right] \geq \mathbb{P}\left[Y_n \leq (2Cn)^{1/\alpha}\right] \geq 1/2,$$

since $X_n \leq X_0 + Y_n = Y_n$. Thus taking $r = (2Cn)^{1/\alpha}$, we have $\mathbb{P}[X_n \leq r] \geq 1/2$, and with this choice of $r$ in (5.28) we obtain $\mathbb{P}[\lambda_x > n] \geq \varepsilon n^{-\delta/\alpha}$, for some $\varepsilon > 0$ and all $n$ sufficiently large. Since $\delta > \beta - \alpha$ was arbitrary, the result follows.

5.3 Oscillating random walk

5.3.1 Introduction

We consider $(X_n, n \in \mathbb{Z}^+)$ a discrete-time, time-homogeneous Markov process on $\mathbb{R}$, with transition kernel $p(x, dz) = w_x(z - x)dz$ so that

$$\mathbb{P}[X_{n+1} \in A \mid X_n = x] = \int_A p(x, dz) = \int_A w_x(z - x)dz, \quad (5.29)$$

for all Borel sets $A \subseteq \mathbb{R}$. Here the local transition densities $w_x : \mathbb{R} \to \mathbb{R}_+$ are Borel functions. We will assume one of several ‘heavy-tailed’ conditions for the $w_x$.

We are interested in the recurrence classification of the non-homogeneous random walk $X_n$, where we say $X_n$ is
5.3. Oscillating random walk

- **recurrent** if \( \lim \inf_{n \to \infty} |X_n| = 0 \), a.s.;
- **transient** if \( \lim_{n \to \infty} |X_n| = \infty \), a.s.

For a probability density (Borel) function \( v : \mathbb{R} \to \mathbb{R}_+ \) and an exponent \( \alpha \in (0, \infty) \), we write \( v \in D_\alpha \) to mean that there exists \( c : \mathbb{R}_+ \to (0, \infty) \) with

\[
\sup_{y \geq 0} c(y) < \infty \quad \text{and} \quad \lim_{y \to \infty} c(y) = c \in (0, \infty)
\]

for which

\[
v(y) := \begin{cases} 
  c(y)y^{-1-\alpha} & \text{if } y > 0, \\
  0 & \text{if } y \leq 0.
\end{cases}
\] (5.30)

Our main interest concerns models in which \( w_x \) depends on \( x \) only through \( \text{sgn} \ x \), the sign of \( x \). Such models are known as oscillating random walks. The first example of this type is as follows.

(Os1) Let \( v_\alpha \in D_\alpha \) and \( v_\beta \in D_\beta \), for some \( \alpha, \beta > 0 \). For \( x, y \in \mathbb{R} \), let

\[
w_x(y) := \begin{cases} 
  v_\alpha(-y) & \text{if } x \geq 0, \\
  v_\beta(y) & \text{if } x < 0.
\end{cases}
\]

This case is known as the one-sided oscillating random walk; in words, the walk always jumps in the direction of (and possibly over) the origin, with tail exponent \( \alpha \) from the positive half-line and exponent \( \beta \) from the negative half-line. The following recurrence classification applies.

**Theorem 5.3.1.** Suppose that (Os1) holds. Then the one-sided oscillating random walk is transient if \( \alpha + \beta < 1 \) and recurrent if \( \alpha + \beta > 1 \).

**Remarks 5.3.2.**
(a) Theorem 5.3.1 does not cover the critical case \( \alpha + \beta = 1 \), which cannot be decided upon without further assumptions on the rate of convergence of \( c(y) \) in (5.30).
(b) An analogous result to Theorem 5.3.1 holds in the discrete-space case, where instead of a density function for the increments we have a probability mass function.

A special case in which \( \alpha = \beta \) and \( v_\alpha = v_\beta \) in (Os1) is the *antisymmetric* case in which

\[
w_x(y) = v_\alpha(-y \text{sgn}(x)), \text{ for } x, y \in \mathbb{R}.
\]

In this case, Theorem 5.3.1 shows that the walk is transient for \( \alpha < 1/2 \) and recurrent for \( \alpha > 1/2 \). The rest of this section is devoted to the proof of Theorem 5.3.1. First in Section 5.3.2 we present some comments on neighbouring models.
5.3.2 Relation to some two-dimensional random walks

Example 5.3.3. Consider \( \xi_n = (\xi_n^{(1)}, \xi_n^{(2)}) \), \( n \in \mathbb{Z}_+ \), a nearest-neighbour random walk on \( \mathbb{Z}^2 \) with transition probabilities

\[
\mathbb{P}[\xi_{n+1} = (y_1, y_2) \mid \xi_n = (x_1, x_2)] = p(x_1, x_2; y_1, y_2).
\]

Suppose that the probabilities are given for \( x_2 \neq 0 \) by,

\[
p(x_1, x_2; x_1, x_2 + 1) = p(x_1, x_2; x_1, x_2 - 1) = \frac{1}{3};
\]

\[
p(x_1, x_2; x_1 + 1, x_2) = \frac{1}{3} 1 \{x_2 < 0\};
\]

\[
p(x_1, x_2; x_1 - 1, x_2) = \frac{1}{3} 1 \{x_2 > 0\};
\]

(5.31)

(the rest being zero) and for \( x_2 = 0 \) by

\[
p(x_1, 0; x_1, 1) = 1 \text{ for all } x_1 > 0,
\]

\[
p(x_1, 0; x_1, -1) = 1 \text{ for all } x_1 < 0, \text{ and } p(0, 0; 0, 1) = p(0, 0; 0, -1) = 1/2.
\]

See Figure 5.1 for an illustration.

Set \( \tau_0 := 0 \) and define recursively \( \tau_{k+1} = \min\{n > \tau_k : \xi_n^{(2)} = 0\} \) for \( k \geq 0 \); consider the embedded Markov chain \( X_n = \xi_{\tau_n}^{(1)} \). We show that \( X_n \) is a discrete version of the oscillating random walk described in Section 5.3.1. Indeed, \( \xi_n^{(2)} \) is a random walk on \( \mathbb{Z} \) with increments taking values \(-1, 0, +1\) each with probability \( 1/3 \).

We then have that for some constant \( c \in (0, \infty) \),

\[
\mathbb{P}[\tau_1 > r] = (c + o(1))r^{-1/2}, \text{ as } r \to \infty;
\]

see e.g. [99, p. 415]. Suppose that \( \xi_0^{(1)} = x > 0 \). Note that since between times \( \tau_0 \) and \( \tau_1 \), \( \xi_n^{(1)} \) is monotone,

\[
\mathbb{P}[\xi_{\tau_1}^{(1)} - \xi_{\tau_0}^{(1)} < -r] \geq \mathbb{P}[\tau_1 > 3r + r^{3/4}] - \mathbb{P}[\xi_{\tau_1}^{(1)} - r^{3/4} \geq -r].
\]

Here, by the Azuma–Hoeffding inequality (Theorem 2.4.14),

\[
\mathbb{P}[\xi_{\tau_1}^{(1)} - r^{3/4} \geq -r] \leq \exp\{-\varepsilon r^{1/2}\},
\]

for some \( \varepsilon > 0 \). Similarly,

\[
\mathbb{P}[\xi_{\tau_1}^{(1)} - \xi_{\tau_0}^{(1)} < -r] \leq \mathbb{P}[\tau_1 > 3r - r^{3/4}] + \mathbb{P}[\xi_{\tau_1}^{(1)} - r^{3/4} \leq -r],
\]

where

\[
\mathbb{P}[\xi_{\tau_1}^{(1)} - r^{3/4} \leq -r] \leq \exp\{-\varepsilon r^{1/2}\}.
\]
Combining these bounds, and using the symmetric argument for \( \{\xi_n^{(1)} > r\} \) when \( \xi_0^{(1)} = x < 0 \), we see that

\[
\begin{align*}
\mathbb{P}[X_{n+1} - X_n < -r \mid X_n = x] &= u(r), \text{ if } x > 0, \text{ and} \\
\mathbb{P}[X_{n+1} - X_n > r \mid X_n = x] &= u(r), \text{ if } x < 0,
\end{align*}
\]

where \( u(r) = (c + o(1))r^{-1/2} \). Thus the process \( X_n \) satisfies a discrete-space analogue of (Os1) with \( \alpha = \beta = 1/2 \). This is the critical case identified in Theorem 5.3.1, and a finer analysis is required to determine the behaviour; we conjecture that the walk is recurrent.

Example 5.3.4. We present two variations on the previous example, which are superficially similar but turn out to be less delicate. First, modify the random walk of the previous example by supposing that (5.31) holds but replacing the behaviour at \( x_2 = 0 \) by \( p(x_1,0;x_1,1) = p(x_1,0;x_1,-1) = 1/2 \) for all \( x_1 \in \mathbb{Z} \). See the left-hand part of Figure 5.2 for an illustration.

As in Example 5.3.3, consider the embedded process \( X_n \). In this case, for all \( x \in \mathbb{Z} \), for \( r \geq 0 \),

\[
\begin{align*}
\mathbb{P}[X_{n+1} - X_n < -r \mid X_n = x] &= \mathbb{P}[X_{n+1} - X_n > r \mid X_n = x] = u(r),
\end{align*}
\]

where \( u(r) = (c/2)(1+o(1))r^{-1/2} \). Thus \( X_n \) is a random walk with symmetric increments, and a result of Shepp [283] implies that the walk is transient.
Next, modify the random walk of Example 5.3.3 by supposing that (5.31) holds but replacing the behaviour at \( x_2 = 0 \) by \( p(x_1, 0; x_1, 1) = p(x_1, 0; x_1, -1) = 1/2 \) if \( x_1 \geq 0 \), and \( p(x_1, 0; x_1, 1) = 1 \) for \( x_1 < 0 \). See the right-hand part of Figure 5.2 for an illustration.

This time the walk takes a symmetric increment as at (5.33) when \( x < 0 \) but a one-sided increment as at (5.32) when \( x \geq 0 \). In this case a result of Rogozin and Foss [273] shows that the walk is transient. △

Figure 5.2: Pictorial representation of the two non-homogeneous nearest-neighbour random walks on \( \mathbb{Z}^2 \) of Example 5.3.4. Each of these walks is transient.

5.3.3 Complexes of half-lines

As an aside, we describe an extension of the model of Section 5.3.1 to a Markov process on a complex of half-lines \( \mathbb{R}_+ \times S \), where \( S \) is finite. On a given half-line, a particle performs a random walk with a heavy-tailed increment distribution, until the point at which it would exit the half-line, when it moves (in general, at random) through the origin to another half-line.

We consider \((Y_n, \xi_n), n \in \mathbb{Z}_+\), a discrete-time, time-homogeneous Markov process with state space \( \mathbb{R}_+ \times S \), where \( S \) is a finite non-empty set. The state space is equipped with the appropriate Borel sets, namely, sets of the form \( B \times A \) where \( B \in \mathcal{B}(\mathbb{R}_+) \) is a Borel set in \( \mathbb{R}_+ \), and \( A \subseteq S \). The Markov process will be described by:

• an irreducible stochastic matrix \( P = (p(i, j), i, j \in S) \); and
5.3. Oscillating random walk

- a collection \((w_i, i \in S)\) of probability density functions, so \(w_i : \mathbb{R} \to \mathbb{R}_+\) is a Borel function with \(\int_{\mathbb{R}} w_i(y)dy = 1\).

We view \(\mathbb{R}_+ \times S\) as a complex of half-lines \(\mathbb{R}_+ \times \{k\}\), or branches, connected at a central origin \(O := \{0\} \times S\); at time \(n\), the coordinate \(\xi_n\) describes which branch the process is on, and \(Y_n\) describes the distance along that branch at which the process sits.

The transition kernel of the process is given for \((y,i) \in \mathbb{R}_+ \times S\) by

\[
P[ (Y_{n+1},\xi_{n+1}) \in B \times \{j\} \mid (Y_n,\xi_n) = (y,i) ] = p(i,j) \int_B w_i(-z-y)dz + 1\{i = j\} \int_B w_i(z-y)dz, \tag{5.34}
\]

for all Borel sets \(B \subseteq \mathbb{R}_+\) and all \(j \in S\).

The dynamics of the process represented by (5.34) can be described algorithmically as follows. Given \((Y_n,\xi_n) = (y,i) \in \mathbb{R}_+ \times S\), generate (independently) a spatial increment \(\varphi_{n+1}\) from the distribution given by \(w_i\) and a random index \(\eta_{n+1} \in S\) according to the distribution \(p(i, \cdot)\). Then,

- if \(y + \varphi_{n+1} \geq 0\), set \((Y_{n+1},\xi_{n+1}) = (y + \varphi_{n+1},i)\); or
- if \(y + \varphi_{n+1} < 0\), set \((Y_{n+1},\xi_{n+1}) = (|y + \varphi_{n+1}|,\eta_{n+1})\).

In words, the walk takes a \(w_\xi\)-distributed step. If this step would bring the walk beyond the origin, it passes through the origin and switches onto branch \(\eta_{n+1}\) (or, if \(\eta_{n+1}\) happens to be equal to \(\xi_n\), it reflects back along the same branch).

The finite irreducible stochastic matrix \(P\) is associated with a (unique) positive invariant probability distribution \((\mu_k, k \in S)\) satisfying

\[
\sum_{j \in S} \mu_j p(j,k) - \mu_k = 0, \text{ for all } k \in S. \tag{5.35}
\]

For future reference, we state the following.

(Os2) Let \(P = (p(i,j), i,j \in S)\) be an irreducible stochastic matrix, and let \((\mu_k, k \in S)\) denote the corresponding invariant distribution.

The \(w_k\) are then described by a collection of positive parameters \((\alpha_k, k \in S)\).

(Os3) Suppose that, for each \(k \in S\), we have an exponent \(\alpha_k \in (0, \infty)\) and a density function \(v_k \in \mathcal{D}_{\alpha_k}\). Then suppose that, for all \(y \in \mathbb{R}\), \(w_k\) is given by \(w_k(y) = v_k(-y)\).
The following recurrence classification, whose proof we omit, is contained in the main result of [229], and contains Theorem 5.3.1 as a special case where \( \mathcal{S} = \{-1, +1\} \), and \( \mathbb{R}_+ \times \mathcal{S} \) is identified with \( \mathbb{R} \) in the natural way.

**Theorem 5.3.5.** Suppose that (Os2) and (Os3) hold.

(a) Suppose that \( \max_{k \in \mathcal{S}} \alpha_k \geq 1 \). Then the process is recurrent.

(b) Suppose that \( \max_{k \in \mathcal{S}} \alpha_k < 1 \).

(i) If \( \sum_{k \in \mathcal{S}} \mu_k \cot(\pi \alpha_k) < 0 \), then the process is recurrent.

(ii) If \( \sum_{k \in \mathcal{S}} \mu_k \cot(\pi \alpha_k) > 0 \), then the process is transient.

### 5.3.4 Lyapunov functions

The proofs of the results in Section 5.3.1 are based on the Lyapunov function approach; our function is roughly of the form \( x \mapsto |x|^\nu \), with two modifications: one is a minor adjustment near \( x = 0 \) to control the case \( \nu < 0 \), and the other is the introduction of an extra parameter, \( \lambda \), to give different weights to the different sides of the origin, which is a crucial technical tool. Precisely, for parameters \( \nu \in \mathbb{R} \) and \( \lambda \in \mathbb{R}_+ \), we consider the function

\[
 f_{\nu}(\cdot; \lambda) : \mathbb{R} \to \mathbb{R}_+ \text{ given by } f_{\nu}(x; \lambda) := \begin{cases} |x|^\nu & \text{if } x \geq 1, \\ \lambda + (1 - \lambda) \frac{1+x}{2} & \text{if } |x| < 1, \\ \lambda|x|^\nu & \text{if } x \leq -1. \end{cases} \tag{5.36}
\]

Then \( x \mapsto f_{\nu}(x; \lambda) \) is continuous. The form given at (5.36) ensures that \( f_{\nu} \) is bounded as a function of \( x \) for \( \nu < 0 \). Note that for \( \lambda > 0 \) and any \( x \in \mathbb{R} \),

\[
 f_{\nu}(x; \lambda) = \lambda f_{\nu}(-x; 1/\lambda). \tag{5.37}
\]

We consider the process \( f_{\nu}(X_n; \lambda) \) for appropriately chosen \( \nu \) and \( \lambda \). By (5.29), the one-step mean increment of \( f_{\nu}(X_n; \lambda) \) is given by

\[
 \mathbb{E} [f_{\nu}(X_{n+1}; \lambda) - f_{\nu}(X_n, \lambda) \mid X_n = x] = \int_{-\infty}^{\infty} (f_{\nu}(x+y; \lambda) - f_{\nu}(x; \lambda)) w_x(y) dy =: F_{\nu}(x; \lambda),
\]

say. Suppose that (Os1) holds. Let \( x > 0 \). Then some manipulation using (5.37) shows that

\[
 F_{\nu}(x; \lambda) = \int_{0}^{\infty} (f_{\nu}(x-y; \lambda) - f_{\nu}(x; \lambda)) v_\alpha(y) dy =: F_{\nu,\alpha}^\text{one}(x; \lambda), \text{ and }
\]
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\[ F_\nu(-x; \lambda) = \lambda \int_0^\infty (f_\nu(x - y; 1/\lambda) - f_\nu(x; 1/\lambda)) v_\beta(y) dy. \]

In other words, (Os1) implies that

\[ F_\nu(x; \lambda) = \begin{cases} 
F_{\nu, \alpha}^\text{one}(|x|; \lambda) & \text{if } x > 0 \\
\lambda F_{\nu, \beta}^\text{one}(|x|; 1/\lambda) & \text{if } x < 0 \end{cases} \tag{5.38} \]

We will study \( F_{\nu, \alpha}^\text{one}(x; \lambda) \) in Section 5.3.6 below; in our analysis the following integrals will appear.

\[ i_{1, \nu, \alpha} := \int_0^1 \frac{(1 - u)^\nu - 1}{u^{1+\alpha}} du; \tag{5.39} \]
\[ i_{2, \nu, \alpha} := \int_1^\infty \frac{(u - 1)^\nu}{u^{1+\alpha}} du. \tag{5.40} \]

We will need to estimate \( i_{1, \nu, \alpha} \) and \( i_{2, \nu, \alpha} \) as \( \nu \to 0 \); this is the purpose of the next subsection.

5.3.5 Technical preliminaries

We collect some facts from [1]. The Euler gamma function \( \Gamma \) satisfies the functional equation \( z \Gamma(z) = \Gamma(z + 1), \) as well as the following ‘reflection formula’, equation 6.1.17 from [1, p. 256]:

\[ \Gamma(z)\Gamma(1 - z) = -\pi \csc \pi z, \tag{5.41} \]
valid for \( z \in (0, 1); \) one consequence of the functional equation is that

\[ \Gamma(z) \sim 1/z, \quad \text{as } z \downarrow 0. \tag{5.42} \]

Moreover, for \( x > 0, \) by Taylor’s formula,

\[ \Gamma(x + h) = \Gamma(x)(1 + h\psi(x) + O(h^2)), \tag{5.43} \]
as \( h \to 0, \) where \( \psi(z) = \frac{\text{d}}{\text{d}z} \log \Gamma(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function, and \( \psi(1) = -\gamma, \) where \( \gamma \approx 0.5772 \) is Euler’s constant. The digamma function also satisfies a reflection formula, namely equation 6.3.7 from [1, p. 259]:

\[ \psi(1 - z) - \psi(z) = \pi \cot \pi z. \tag{5.44} \]

The beta function is given for \( a > 0, b > 0 \) by

\[ B(a, b) = \int_0^1 u^{a-1}(1 - u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}. \tag{5.45} \]
The Gauss hypergeometric function is defined via the series
\[ 2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} z^n, \] (5.46)
which is convergent for \(|z| < 1\). A consequence of (5.46) is that \(2F_1(a, b; c; z) = 1 + O(z)\) as \(z \to 0\). For \(c > b > 0\) there is the integral representation
\[ 2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}du. \] (5.47)

We will need the following linear transformation formula, equation 15.3.6 from [1, p. 559],
\[ 2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} 2F_1(a, b; a+b-c+1; 1-z) \]
\[ + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} 2F_1(c-a, c-b; c-a-b+1; 1-z). \] (5.48)

**Lemma 5.3.6.** The following hold.

(i) For \(\alpha \in (0, 1)\) and \(\nu \in (-1, \alpha)\),
\[ i_1^{\nu, \alpha} = \frac{1}{\alpha} \left( 1 - \frac{\Gamma(1+\nu)\Gamma(1-\alpha)}{\Gamma(1+\nu-\alpha)} \right). \] (5.49)

(ii) For \(\alpha > 0\) and \(\nu \in (-1, \alpha)\),
\[ i_2^{\nu, \alpha} = \frac{\Gamma(1+\nu)\Gamma(\alpha-\nu)}{\Gamma(1+\alpha)}. \] (5.50)

**Proof.** For part (i), we compute the integral via
\[ \int_0^1 ((1-u)^{\nu} - 1)u^{-1-\alpha}du = \lim_{t \downarrow 0} \int_t^1 ((1-u)^{\nu} - 1)u^{-1-\alpha}du. \]
For the last integral, we get, for \(\alpha > 0\) and \(t > 0\),
\[ \int_t^1 ((1-u)^{\nu} - 1)u^{-1-\alpha}du = \int_t^1 (1-u)^{\nu}u^{-1-\alpha}du + \frac{1-t^{-\alpha}}{\alpha}. \]
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With the substitution \( v = \frac{1-u}{1+t} \), the last integral becomes

\[
\int_t^1 (1-u)\nu u^{-\alpha} du = (1-t)^{1+\nu} \int_0^1 \nu' (1-(1-t)\nu')^{-\alpha-1} dv
\]

\[
= (1-t)^{1+\nu} \frac{2F_1(1+\alpha, 1+\nu; 2+\nu; 1-t)}{1+\nu},
\]

provided \( \nu > -1 \), by (5.47). Now, by (5.48),

\[
2F_1(1+\alpha, 1+\nu; 2+\nu; 1-t) = \frac{\Gamma(2+\nu)\Gamma(-\alpha)}{\Gamma(1-\alpha+\nu)} 2F_1(1+\alpha, 1+\nu; 1+\alpha; t)
\]

\[
+ t^{-\alpha} \frac{1+\nu}{\alpha} 2F_1(1-\alpha+\nu, 1; 1+\alpha; t)
\]

\[
= \frac{\Gamma(2+\nu)\Gamma(-\alpha)}{\Gamma(1-\alpha+\nu)} + \frac{1+\nu}{\alpha} t^{-\alpha} + O(t^{1-\alpha}),
\]

as \( t \downarrow 0 \), provided \( \alpha \in (0, 1) \). Combining these results we get

\[
\int_t^1 ((1-u)\nu - 1)u^{-\alpha} du = \frac{(1-t)^{1+\nu}}{1+\nu} \left[ \frac{\Gamma(2+\nu)\Gamma(-\alpha)}{\Gamma(1-\alpha+\nu)} + \frac{1+\nu}{\alpha} t^{-\alpha} + O(t^{1-\alpha}) \right]
\]

\[
+ \frac{1-t^{-\alpha}}{\alpha} \frac{\Gamma(1+\nu)\Gamma(-\alpha)}{\Gamma(1-\alpha+\nu)} + O(t^{1-\alpha}).
\]

Hence

\[
\int_0^1 ((1-u)\nu - 1)u^{-\alpha} du = \frac{1}{\alpha} + \frac{\Gamma(1+\nu)\Gamma(-\alpha)}{\Gamma(1-\alpha+\nu)}
\]

\[
= \frac{1}{\alpha} \left( 1 - \frac{\Gamma(1+\nu)\Gamma(1-\alpha)}{\Gamma(1-\alpha+\nu)} \right),
\]

using (5.41). Thus we obtain (5.49).

For part (ii),

\[
\int_1^\infty (u-1)\nu u^{-\alpha} du = \int_0^1 (1-v)\nu v^{\alpha-\nu-1} dv = \frac{\Gamma(1+\nu)\Gamma(\alpha-\nu)}{\Gamma(1+\alpha)},
\]

by (5.45), provided \( \nu > -1 \) and \( \alpha - \nu > 0 \). Hence we obtain (5.50).

\[ \square \]

Lemma 5.3.7. \( \text{(i) For } \alpha \in (0, 1), \text{ as } \nu \to 0, \]

\[
i_1^{\nu, \alpha} = \frac{\nu}{\alpha} (\psi(1-\alpha) - \psi(1)) + O(\nu^2). \quad (5.51)
\]
(ii) For $\alpha > 0$, as $\nu \to 0$,

$$i_{3}^{\nu,\alpha} = \frac{1}{\alpha} + \frac{\nu}{\alpha} (\psi(1) - \psi(\alpha)) + O(\nu^2). \quad (5.52)$$

**Proof.** For part (i), we have from (5.49) with (5.43) that

$$i_{1}^{\nu,\alpha} = \frac{1}{\alpha} \left( 1 - (1 + \nu \psi(1) + O(\nu^2))(1 - \nu \psi(1 - \alpha) + O(\nu^2)) \right),$$

which gives (5.51).

For part (ii), we have from (5.50) with (5.43) that

$$i_{3}^{\nu,\alpha} = \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} (1 + \nu \psi(1) + O(\nu^2))(1 - \nu \psi(\alpha) + O(\nu^2)),$$

which gives (5.52). \qed

### 5.3.6 Increment estimates

The following result gives an estimate for $F_{\nu,\alpha}^{\text{one}}(x; \lambda)$ for $\alpha \in (0, 1)$.

**Lemma 5.3.8.** Suppose that $v_\alpha \in D_\alpha$ with $\alpha \in (0, 1)$. Then for any $\nu \in (-1, \alpha)$ and $\lambda \in \mathbb{R}_+$, as $x \to +\infty$,

$$F_{\nu,\alpha}^{\text{one}}(x; \lambda) = cx^{\nu-\alpha} R_{\nu,\alpha}(\lambda) + o(x^{\nu-\alpha}), \quad (5.53)$$

where

$$R_{\nu,\alpha}(\lambda) = i_{1}^{\nu,\alpha} + \lambda i_{2}^{\nu,\alpha} - \frac{1}{\alpha} \quad (5.54)$$

Moreover, suppose that for $\theta \in \mathbb{R}$, $\lambda(\theta, \nu) > 0$ is such that

$$\lim_{\nu \to 0} \frac{1 - \lambda(\theta, \nu)}{\nu} = \theta. \quad (5.55)$$

Then, as $\nu \to 0$,

$$R_{\nu,\alpha}(\lambda(\theta, \nu))) = -\frac{\nu}{\alpha} r(\theta, \alpha) + o(\nu), \quad (5.56)$$

where $r(\theta, \alpha) = \theta - \pi \cot \pi \alpha$.

For $\alpha \geq 1$ we have the following result.

**Lemma 5.3.9.** Suppose that $v_\alpha \in D_\alpha$ with $\alpha \geq 1$. Then for any $\nu \in (0, \alpha)$ and $\lambda \in \mathbb{R}_+$, there exists $\varepsilon > 0$ such that, as $x \to \infty$,

$$F_{\nu,\alpha}^{\text{one}}(x; \lambda) \leq -\varepsilon x^{\nu-1} + o(x^{\nu-1}).$$
Before giving the proofs of these two lemmas, we introduce the notation

\[ I^\nu,\alpha_1(x) := \int_0^{x-1} (x-y)^\nu - x^\nu \, v_\alpha(y)dy; \]
\[ I^\nu,\alpha_2(x; \lambda) := \int_{x+1}^\infty (\lambda(y-x)^\nu - x^\nu) \, v_\alpha(y)dy; \]
\[ I^\nu,\alpha_3(x; \lambda) := \int_{x-1}^{x+1} \left( \lambda + (1-\lambda) \frac{1+y-x}{2} - x^\nu \right) v_\alpha(y)dy. \]

From (5.36), we obtain the decomposition

\[ F^\nu_{\nu,\alpha}(x; \lambda) = I^\nu,\alpha_1(x) + I^\nu,\alpha_2(x; \lambda) + I^\nu,\alpha_3(x; \lambda). \quad (5.57) \]

We estimate the integrals on the right-hand side of (5.57) individually via a series of lemmas.

**Lemma 5.3.10.** Suppose that \( v_\alpha \in \mathcal{D}_\alpha \) with \( \alpha > 0 \).

(i) If \( \alpha \in (0,1) \) then, for any \( \nu \in (-1, \alpha) \), as \( x \to \infty \),

\[ I^\nu,\alpha_1(x) = cx^{\nu-\alpha} + o(x^{\nu-\alpha}). \]

(ii) If \( \alpha = 1 \) then, for any \( \nu > 0 \), there exist \( \varepsilon > 0 \) and \( x_0 \in \mathbb{R}_+ \) such that

\[ I^\nu,\alpha_1(x) \leq -\varepsilon x^{\nu-1} \log x, \text{ for all } x \geq x_0. \]

(iii) If \( \alpha > 1 \) then, for any \( \nu > 0 \), there exist \( \varepsilon > 0 \) and \( x_0 \in \mathbb{R}_+ \) such that

\[ I^\nu,\alpha_1(x) \leq -\varepsilon x^{\nu-1}, \text{ for all } x \geq x_0. \]

**Proof.** First suppose that \( \alpha \in (0,1) \). Then

\[ I^\nu,\alpha_1(x) = x^{\nu-\alpha} \int_0^{1-(1/x)} g(u)c(ux)du, \]

where \( g(u) = ((1-u)^\nu - 1)u^{-1-\alpha} \). Note that \( g(u) \sim \nu u^{-\alpha} \) as \( u \downarrow 0 \), so that \( g(u) \) is integrable over \( (0,1) \) provided \( \alpha < 1 \) and \( \nu > -1 \). Then for any \( \varepsilon > 0 \),

\[ \left| \int_\varepsilon^{1-(1/x)} g(u)c(ux)du - c \int_\varepsilon^{1-(1/x)} g(u)du \right| \leq \left( \sup_{y \geq \varepsilon x} |c(y) - c| \right) \int_0^1 |g(u)|du, \]
which tends to 0 as \( x \to \infty \), since \( c(y) \to c \). Moreover,
\[
\int_0^\varepsilon |g(u)|c(ux)du \leq \left( \sup_{y \geq 0} c(y) \right) \int_0^\varepsilon |g(u)|du,
\]
which tends to 0 as \( \varepsilon \downarrow 0 \). It follows from (5.39) that
\[
\int_1^{1-(1/x)} g(u)c(ux)du = ci_{1,\alpha}^{\nu} + o(1),
\]
yielding part (i).

Now suppose that \( \alpha \geq 1 \) and \( \nu > 0 \). For any \( \nu > 0 \), there exists \( \delta \in (0,1) \) such that \((1-u)^\nu - 1 \leq -(\nu/2)u \) for all \( u \in [0, \delta] \). Moreover, \( c(y) \geq c/2 > 0 \) for all \( y \geq y_0 \) sufficiently large. Hence, since the integrand is non-positive, for \( x > y_0/\delta \),
\[
I_{1}^{\nu,\alpha}(x) = x^{\nu} \int_0^{x^{-1}} ((1 - y/x)^\nu - 1) c(y)y^{-\alpha}dy
\leq x^{\nu} \frac{c}{2} \int_{y_0}^{\delta x} ((1 - y/x)^\nu - 1) c(y)y^{-\alpha}dy
\leq -\frac{c\nu}{4} x^{\nu-1} \int_{y_0}^{\delta x} y^{-\alpha}dy,
\]
which yields parts (ii) and (iii) of the lemma.

**Lemma 5.3.11.** Suppose that \( v_\alpha \in \mathcal{D}_\alpha \) with \( \alpha > 0 \). Then for any \( \nu \in (-1, \alpha) \) and any \( \lambda \in \mathbb{R}_+ \), as \( x \to \infty \),
\[
I_2^{\nu,\alpha}(x; \lambda) = cx^{\nu-\alpha} \left( \lambda x_2^{\nu,\alpha} - \frac{1}{\alpha} \right) + o(x^{\nu-\alpha}).
\]

**Proof.** The substitution \( u = y/x \) shows that
\[
I_2^{\nu,\alpha}(x; \lambda) = x^{\nu-\alpha} \int_{1+(1/x)}^{\infty} g(u)c(ux)du,
\]
where \( g(u) = (\lambda(u-1)^\nu - 1)u^{-1-\alpha} \). Here
\[
\left| \int_{1+(1/x)}^{\infty} g(u)c(ux)du - c \int_{1+(1/x)}^{\infty} g(u)du \right| \leq \left( \sup_{y \geq x} |c(y) - c| \right) \int_1^\infty |g(u)|du,
\]
which tends to 0 as \( x \to \infty \) since the last integral is finite provided \( \nu \in (-1, \alpha) \). The statement in the lemma follows from (5.40). \( \square \)
Lemma 5.3.12. Suppose that \( \alpha > 0, \nu \in \mathbb{R}, \) and \( \lambda \in \mathbb{R}. \) Then \( I_3^{\nu,\alpha}(x; \lambda) = o(x^{\nu-\alpha}) \) as \( x \to \infty. \)

Proof. Note that

\[
|I_3^{\nu,\alpha}(x; \lambda)| \leq \left( \sup_{y \geq 0} c(y) \right) \int_{x-1}^{x+1} (|\lambda| + |1 - \lambda| + x^\nu) (x - 1)^{-1-\alpha} dy,
\]

which gives the result. \( \square \)

Now we can complete the proofs of Lemmas 5.3.8 and 5.3.9.

Proof of Lemma 5.3.8. From (5.57) with the expressions for the integrals given in Lemmas 5.3.10(i), 5.3.11 and 5.3.12, we obtain (5.53).

From (5.53) with the estimates for \( i_1^{\nu,\alpha} \) and \( i_2^{\nu,\alpha} \) given at (5.51) and (5.52) respectively, we obtain

\[
R^{\nu,\alpha}(\lambda) = \frac{1}{\alpha} \left( \lambda - 1 - \nu (\lambda \psi(\alpha) - \psi(1 - \alpha) + (\lambda - 1) \psi(1)) + O(\nu^2) \right),
\]

as \( \nu \to 0, \) uniformly for \( |\lambda| \) bounded away from infinity. In particular, if \( \lambda = \lambda(\theta, \nu) = 1 - \theta \nu + o(\nu) \) we obtain

\[
R^{\nu,\alpha}(\lambda(\theta, \nu)) = -\frac{\nu}{\alpha} (\theta + \psi(\alpha) - \psi(1 - \alpha)) + o(\nu),
\]

and then (5.56) follows from the digamma reflection formula (5.44). \( \square \)

Proof of Lemma 5.3.9. Recall the decomposition (5.57). Suppose \( \nu \in (0, \alpha). \)

If \( \alpha = 1, \) then Lemmas 5.3.10(ii), 5.3.11 and 5.3.12 show that

\[
F^{\text{one}}_{\nu,\alpha}(x) \leq -\varepsilon x^{\nu-1} \log x + O(x^{\nu-1}) \leq -\varepsilon x^{\nu-1},
\]

for all \( x \) sufficiently large. If \( \alpha > 1, \) then Lemmas 5.3.10(iii), 5.3.11 and 5.3.12 show that

\[
F^{\text{one}}_{\nu,\alpha}(x) \leq -\varepsilon x^{\nu-1} + O(x^{\nu-\alpha}),
\]

as required. \( \square \)

5.3.7 Proof of recurrence classification

In this section we complete the proof of Theorem 5.3.1. First we show the following non-confinement result.

Lemma 5.3.13. Suppose that (Os1) holds. Then \( \limsup_{n \to \infty} |X_n| = \infty, \) a.s.
Proof. We claim that for each \( x \in \mathbb{R}_+ \), there exists \( \varepsilon_x > 0 \) such that

\[
P[|X_{n+1} - X_n| \geq 1 \mid X_n = y] \geq \varepsilon_x, \quad \text{for all } y \in [-x, x].
\]  

(5.58)

Indeed, given \( x \in \mathbb{R}_+ \), we may choose \( z_0 \geq x \) sufficiently large so that, for some \( \varepsilon > 0 \), \( v_\alpha(z) \geq \varepsilon z^{1-\alpha} \) and \( v_\alpha(z) \geq \varepsilon z^{1-\beta} \) for all \( z \geq z_0 \). Then if \( y \in [0,x] \),

\[
P[X_{n+1} < -y - 1 \mid X_n = y] \geq \int_{2z_0+1}^{\infty} \varepsilon z^{1-\alpha} \, dz,
\]

which implies (5.58); a similar argument applies when \( y \in [-x,0) \). The property (5.58) implies the result by an application of Proposition 3.3.4. \qed

Lemma 5.3.14. Suppose that (Os1) holds.

(i) If \( \alpha + \beta > 1 \), then there exist \( \nu > 0 \), \( \lambda \in \mathbb{R} \), and a bounded set \( A \subseteq \mathbb{R} \), such that

\[
F_\nu(x; \lambda) < 0, \quad \text{for all } x \notin A.
\]

(5.59)

(ii) If \( \alpha + \beta < 1 \), then there exist \( \nu < 0 \), \( \lambda \in \mathbb{R} \), and a bounded set \( A \subseteq \mathbb{R} \), such that

\[
F_\nu(x; \lambda) < 0, \quad \text{for all } x \notin A.
\]

(5.60)

Proof. First we prove part (i). First suppose that \( \beta \geq 1 \). If \( \alpha \geq 1 \) as well, then Lemma 5.3.9 applied twice shows that, for all \( x \) sufficiently large,

\[
F^{\text{one}}_{\nu,\alpha}(x; \lambda) < 0, \quad \text{and } \lambda F^{\text{one}}_{\nu,\beta}(x; 1/\lambda) < 0,
\]

regardless of choice of \( \nu > 0 \) and \( \lambda \), giving (5.59). If \( \beta \geq 1 \) but \( \alpha \in (0,1) \), we choose \( \lambda = \lambda(\theta, \nu) = 1 - \theta \nu \) satisfying (5.55); since \( \alpha \in (0,1) \) we may choose \( \theta \in \mathbb{R} \) so that \( r(\theta, \alpha) > 0 \). Thus we may choose \( \nu > 0 \) sufficiently small and \( \lambda = \lambda(\theta, \nu) \) such that \( R_{\nu,\alpha}(\lambda) < 0 \), giving the first inequality in (5.61). The second inequality in (5.61) follows from another application of Lemma 5.3.9. A similar argument applies if \( \alpha \geq 1 \).

It remains to suppose that \( \alpha + \beta > 1 \) with \( \alpha, \beta \in (0,1) \). We choose \( \lambda = \lambda(\theta, \nu) = 1 - \theta \nu \) satisfying (5.55), so that \( 1/\lambda = 1 + \theta \nu + o(\nu) \) as \( \nu \to 0 \). We aim to choose \( \theta \) so that, for \( \nu > 0 \) sufficiently small, both \( R_{\nu,\alpha}(\lambda) < 0 \) and \( R_{\nu,\beta}(1/\lambda) < 0 \). Thus we require to choose \( \theta \in \mathbb{R} \) so that \( r(\theta, \alpha) > 0 \) and \( r(-\theta, \beta) > 0 \), i.e.,

\[
\pi \cot (\pi \alpha) - \theta < 0, \quad \text{and } \pi \cot (\pi \beta) + \theta < 0.
\]
5.3. Oscillating random walk

The desired \( \theta \) exists provided \( \cot \pi \alpha + \cot \pi \beta < 0 \), which using the identity
\[
\cot \pi \alpha + \cot \pi \beta = \frac{\sin \pi (\alpha + \beta)}{\sin \pi \alpha \cdot \sin \pi \beta},
\]
is equivalent to \( \alpha + \beta > 1 \).

Next we prove part (ii). Suppose \( \alpha, \beta \in (0, 1) \). We again choose \( \lambda = \lambda(\theta, \nu) = 1 - \theta \nu \) satisfying (5.55), so that \( 1/\lambda = 1 + \theta \nu + o(\nu) \) as \( \nu \to 0 \). We aim to choose \( \theta \) so that, for \( \nu < 0 \) sufficiently close to 0, both \( R_{\nu, \alpha} (1/\lambda) < 0 \). Thus we require to choose \( \theta \in \mathbb{R} \) so that \( r(\theta, \alpha) < 0 \) and \( r(-\theta, \beta) < 0 \), i.e.,
\[
\pi \cot (\pi \alpha) - \theta > 0, \quad \text{and} \quad \pi \cot (\pi \beta) + \theta > 0.
\]
The desired \( \theta \) exists provided \( \alpha + \beta < 1 \).

To establish recurrence, we use the following analogue of Theorem 3.5.8.

**Lemma 5.3.15.** Let \( X_n \) be an \( \mathcal{F}_n \)-adapted process taking values in \( \mathbb{R} \). Let \( f : \mathbb{R} \to \mathbb{R}_+ \) be such that \( \lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} = \infty \), and \( \mathbb{E} f(X_0) < \infty \). Suppose that there exist \( x_0 \in \mathbb{R}_+ \) and \( C < \infty \) for which, for all \( n \geq 0 \),
\[
\mathbb{E} [f(X_{n+1}) - f(X_n) | \mathcal{F}_n] \leq 0, \quad \text{on} \quad \{|X_n| > x_0\}, \quad \text{a.s.} \\
\mathbb{E} [f(X_{n+1}) - f(X_n) | \mathcal{F}_n] \leq C, \quad \text{on} \quad \{|X_n| \leq x_0\}, \quad \text{a.s.}
\]
Then
\[
\mathbb{P} \left[ \{ \limsup_{n \to \infty} |X_n| < \infty \} \cup \{ \liminf_{n \to \infty} |X_n| \leq x_0 \} \right] = 1.
\]

**Proof.** The proof is analogous to the proof of Theorem 3.5.8, using the non-negative supermartingale \( Y_m = f(X_{m \wedge \lambda}) \) where \( \lambda = \min\{m \geq n : |X_m| \leq x_0\} \). Writing \( Y_\infty = \lim_{m \to \infty} Y_m \), this means that
\[
\lim_{m \to \infty} \sup f(X_m) \leq Y_\infty, \quad \text{on} \quad \{ \lambda = \infty \}.
\]
Setting \( \zeta_+ = \sup\{x \geq 0 : f(x) \leq Y_\infty\} \) and \( \zeta_- = \sup\{x \geq 0 : f(-x) \leq Y_\infty\} \), it follows that \( \limsup_{m \to \infty} |X_m| \leq \zeta_+ \vee \zeta_- < \infty \) on \( \{ \lambda = \infty \} \). Hence
\[
\mathbb{P} \left[ \{ \limsup_{n \to \infty} |X_n| < \infty \} \cup \{ \inf_{m \geq n} |X_m| \leq x_0 \} \right] = 1.
\]
Since \( n \in \mathbb{Z}_+ \) was arbitrary, it follows that
\[
\mathbb{P} \left[ \{ \limsup_{n \to \infty} |X_n| < \infty \} \cup \bigcap_{n \geq 0} \{ \inf_{m \geq n} |X_m| \leq x_0 \} \right] = 1,
\]
which gives the result. \( \square \)
To establish transience, we use the following analogue of Lemma 3.5.7.

**Lemma 5.3.16.** Let $X_n$ be an $\mathcal{F}_n$-adapted process taking values in $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}_+$ be such that $\sup_x f(x) < \infty$ and $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0$. Suppose that there exists $x_1 \in \mathbb{R}_+$ for which $\inf_{|y| \leq x_1} f(y) > 0$ and, for all $n \geq 0$,

$$E[f(X_{n+1}) - f(X_n) | \mathcal{F}_n] \leq 0, \text{ on } \{|X_n| > x_1\}.$$  

Then for any $\varepsilon > 0$ there exists $x \in (x_1, \infty)$ for which, for all $n \geq 0$,

$$P\left[ \inf_{m \geq n} |X_m| \geq x_1 \left| \mathcal{F}_n \right. \right] \geq 1 - \varepsilon, \text{ on } \{|X_n| > x\}.$$  

**Proof.** The proof is analogous to the proof of Lemma 3.5.7, using the non-negative supermartingale $Y_m = f(X_{m\wedge \lambda})$ where $\lambda = \min\{m \geq n : |X_m| \leq x_1\}$. 

Now we can complete the proof of Theorem 5.3.1.

**Proof of Theorem 5.3.1.** First suppose that $\alpha + \beta > 1$. Then Lemma 5.3.14 shows that there exist $\nu > 0$, $\lambda \in \mathbb{R}$, and $x_0 \in \mathbb{R}_+$, such that

$$E[f_\nu(X_{n+1}; \lambda) - f_\nu(X_n; \lambda) \mid X_n = x] \leq 0, \text{ for } |x| > x_0.$$  

Thus we may apply Lemma 5.3.15, which together with Lemma 5.3.13 shows that $\lim \inf_{n \to \infty} |X_n| \leq x_0$, a.s. Thus there exists an interval $I \subseteq [0, x_0 + 1]$ such that $|X_n| \in I$ i.o.; let $\tau_0 := 0$ and for $k \in \mathbb{N}$ define $\tau_k = \min\{n > \tau_{k-1} + 1 : |X_n| \in I\}$. Fix $\varepsilon \in (0, 1)$ and $z_0 \in \mathbb{R}_+$ such that $v_{\alpha}(z) > \varepsilon z^{-1-\alpha}$ and $v_{\beta}(z) > \varepsilon z^{-1-\beta}$ for all $z \geq z_0$. Then

$$P[|X_{\tau_k+2}| < \varepsilon \mid X_{\tau_k}] \geq P[|X_{\tau_k+2}| < \varepsilon \mid X_{\tau_k+1} = -\text{sgn}(X_{\tau_k})(z_0 + 1, z_0 + 2)] \times P[X_{\tau_k+1} \not\in -\text{sgn}(X_{\tau_k})(z_0 + 1, z_0 + 2) \mid X_{\tau_k}] \geq 2\varepsilon(z_0 + 3)^{-1-\alpha\vee\beta} (x_0 + z_0 + 3)^{-1-\alpha\vee\beta},$$

uniformly in $k$. Thus an application of Lévy’s extension of the Borel–Cantelli lemma (Theorem 2.3.19) shows that $|X_n| < \varepsilon$, i.o. Since $\varepsilon \in (0, 1)$ was arbitrary, we obtain $\lim \inf_{n \to \infty} |X_n| = 0$, a.s.

Next suppose that $\alpha + \beta < 1$. Then Lemma 5.3.14 shows that there exist $\nu < 0$, $\lambda \in \mathbb{R}$, and $x_1 \in \mathbb{R}_+$, such that

$$E[f_\nu(X_{n+1}; \lambda) - f_\nu(X_n; \lambda) \mid X_n = x] \leq 0, \text{ for } |x| > x_1.$$  


Thus we may apply Lemma 5.3.16, which shows that for any \( \varepsilon > 0 \) there exists \( x \in (x_1, \infty) \) for which, for all \( n \geq 0 \),

\[
P\left[ \inf_{m \geq n} |X_m| \geq x_1 \, \middle| \, F_n \right] \geq 1 - \varepsilon, \text{ on } \{|X_n| > x\}.
\]

As usual, let \( \sigma_x = \min\{n \geq 0 : |X_n| \geq x\} \). Then, on \( \{\sigma_x < \infty\} \),

\[
P\left[ \inf_{m \geq \sigma_x} |X_m| > x_1 \, \middle| \, F_{\sigma_x} \right] \geq 1 - \varepsilon, \text{ a.s.}
\]

But on \( \{\sigma_x < \infty\} \cap \{\inf_{m \geq \sigma_x} |X_m| > x_1\} \) we have \( \liminf_{m \to \infty} |X_m| \geq x_1 \), so

\[
P\left[ \liminf_{m \to \infty} |X_m| \geq x_1 \right] \geq \mathbb{E}\left[ P\left[ \inf_{m \geq \sigma_x} |X_m| > x_1 \, \middle| \, F_{\sigma_x} \right] 1\{\sigma_x < \infty\} \right]
\]

\[
\geq (1 - \varepsilon) P[\sigma_x < \infty] = (1 - \varepsilon),
\]

by Lemma 5.3.13. Since \( \varepsilon > 0 \) was arbitrary, we get \( \liminf_{m \to \infty} |X_m| \geq x_1 \), a.s., and since \( x_1 \) was arbitrary we get \( \lim_{m \to \infty} |X_m| = \infty \), a.s. \( \square \)

Bibliographical notes

Section 5.2

There is an extensive and rich theory of sums of independent, identically distributed (i.i.d.) random variables (classical ‘random walks’): see for instance the books of Kallenberg [150, Chapter 9], Loève [201, §26.2], or Stout [295, §3.2]. When the summands are integrable, the (first-order) asymptotic behaviour is governed by the mean. Completely different phenomena occur when the mean does not exist: see for example the book of Foss et al. [105].

To compare with the Markov chain setting of Section 5.2, it is helpful to keep in mind the classical independent-increments case, where \( S_n := \sum_{m=1}^{n} \zeta_m \) for a sequence of independent (often, i.i.d.) \( \mathbb{R} \)-valued random variables \( \zeta_1, \zeta_2, \ldots \). Thus we give a brief summary of some known results in that setting. Many of the results that we discuss for random walks have analogues for suitable Lévy processes: see e.g. the book of Sato [278], particularly Sections 37 and 48.

A classical result of Kesten [163, Corollary 3] states that if \( \zeta_1, \zeta_2, \ldots \) are i.i.d. random variables with \( \mathbb{E}[\zeta_1] = \infty \), then as \( n \to \infty \), \( n^{-1} S_n \) either: (i) tends to \( +\infty \) a.s.; (ii) tends to \( -\infty \) a.s.; or (iii) satisfies

\[
-\infty = \liminf_{n \to \infty} n^{-1} S_n < \limsup_{n \to \infty} n^{-1} S_n = +\infty, \text{ a.s.} \quad (5.62)
\]
Erickson [89] gives criteria for classifying such behaviour. Note that Kesten’s result implies the non-confinement result (5.1) in the i.i.d. case. Other classical results deal with the growth rate of the upper envelope of $S_n$, i.e., determining sequences $a_n$ for which $|S_n| \geq a_n$ infinitely often (or not), or $S_n \geq a_n$ infinitely often; here we mention the work of Feller [97], as well as results related to the Marcinkiewicz–Zygmund strong law of large numbers (see e.g. [166, Theorem 1]). The lower envelope behaviour, i.e., when $|S_n| \geq a_n$ all but finitely often, is considered by Griffin [121] (particularly Theorem 3.5); see also Pru90 [261].

Note that (5.62) can hold and $S_n$ be transient (with respect to bounded sets); Lo`eve [201, §26.2] gives the example of a symmetric stable random walk without a mean. The general criterion for deciding between transience and recurrence is due to Chung and Fuchs (see e.g. [150, Theorem 9.4] or [201, §26.2]), and is rather subtle: Shepp showed [284] that there exist distributions for $\zeta_1$ with arbitrarily heavy tails but for which $S_n$ is still recurrent. By assuming additional regularity for the distribution of $\zeta_1$, one can obtain more tractable criteria for recurrence; Shepp gives a criterion when the distribution of $\zeta_1$ is symmetric [283, Theorem 5].

The results of Section 5.2 are based on [133], which considered the more general setting where $X_n$ is adapted to a filtration $F_n$, and not necessarily Markov (i.e., the analogue of the general Lamperti setting of Chapter 3).

Theorem 5.2.2 is contained in Theorem 2.2 of [133]. In Theorem 5.2.2, condition (H2) is natural. For $\gamma \leq 1$, $(\Delta_n^{+})_n^{\gamma} \geq x^{\gamma-1} \Delta_n^{+} \{\Delta_n^{+} \leq x\}$ for any $x > 0$, so (H2) implies that $\mathbb{E}[(\Delta_n^{+})_n^{\gamma} \mid X_n = x] = \infty$ for any $\gamma > \alpha$. A counterexample due to K.L. Chung (see the Mathematical Reviews entry for [74]; also Baum [16]) shows that (H2) cannot be replaced by a condition on the moments of the increments, even in the case of a sum of i.i.d. random variables. Chung’s example has, for $\alpha \in (0, 1)$ and $\beta > \alpha$, $\mathbb{E}[(\zeta_1^{-})^{\beta}] < \infty$ and $\mathbb{E}[(\zeta_1^{+})^{\alpha}] = \infty$, but $\mathbb{E}[\zeta_1^{+} \{\zeta_1^{+} \leq x\}] = o(x^{1-\alpha})$ along a subsequence, so (H2) does not hold. For $X_n = S_n$ as in Chung’s example, $\liminf_{n \to \infty} X_n = -\infty$, a.s.

Theorems 5.2.4 and 5.2.5 are contained in Theorems 2.4 and 2.6 of [133], respectively. In the case of a sum of independent random variables, Theorem 5.2.4 is slightly weaker than optimal. Suppose that $\zeta_1, \zeta_2, \ldots$ are independent, and that for some $\theta \in (0, 1)$ and $\varphi \in \mathbb{R}$,

$$\sup_{k \in \mathbb{N}} \limsup_{x \to \infty} (x^\theta \log x)^{-\varphi} \mathbb{P}[|\zeta_k| \geq x] < \infty.$$ 

Then, with $S_n = \sum_{k=1}^{n} \zeta_k$, for any $\varepsilon > 0$, a.s., for all but finitely many
\( n \geq 0, \)
\[ |S_n| \leq n^{1/\theta} (\log n)^{\varphi + 1 + \varepsilon}. \]  
(5.63)

The bound (5.63) belongs to a family of classical results with a long history; the case \( \varphi = 0 \) is due to Lévy and Marcinkiewicz (quoted by Feller [97, p. 257]), and the general case of (5.63) follows for example from a result of Loève [201, p. 253]. Under the additional condition that the summands are identically distributed, sharp results are given by Feller [97, Theorem 2]; for a recent reference, see [185]. Related results in the i.i.d. case are also given by Chow and Zhang [42] (see also [166, Theorem 2]).

Theorems 5.2.2 and 5.2.5 together can be viewed as an analogue of Erickson’s [89] result in the case of a sum of i.i.d. random variables; in the i.i.d. case the conclusion of Theorem 5.2.2 follows from [89, Corollary 1]. The results of [89] show that the conditions in Theorem 5.2.2 are close to optimal.

Note that (H1) implies that, a.s.,
\[ \mathbb{E}[(\Delta_n^+)^\alpha \mid X_n = x] = \int_0^\infty \mathbb{P}[\Delta_n^+ > y^{1/\alpha} \mid X_n = x]dy \geq c \int_{x_0}^{\infty} y^{-1}dy = \infty. \]

Conditions (H2) and (H1) are closely related, but neither implies the other. However, if one replaces the inequalities by equalities, the former implies the latter. In the case where \( X_n = S_n \) is a sum of i.i.d. random variables, a weaker version of Theorem 5.2.5 was obtained by Derman and Robbins [74] and stated in a stronger form by Stout [295, Theorem 3.2.6]; although Stout’s statement is still weaker than our Theorem 5.2.5, his proof gives essentially the same result (in the i.i.d. case). Also relevant in the i.i.d. case is a result of Chow and Zhang [42, Theorem 1]. Chung’s counterexample (see above) shows that the condition (H1) cannot be replaced by a moments condition, for instance.

The results on first passage times, Theorems 5.2.9 and 5.2.10 are contained in Theorems 2.9 and 2.10 of [133], respectively. The conditions in Theorems 5.2.9 and 5.2.10 are not far from optimal; in the i.i.d. case, sharp results on the existence or non-existence of moments for \( \rho_x \) are given by Kesten and Maller [165, Theorem 2.1]; see [165] for references to earlier work. Lemma 5.2.12 is essentially a large deviations result of the same kind as (but more general than) those obtained in [137] for the case \( X_n = S_n \), a sum of i.i.d. nonnegative random variables; indeed, the results in [137] show that Lemma 5.2.12 is close to best possible.

The results on last exit times, Theorems 5.2.13 and 5.2.14, are contained in Theorems 2.11 and 2.12 of [133], respectively. Again, in the i.i.d. case sharp results are given by Kesten and Maller [165, Theorem 2.1].
Section 5.3

The material in this section is based on [229]. The oscillating random walk was studied in the case of a discrete state-space by Kemperman [157], to whom the model was suggested in 1960 by Anatole Joffe and Peter Ney (see [157, p. 29]). The model was studied subsequently by Rogozin and Foss [273] and in a series of papers by Sandrić [276, 277]. Kemperman [157] and Rogozin and Foss [273] obtain their results using some sophisticated classical methods for the analysis of random walks, primarily Wiener–Hopf theory together with an appeal to various deep results from renewal theory. The Lyapunov function approach presented here follows [229], and has similarities to the approach of Sandrić [].

Theorem 5.3.1 on the one-sided oscillating random walk was obtained in the case of a discrete state-space by Kemperman [157, p. 21].

The first two-dimensional random walk of Example 5.3.4 walk was studied by Campanino and Petritis [34], who showed that it is transient using completely different methods to those presented here.
Chapter 6

Further applications

6.1 Random walk in random environment

6.1.1 Introduction

The first two sections of this chapter give some applications to processes in random environments. The law of a time-homogeneous Markov chain is determined by its collection of one-step transition probabilities, which describe how the process moves around the state space. Viewing the process as describing the dynamics of a particle, the collection of transition probabilities can be seen as describing the environment in which the particle moves. Motivated by dynamics in disordered systems, the environment can itself be random, i.e., the transition probabilities are chosen according to some probabilistic rule, before the process starts. The prototypical model of this type is the random walk in random environment, which we discuss in this section.

Given an infinite sequence \( \omega = (p_0, p_1, p_2, \ldots) \) such that \( p_x \in (0, 1) \) for all \( x \in \mathbb{Z}_+ \), we consider \((X_n, n \geq 0)\) a nearest-neighbour random walk on \( \mathbb{Z}_+ \) defined as follows. Set \( X_0 = 0 \), and for \( x \geq 1 \),

\[
\begin{align*}
P_\omega[X_{n+1} = x - 1 \mid X_n = x] &= p_x, \\
P_\omega[X_{n+1} = x + 1 \mid X_n = x] &= 1 - p_x =: q_x, \\
\end{align*}
\]

and \( P_\omega[X_{n+1} = 0 \mid X_n = 0] = p_0, \ P_\omega[X_{n+1} = 1 \mid X_n = 0] = 1 - p_0 =: q_0 \). The assumption that each \( p_x \) be in \((0, 1)\), with the given form for the reflection at the origin, ensures that \( X_n \) is an irreducible, aperiodic Markov chain on \( \mathbb{Z}_+ \) under the quenched law \( P_\omega \).

The sequence of jump probabilities \( \omega \) specifies the environment for the random walk. We take \( \omega \) itself to be random. Specifically, \( p_0, p_1, \ldots \) will
be a sequence of i.i.d. \((0,1)\)-valued random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We use \(\mathbb{E}\) to denote expectation with respect to the (environment) probability measure \(\mathbb{P}\), and \(\mathbb{E}_\omega\) to denote expectation with respect to the (quenched) probability measure \(\mathbb{P}_\omega\).

In Section 6.2 we consider \textit{random strings in random environment}. The random strings model includes the random walk in random environment model given here as a special case, and the proofs in Sections 6.1 and 6.2 have several common strands. As we shall see, studying recurrence of the random walk requires analysis of products of independent random variables (compare Section 2.2), while in the case of strings one must study products of random matrices. Via a simple transformation, one can study products of random variables by instead studying \textit{sums}, and so a wealth of classical theory is available; for matrices, we must appeal to some more sophisticated theory. For these reasons, it is recommended to read Section 6.1 before Section 6.2.

### 6.1.2 Recurrence classification

For any realization of the environment \(\omega\), the irreducible Markov chain \(X_n\) is either recurrent or transient under \(\mathbb{P}_\omega\). The following recurrence classification, which is essentially due to Solomon [291], shows that the crucial quantity is \(\zeta_x := \log(p_x/q_x)\). In this and subsequent results, statements are made that hold true ‘for \(\mathbb{P}\)-a.e. \(\omega\)’, that is, for almost every environment.

\textbf{Theorem 6.1.1.} Suppose that \(\mathbb{E}|\zeta_0| < \infty\).

(i) If \(\mathbb{E}\zeta_0 < 0\), then \(X_n\) is transient for \(\mathbb{P}\)-a.e. \(\omega\).

(ii) If \(\mathbb{E}\zeta_0 = 0\), then \(X_n\) is null recurrent for \(\mathbb{P}\)-a.e. \(\omega\).

(iii) If \(\mathbb{E}\zeta_0 > 0\), then \(X_n\) is positive recurrent for \(\mathbb{P}\)-a.e. \(\omega\).

For fixed \(\omega\), \(X_n\) is the nearest-neighbour random walk of Section 2.2. Similarly to that section, we use the notation

\[
d_x := d_x(\omega) := \prod_{y=0}^{x-1} \frac{p_y}{q_y} = \exp \left\{ \sum_{y=0}^{x-1} \zeta_y \right\}.
\]  

(6.2)

\textbf{Proof of Theorem 6.1.1.} For fixed \(\omega\), Theorem 2.2.5 shows that \(X_n\) is recurrent if and only if \(\sum_{x=1}^{\infty} d_x = \infty\). If \(\mathbb{E}\zeta_0 < 0\), then the strong law of large numbers shows that \(\sum_{y=0}^{x-1} \zeta_y \to -\infty\) as \(x \to \infty\), and hence \(\sum_{x=1}^{\infty} d_x < \infty\) for \(\mathbb{P}\)-a.e. \(\omega\), proving part (i).
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For part (iii), Theorem 2.2.5 shows that $X_n$ is positive-recurrent if and only if $\sum_{x=1}^{\infty} d_x^{-1} < \infty$. If $E \zeta_0 > 0$, then the strong law of large numbers shows that $\sum_{y=0}^{x-1} \zeta_y \to +\infty$ as $x \to \infty$, and hence $\sum_{x=1}^{\infty} d_x^{-1} = \sum_{x=1}^{\infty} \exp\{- \sum_{y=0}^{x-1} \zeta_y\} < \infty$ for $P$-a.e. $\omega$, proving part (iii).

Finally, for part (ii), suppose that $E \zeta_0 = 0$. Then either $P[\zeta_0 = 0] = 1$ (the degenerate case of symmetric simple random walk with reflection at 0) and $d_x = d_x^{-1} = 1$ for all $x$, or else $P[\zeta_0 = 0] < 1$ and then

$$\limsup_{x \to \infty} \sum_{y=0}^{x-1} \zeta_y = \limsup_{x \to \infty} \sum_{y=0}^{x-1} (-\zeta_y) = +\infty,$$

for $P$-a.e. $\omega$ (see e.g. [150, p. 167]). In particular, in either case we have $d_x \geq 1$ for infinitely many $x$ for $P$-a.e. $\omega$, and we obtain recurrence, and $d_x^{-1} \geq 1$ for infinitely many $x$ for $P$-a.e. $\omega$, so the walk is not positive-recurrent. This proves part (ii). \hfill \Box

6.1.3 Almost-sure upper bounds

The null-recurrent case in which $E[\zeta_0^2] \in (0, \infty)$ and $E \zeta_0 = 0$ is known as Sinai’s regime; it is this case that we study in detail for the rest of this section. First we give an almost-sure upper bound which demonstrates the very slow movement of the random walk. See Figure 6.1 below for a simulation.

**Theorem 6.1.2.** Suppose that $E[\zeta_0^2] \in (0, \infty)$ and $E \zeta_0 = 0$. Then for any $\varepsilon > 0$, for $P$-a.e. $\omega$, $P_\omega$-a.s., for all but finitely many $n \geq 0$,

$$X_n \leq (\log n)^2 (\log \log n)^{2+\varepsilon}.$$

We prove this result using the Lyapunov function $t(x)$ as defined at (2.8), which for the present purposes is most usefully expressed as

$$t(x) := t(x, \omega) := \sum_{y=0}^{x-1} \sum_{z=0}^{y} q_{y-z}^{-1} \exp \left\{ \sum_{y=y-z+1}^{y} \zeta_w \right\}.$$

(6.3)

We will use the following lower bound on $t(x)$.

**Lemma 6.1.3.** Let $t$ be as defined at (6.3). For any $\omega$ and any $x \in \mathbb{Z}_+$,

$$t(x) \geq \exp \left\{ \max_{0 \leq y \leq x} \sum_{z=0}^{y-1} \zeta_z \right\}.$$

(6.4)
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Proof. Since \( q_{y-z}^{-1} = 1 + (p_{y-z}/q_{y-z}) \), we have from (6.3) that

\[
\begin{align*}
\sum_{y=0}^{x-1} \sum_{z=0}^y \exp \left\{ \sum_{w=y-z}^y \zeta_w \right\} + \sum_{y=0}^{x-1} \sum_{z=0}^y \exp \left\{ \sum_{w=y-z}^y \zeta_w \right\}.
\end{align*}
\] (6.5)

For the lower bound, (6.5) shows that

\[
\begin{align*}
t(x) &\geq \max_{0 \leq y \leq x-1} \max_{0 \leq z \leq y} \exp \left\{ \sum_{w=y-z}^y \zeta_w \right\} \\
&\geq \max_{0 \leq y \leq x-1} \exp \left\{ \sum_{w=0}^y \zeta_w \right\},
\end{align*}
\]

which yields the result in (6.4). \( \square \)

The next result, which is a consequence of Theorem 2.8.1, shows how to deduce from lower bounds for \( t(x) \) an upper bound for \( X_n \).

**Lemma 6.1.4.** Let \( w: \mathbb{Z}_+ \to \mathbb{R}_+ \) be increasing. Suppose that for \( \mathbb{P}\text{-a.e.} \omega \), \( t(x) \geq w(x) \) for all but finitely many \( x \in \mathbb{Z}_+ \). Then for any \( \varepsilon > 0 \), for \( \mathbb{P}\text{-a.e.} \omega, \mathbb{P}_\omega\text{-a.s.}, \) for all but finitely many \( n \geq 0 \),

\[
X_n \leq w^{-1}(n^{1+\varepsilon}).
\]

**Proof.** Let \( \varepsilon > 0 \). For a fixed \( \omega \), Lemma 2.2.4 shows that we may apply Theorem 2.8.1 with \( f = t \) (and \( B = 1 \)) to obtain, \( \mathbb{P}_\omega\text{-a.s.}, \) for all but finitely many \( n \geq 0 \), \( X_n \leq t^{-1}((2n)^{1+\varepsilon}) \).

Under the assumptions of the lemma, for \( \mathbb{P}\text{-a.e.} \omega \) there exists \( x_\omega \in \mathbb{Z}_+ \) such that \( t(x) \geq w(x) \) for all \( x \geq x_\omega \), so that \( t^{-1}(y) \leq w^{-1}(y) \) for all \( y \geq y_\omega \), where \( y_\omega = t(x_\omega) \). For such \( \omega, \mathbb{P}_\omega\text{-a.s.}, \)

\[
X_n \leq t^{-1}((2n)^{1+\varepsilon}) \leq w^{-1}((2n)^{1+\varepsilon}),
\]

for all \( n \) sufficiently large, and the result follows. \( \square \)

The final ingredient is to obtain lower bounds for the right-hand side of (6.4). We will use the following result, due to Csáki [61], which extended a result of Hirsch [128].

**Lemma 6.1.5.** Suppose that \( \zeta_0, \zeta_1, \ldots \) are i.i.d. with \( \mathbb{E}[\zeta_0^2] \in (0, \infty) \) and \( \mathbb{E}\zeta_0 = 0 \). Then, for any \( \varepsilon > 0, \) \( \mathbb{P}\text{-a.e.}, \) for all but finitely many \( n \geq 0 \),

\[
\max_{0 \leq i \leq n} \sum_{j=0}^i \zeta_j \geq n^{1/2}(\log n)^{-1-\varepsilon}.
\]
Now we can complete the proof of Theorem 6.1.2.

**Proof of Theorem 6.1.2.** Let $\varepsilon > 0$. We have from (6.4) and Lemma 6.1.5 that for $\mathbb{P}$-a.e. $\omega$, for all but finitely many $x \in \mathbb{Z}_+$,

$$t(x) \geq \exp \left\{ x^{1/2}(\log x)^{-1-\varepsilon} \right\}.$$ 

Thus we may apply Lemma 6.1.4 with

$$w(x) = \exp \left\{ x^{1/2}(\log x)^{-1-\varepsilon} \right\},$$

so that $w^{-1}(y) \leq (\log y)^2(\log \log y)^{2+3\varepsilon}$,

for $y$ sufficiently large, to obtain that for $\mathbb{P}$-a.e. $\omega$, $\mathbb{P}_\omega$-a.s.,

$$X_n \leq w^{-1}(n^{1+\varepsilon}) \leq (\log n)^2(\log \log n)^{2+4\varepsilon},$$

for all but finitely many $n \geq 0$, which yields the result. \qed

### 6.1.4 Almost-sure lower bounds

Next we give an almost-sure lower bound that is attained infinitely often.

**Theorem 6.1.6.** Suppose that $\mathbb{E}[\zeta_0^2] = \sigma^2 \in (0, \infty)$ and $\mathbb{E}\zeta_0 = 0$. Then for any $\varepsilon > 0$, for $\mathbb{P}$-a.e. $\omega$, $\mathbb{P}_\omega$-a.s., for infinitely many $n \geq 0$,

$$X_n \geq (1 - \varepsilon) \frac{2}{\pi^2 \sigma^2} (\log n)^2 \log \log n.$$ 

The following upper bound on $t(x)$ is a companion to Lemma 6.1.3

**Lemma 6.1.7.** Let $t$ be as defined at (6.3). For any $\omega$ and any $x \in \mathbb{Z}_+$,

$$t(x) \leq 2x^2 \exp \left\{ 2 \max_{0 \leq y \leq x-1} \left| \sum_{z=0}^{y} \zeta_w \right| \right\}.$$

**Proof.** For the first term on the right-hand side of (6.5) we have

$$\sum_{y=0}^{x-1} \sum_{z=0}^{y} \exp \left\{ \sum_{w=y-z+1}^{y} \zeta_w \right\} \leq \sum_{y=0}^{x-1} (y+1) \max_{0 \leq z \leq y} \left\{ \sum_{w=y-z+1}^{y} \zeta_w \right\} \leq x^2 \exp \left\{ \max_{0 \leq y \leq x-1} \max_{0 \leq z \leq y} \sum_{w=y-z+1}^{y} \zeta_w \right\}.$$
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Here we have that

\[
\max_{0 \leq y \leq x-1} \max_{0 \leq z \leq y} \sum_{w=y-z+1}^{y} \zeta_w = \max_{0 \leq y \leq x-1} \left( \sum_{w=0}^{y} \zeta_w + \max_{0 \leq z \leq y} \sum_{w=0}^{y-z} (-\zeta_w) \right)
\]

\[
\leq \max_{0 \leq y \leq x-1} \sum_{w=0}^{y} \zeta_w + \max_{0 \leq y \leq x-1} \sum_{w=0}^{y} (-\zeta_w).
\]

Hence

\[
\sum_{y=0}^{x-1} \sum_{z=0}^{y} \exp \left\{ \sum_{w=y-z+1}^{y} \zeta_w \right\} \leq x^2 \exp \left\{ \max_{0 \leq y \leq x-1} \sum_{z=0}^{y} \zeta_z + \max_{0 \leq y \leq x-1} \sum_{z=0}^{y} (-\zeta_z) \right\}.
\]

A similar calculation yields the same upper bound for the second term on the right-hand side of (6.5), and (6.6) follows.

The key to the proof of Theorem 6.1.6 is the following technical result.

**Lemma 6.1.8.** Suppose that there are non-negative, increasing functions \(u\) and \(v\) such that

1. for \(P\)-a.e. \(\omega\), \(t(x) \leq u(x)\) for all but finitely many \(x \in \mathbb{N}\);
2. for \(P\)-a.e. \(\omega\), \(t(x) \leq v(x)\) for infinitely many \(x \in \mathbb{N}\).

Suppose also that

\[
\sum_{x=1}^{\infty} \frac{u(x)}{v(x^{3/2})} < \infty, \quad \text{and} \quad \lim_{x \to \infty} \left( \frac{v(x^{3/4})}{v(x)} \right) = 0. \quad (6.7)
\]

Then for any \(\varepsilon > 0\), for \(P\)-a.e. \(\omega\), \(P_\omega\)-a.s., for infinitely many \(n \geq 0\),

\[X_n \geq v^{-1}((1-\varepsilon)n).\]

**Proof.** Recall that \(\tau_x = \min\{n \geq 0 : X_n = x\}\), and, from Lemma 2.2.3, that \(E_\omega \tau_x = t(x)\) given that \(X_0 = 0\). First we obtain a rough upper bound on \(\tau_x\) (equation (6.8) below). Condition (i) and Markov’s inequality yield, for \(P\)-a.e. \(\omega\),

\[
P_\omega[\tau_x > v(x^{3/2})] \leq \frac{t(x)}{v(x^{3/2})} \leq \frac{u(x)}{v(x^{3/2})},
\]

for all \(x \geq x_0\), where \(x_0 := x_0(\omega) < \infty\). Hence, for \(P\)-a.e. \(\omega\),

\[
\sum_{x \geq 1} P_\omega[\tau_x > v(x^{3/2})] \leq x_0 + \sum_{x \geq 1} \frac{u(x)}{v(x^{3/2})} < \infty,
\]
by the first condition in (6.7). By the Borel–Cantelli lemma, for $\mathbb{P}$-a.e. $\omega$,

$$\tau_x \leq v(x^{3/2}),$$  \hspace{1cm} (6.8)

for all $x \geq N$ where $N := N(\omega)$ (a random variable for each $\omega$) has $\mathbb{P}_\omega[N(\omega) < \infty] = 1$ for $\mathbb{P}$-a.e. $\omega$.

Now we use (6.8) and condition (ii) to show that $\tau_x$ is in fact much smaller than the bound in (6.8) for infinitely many $x$. Let $\varepsilon > 0$. Condition (ii) implies that for $\mathbb{P}$-a.e. $\omega$ there exist $x_i := x_i(\omega), i \in \mathbb{N},$ such that $x_{i+1} > x_i^2$ for all $i$ and $t(x_i) \leq v(x_i)$ for all $i$ (the $x_i$ are a subsequence of that specified by (ii) chosen so as to have very large spacings). By Markov’s inequality and the fact that $E_\omega \tau_{x_{i+1}} = t(x_{i+1}) \leq v(x_{i+1})$,

$$\mathbb{P}_\omega[\tau_{x_{i+1}} - \tau_{x_i} > (1 + \varepsilon)v(x_{i+1})] \leq \mathbb{P}_\omega[\tau_{x_{i+1}} > (1 + \varepsilon)v(x_{i+1})] \leq (1 + \varepsilon)^{-1}.$$

It follows that, for all $i \in \mathbb{N}$,

$$\mathbb{P}_\omega[\tau_{x_{i+1}} - \tau_{x_i} \leq (1 + \varepsilon)v(x_{i+1})] \geq \varepsilon',$$  \hspace{1cm} (6.9)

where $\varepsilon' > 0$ depends only on $\varepsilon$. So by (6.9), for $\mathbb{P}$-a.e. $\omega$, for any $\varepsilon > 0$,

$$\sum_{i \in \mathbb{N}} \mathbb{P}_\omega[\tau_{x_{i+1}} - \tau_{x_i} \leq (1 + \varepsilon)v(x_{i+1})] = \infty.$$  \hspace{1cm} (6.10)

Under $\mathbb{P}_\omega$, the random variables $\tau_{x_{i+1}} - \tau_{x_i}, i \in \mathbb{N},$ are independent, by the strong Markov property. Hence (6.10) and the Borel–Cantelli lemma imply that, for $\mathbb{P}$-a.e. $\omega$, $\mathbb{P}_\omega$-a.s., for infinitely many $i$,

$$\tau_{x_{i+1}} - \tau_{x_i} \leq (1 + \varepsilon)v(x_{i+1}).$$

With (6.8) this implies that, for $\mathbb{P}$-a.e. $\omega$, $\mathbb{P}_\omega$-a.s., for infinitely many $i$,

$$\tau_{x_{i+1}} \leq (1 + \varepsilon)v(x_{i+1}) + v(x_{i+1}^{3/2}) \leq (1 + \varepsilon)v(x_{i+1}) + v(x_{i+1}^{3/4}),$$

since $x_i < x_{i+1}^{1/2}$ and $v$ is increasing. Hence by the second condition in (6.7), we obtain that, for any $\varepsilon > 0$, for $\mathbb{P}$-a.e. $\omega$, $\mathbb{P}_\omega$-a.s., for infinitely many $x$,

$$\tau_x \leq (1 + 2\varepsilon)v(x) = (1 + 2\varepsilon)v(X_{\tau_x}).$$

Since $\tau_x \neq \tau_y$ for any $x \neq y$, it follows that, for any $\varepsilon > 0$, for $\mathbb{P}$-a.e. $\omega$, $\mathbb{P}_\omega$-a.s., $n \leq (1 + 2\varepsilon)v(X_n)$ for infinitely many $n$, which gives the result. \qed

We will use the so-called ‘other’ law of the iterated logarithm due to Chung [44] (under a 3rd moments condition) and Jain and Pruitt [141]:
Lemma 6.1.9. Suppose that $\zeta_0, \zeta_1, \ldots$ are i.i.d. with $\mathbb{E}[\zeta_0^2] = \sigma^2 \in (0, \infty)$ and $\mathbb{E}\zeta_0 = 0$. Then, $\mathbb{P}$-a.s.,

$$\liminf_{n \to \infty} \left( n^{-1/2} (\log \log n)^{1/2} \max_{0 \leq i \leq n} \left| \sum_{j=0}^{i} \zeta_j \right| \right) = \frac{\pi \sigma}{\sqrt{8}}.$$ 

Proof of Theorem 6.1.6. For $\varepsilon \in (0, 1/4)$ set

$$u(x) = 2x^2 \exp \left\{ 2x^{(1/2)} + \varepsilon \right\}, \quad \text{and} \quad v(x) = 2x^2 \exp \left\{ (1 + \varepsilon)2^{-1/2} \sigma x^{1/2} (\log \log x)^{-1/2} \right\}.$$ 

Consider the upper bound for $t(x)$ given in (6.6). Then for $\mathbb{P}$-a.e. $\omega$, $t(x) \leq u(x)$ for all but finitely many $x$, as follows easily from the law of the iterated logarithm, or from Theorem 2.8.1. Moreover, Lemma 6.1.9 shows that, for $\mathbb{P}$-a.e. $\omega$, $t(x) \leq v(x)$ for infinitely many $x$.

It is straightforward to verify that $u$ and $v$ also satisfy (6.7). Thus we may apply Lemma 6.1.8 with this choice of $u$ and $v$ to obtain the result. \qed

6.1.5 Perturbation of Sinai’s regime

We finish this section by studying a non-i.i.d. environment. Again suppose that $X_n$ has quenched transition probabilities as defined at (6.1), but now suppose that there exist constants $x_0 \in \mathbb{N}$, $a \in \mathbb{R}$, and $\beta > 0$ such that $p_x \in (0, 1)$ for all $x \geq 0$, and

$$p_x = \xi_x + ax^{-\beta}, \quad \text{and} \quad q_x = 1 - \xi_x - ax^{-\beta}, \quad \text{for all} \quad x \geq x_0, \quad (6.11)$$

where $\xi_0, \xi_1, \ldots$ are i.i.d. random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that there exists $\delta > 0$ for which

$$\mathbb{P}[\delta \leq \xi_0 \leq 1 - \delta] = 1; \quad (6.12)$$

this is a uniform ellipticity condition. In this section we define

$$\zeta_x := \log \left( \frac{\xi_x}{1 - \xi_x} \right).$$

We assume that $\mathbb{E}[\zeta_0^2] > 0$ and $\mathbb{E}\zeta_0 = 0$. In the case $a = 0$, this model reduces to the random walk in i.i.d. random environment in Sinai’s regime, as considered earlier in this section, and in that case the present definition of $\zeta_x$ reduces to the usage in Section 6.1.2. For $a \neq 0$, this model represents
a perturbation of Sinai’s regime, in which the \( p_x \) are independent but no longer i.i.d.

This perturbation of the homogeneous (i.i.d.) environment is motivated by analogy with the problems studied in Chapters 3 and 4 in which the homogeneous (zero-drift) regime is perturbed by an asymptotically zero drift. Our first result for this model is the following recurrence classification; whereas in the context of asymptotically zero-drift random walks, a perturbation of order \( x^{-1} \) is required to effect a phase transition, the random environment is seen to be more stable to perturbations, with \( x^{-1/2} \) being the critical order.

**Theorem 6.1.10.** Suppose that \( x_0 \in \mathbb{N}, a \in \mathbb{R}, \delta > 0 \) and \( \beta > 0 \) the conditions (6.11) and (6.12) hold, and that \( \mathbb{E}[\zeta_0^2] > 0 \) and \( \mathbb{E}\zeta_0 = 0 \).

(i) If \( a < 0 \) and \( \beta \in (0, 1/2) \), then for \( \mathbb{P}\)-a.e. \( \omega \), \( X_n \) is transient.

(ii) If \( a > 0 \) and \( \beta \in (0, 1/2) \), then for \( \mathbb{P}\)-a.e. \( \omega \), \( X_n \) is positive-recurrent.

(iii) If \( \beta \geq 1/2 \), then for \( \mathbb{P}\)-a.e. \( \omega \), \( X_n \) is null-recurrent.

We also present almost-sure bounds giving the rate of escape in the transient case.

**Theorem 6.1.11.** Suppose that \( x_0 \in \mathbb{N}, a < 0, \delta > 0 \) and \( \beta \in (0, 1/2) \) the conditions (6.11) and (6.12) hold, and that \( \mathbb{E}[\zeta_0^2] > 0 \) and \( \mathbb{E}\zeta_0 = 0 \). Then for any \( \varepsilon > 0 \), for \( \mathbb{P}\)-a.e. \( \omega \), \( P_\omega\)-a.s., for all but finitely many \( n \geq 0 \),

\[
(\log \log n)^{-(1/\beta) - \varepsilon} \leq \frac{X_n}{(\log n)^{1/\beta}} \leq (\log \log n)^{(2/\beta) + \varepsilon}.
\]

See Figure 6.1 for a simulation.

The rest of this section is devoted to the proofs of these two theorems. First we use a similar idea to the proof of Theorem 6.1.1 to give the proof of Theorem 6.1.10.

**Proof of Theorem 6.1.10.** From Taylor’s theorem for \( \log(1+x) \), we have that

\[
\log p_y = \log \xi_y + \log(1 + ay^{\beta}\xi_y^{-1})
= \log \xi_y + ay^{\beta}\xi_y^{-1} + O(y^{-2\beta}), \quad \text{as } y \to \infty,
\]

where we have used (6.12) to obtain the non-random error term. Similarly,

\[
\log q_y = \log(1 - \xi_y) - ay^{\beta}(1 - \xi_y)^{-1} + O(y^{-2\beta}).
\]
Figure 6.1: Simulations of $10^5$ steps of random walk in random environment in Sinai’s regime (left) and the perturbation with $a < 0$ and $\beta = 1/5$ (right). Also shown are curves proportional to $(\log n)^2$ (left) and $(\log n)^5$ (right).

Hence, as $y \to \infty$,

$$\log(p_y/q_y) = \zeta_y + ay^{-\beta} \xi_y^{-1}(1 - \xi_y)^{-1} + O(y^{-2\beta}). \tag{6.13}$$

It follows from (6.13) that, with the definition of $d_x = d_x(\omega)$ from (6.2),

$$d_x = \exp \left\{ \sum_{y=1}^{x-1} \left( \zeta_y + ay^{-\beta} \xi_y^{-1}(1 - \xi_y)^{-1} + O(y^{-2\beta}) \right) \right\}. \tag{6.14}$$

For fixed $\omega$, Theorem 2.2.5 shows that $X_n$ is recurrent if and only if $\sum_{x=1}^{\infty} d_x = \infty$. Suppose that $a < 0$ and $\beta \in (0, 1/2)$. Then, from (6.14),

$$\log d_x \leq \sum_{y=1}^{x-1} \zeta_y - Cx^{1-\beta} + O(x^{1-2\beta}),$$

where $C = C(\delta, \beta)$ is a positive constant that does not depend on $\omega$. Here, by the law of the iterated logarithm (or Theorem 2.8.1), $P$-a.s., $\sum_{y=1}^{x-1} \zeta_y \leq x^{1/2} \log x$ for all but finitely many $x$. Thus, since $1 - \beta > 1/2$, we have $\log d_x \to -\infty$, $P$-a.s., and hence $X_n$ is transient, proving part (i).

For part (ii), Theorem 2.2.5 shows that $X_n$ is positive recurrent if and only if $\sum_{x=1}^{\infty} d_x^{-1} < \infty$. Suppose that $a > 0$ and $\beta \in (0, 1/2)$. Then,
from (6.14),
\[ \log d_x^{-1} \leq \sum_{y=1}^{x-1} (-\zeta_y) - Cx^{1-\beta} + O(x^{1-2\beta}), \]
where \( C = C(\delta, \beta) \) is a positive constant that does not depend on \( \omega \). Hence, similarly to part (i), \( \log d_x^{-1} \rightarrow -\infty \), \( \mathbb{P} \)-a.s., and \( X_n \) is positive recurrent, proving part (ii).

Finally, for part (iii), suppose that \( \beta \geq 1/2 \). Then, from (6.14),
\[ \log d_x \geq x - 1 \sum_{y=1}^{x} \zeta_y - Cx^{1/2} + O(1), \]
and, by the law of the iterated logarithm, there is a constant \( c > 0 \) depending only on \( \mathbb{E}[\zeta_0^2] \) such that, \( \mathbb{P} \)-a.s.,
\[ \sum_{1 \leq j \leq b(i)} \left( \sum_{k=i-j+1}^{i} Y_k \right) \geq (b(n/2))^{1/2}(\log n)^{-1-\varepsilon}. \]
Proof. First we prove part (i). Set \( Z^i_j = \sum_{k=i-j+1}^{i} Y_k \). Then \( Z^i_{j+1} - Z^i_j = Y_{i-j} \); so for fixed \( i \), \( Z^i_j \) is a martingale over \( j \in \{1, 2, \ldots, i\} \) with uniformly bounded increments. Hence the Azuma–Hoeffding inequality (Theorem 2.4.14) implies that for some \( c \in \mathbb{R}_+ \), for all \( j \in \{1, \ldots, i\} \), for \( t > 0 \),

\[
P[|Z^i_j| \geq t] \leq 2 \exp(-ct^2).
\]

Thus for a suitable \( C \in \mathbb{R}_+ \), for \( j \leq i \), \( \mathbb{P}[|Z^i_j| \geq C(j^{1/2}(\log i)^{1/2})] \leq i^{-3} \). Then

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{i} \mathbb{P}[|Z^i_j| \geq C(j^{1/2}(\log i)^{1/2})] \leq \sum_{i=1}^{\infty} i^{-2} < \infty,
\]

and the Borel–Cantelli lemma implies that, a.s., there are only finitely many pairs \((i, j)\) (with \( j \leq i \)) for which \( |Z^i_j| \geq C(j^{1/2}(\log i)^{1/2}) \). This proves (i).

Next we prove part (ii). For fixed \( i \), note that

\[
\max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} Y_k \overset{d}{=} \max_{1 \leq j \leq b(i)} \sum_{k=1}^{j} W_k,
\]

where \( W_1, W_2, \ldots \) are independent random variables with \( W_k \overset{d}{=} Y_{i+1-k} \) for each \( k \). Fix \( \varepsilon > 0 \). Define the event

\[
E_i := \left\{ \max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} Y_k \leq (b(i))^{1/2}(\log b(i))^{-1-\varepsilon} \right\}.
\]

Then Corollary 2 of Hirsch [128, p. 118] implies that there are absolute constants \( C, C' \in (0, \infty) \) such that for all \( i \geq x_0 \),

\[
\mathbb{P}[E_i] \leq C(\log b(i))^{-1-\varepsilon} \leq C'(\log i)^{-1-\varepsilon},
\]

since \( b(i) \geq i^{\gamma} \). Consider the subsequence \( i = 2^m \) for \( m = 1, 2, \ldots \). Then

\[
\sum_{m=1}^{\infty} \mathbb{P}[E_{2^m}] \leq C \sum_{m=1}^{\infty} m^{-1-\varepsilon} < \infty.
\]

Hence by the Borel–Cantelli lemma, \( E_{2^m} \) occurs for only finitely many \( m \), a.s. Each \( n \geq 2 \) satisfies \( 2^m(n) \leq n < 2^m(n)+1 \) with \( m(n) \in \mathbb{N} \) such that \( m(n) \to \infty \) as \( n \to \infty \). Then,

\[
\max_{1 \leq i \leq n} \max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} Y_k \geq \max_{1 \leq i \leq 2^m(n)} \max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} Y_k
\]
6.1. Random walk in random environment

\[ \geq \max_{1 \leq j \leq \beta(2^{m(n)})} \sum_{k=2^{m(n)}-j+1}^{2^{m(n)}} Y_k \geq \left( \beta(2^{m(n)}) \right)^{1/2} (\log(2^{m(n)}))^{-1-\varepsilon}, \]

a.s., for all \( n \) large enough. Since \( n \geq 2^{m(n)} > n/2 \), part (ii) follows. \( \square \)

**Proof of Lemma 6.1.12.** First we prove the upper bound in (6.15). Throughout this proof, \( C \) and \( y_0 \) will denote finite positive constants that depend only on \( \delta \) and \( \beta \), and whose value may change from line to line. Since \( a < 0 \) and \( \mathbb{E} \zeta_0 = 0 \), we have from (6.13) with (6.12) that

\[ \mathbb{E} \log(p_y/q_y) \leq -C y^{-\beta}, \text{ for all } y \geq y_0. \]

Using the simple inequality

\[ \sum_{w=y-z+1}^{y} w^{-\beta} \geq z \min_{y-z+1 \leq w \leq y} w^{-\beta} = z y^{-\beta}, \] (6.16)

we obtain that for all \( y, z \) with \( y-z \geq y_0 \),

\[ \mathbb{E} \sum_{w=y-z+1}^{y} \log(p_w/q_w) \leq -C z y^{-\beta}. \] (6.17)

By Lemma 6.1.13(i) we have that, for some \( C \in \mathbb{R}_+ \), for \( \mathbb{P} \)-a.e. \( \omega \), all but finitely many \( y \), and all \( z \in \{1, 2, \ldots, y\} \),

\[ \sum_{w=y-z+1}^{y} (\log(p_w/q_w) - \mathbb{E} \log(p_w/q_w)) \leq C z^{1/2} (\log y)^{1/2}. \] (6.18)

For \( \varepsilon > 0 \) set \( r(y) := \lceil y^{2\beta} (\log y)^{1+\varepsilon} \rceil \). Then from (6.18) with (6.17), for \( \mathbb{P} \)-a.e. \( \omega \), for all but finitely many \( y \in \mathbb{N} \) and all \( z \) with \( r(y) \leq z \leq y \),

\[ \sum_{w=y-z+1}^{y} \log(p_w/q_w) \leq -C z y^{-\beta} + C z^{1/2} (\log y)^{1/2} \leq -(C/2) z y^{-\beta}. \] (6.19)

On the other hand, for all but finitely many \( y \in \mathbb{N} \) and all \( z \) with \( z \leq r(y) \),

\[ \sum_{w=y-z+1}^{y} \log(p_w/q_w) \leq C z^{1/2} (\log y)^{1/2}. \] (6.20)
So from (6.3) with (6.19) and (6.20) we obtain, for \( P \)-a.e. \( \omega \), for any \( \varepsilon > 0 \), for all but finitely many \( y \),

\[
t(x) \leq \sum_{y=0}^{x-1} \sum_{z=0}^{r(y)} \exp(Cz^{1/2}(\log y)^{1/2}) + \sum_{y=0}^{x-1} \sum_{z=r(y)}^{y} \exp(-Czy^{-\beta}) \\
\leq \sum_{y=0}^{x-1} \exp(Cy^{\beta}(\log y)^{1+\varepsilon}),
\]

which gives the upper bound for \( t(x) \) in (6.15).

We now prove the lower bound for \( t(x) \) in (6.15). For \( \varepsilon > 0 \) define \( s(y) := \lfloor y^{2\alpha}(\log y)^{-2-\varepsilon} \rfloor \); since \( \beta < 1/2 \) we have \( s(y) < y \). Then,

\[
t(x) \geq \max_{1 \leq y \leq x-1} \max_{1 \leq z \leq s(y)} \sum_{w=y-z+1}^{y} \log(p_w/q_w).
\]

(6.21)

Now we have from (6.13) with (6.12) that, for \( P \)-a.e. \( \omega \),

\[
\log(p_y/q_y) \geq \zeta_y - C y^{-\beta}, \text{ for all } y \geq y_0.
\]

Hence, for \( y - z \geq y_0 \),

\[
\sum_{w=y-z+1}^{y} \log(p_w/q_w) \geq -C(y^{1-\beta} - (y - z)^{1-\beta}).
\]

(6.22)

Taylor’s theorem implies that for \( \beta \in (0,1) \),

\[
y^{1-\beta} - (y - z)^{1-\beta} \leq Czy^{-\beta}, \text{ for all } y - z \geq y_0.
\]

(6.23)

It follows from (6.21) with (6.22) and (6.23) that for all \( x \geq 1 \),

\[
t(x) \geq \exp\left\{ \max_{1 \leq y \leq x-1} \max_{1 \leq z \leq s(y)} \sum_{w=y-z+1}^{y} \zeta_w - Cs(x) x^{-\beta} \right\},
\]

(6.24)

using the fact that \( y^{-\beta}s(y) \) is eventually increasing for \( \beta > 0 \). We have by Lemma 6.1.13(ii) that for any \( \varepsilon > 0 \), for \( P \)-a.e. \( \omega \),

\[
\max_{1 \leq y \leq x-1} \max_{1 \leq z \leq s(y)} \sum_{w=y-z+1}^{y} \zeta_w \geq (s(x/2))^{1/2}(\log x)^{-1-(\varepsilon/4)} \\
\geq C x^{\beta}(\log x)^{-2-(3\varepsilon/4)},
\]

for all but finitely many \( x \), while \( s(x)x^{-\beta} \leq x^\beta(\log x)^{-2-\varepsilon} \). Hence (6.24) implies the lower bound in (6.15). □
Now we can complete the proof of Theorem 6.1.11. For the upper bound in the theorem, we once more use Lemma 6.1.4. The lower bound needs an additional idea. The idea of Lemma 6.1.8 was to obtain an upper bound on hitting times, which could be translated to a lower bound on \( X_n \) that was valid infinitely often. In order to extend this technique to the transient case, and obtain a lower bound for \( X_n \) valid \textit{all but finitely often}, we show in addition that (roughly speaking), in the present case, the time of the last visit of the random walk to a site is not too much greater than the first hitting time; a similar approach was used in Theorem 3.10.1 for the transient Lamperti problem.

**Proof of Theorem 6.1.11.** The lower bound in (6.15) implies that for any \( \varepsilon > 0 \), for \( P \)-a.e. \( \omega \), for all \( x \) sufficiently large,

\[
t(x) \geq \exp(x^\beta (\log x)^{-2-\varepsilon}) =: w(x).
\]

Hence Lemma 6.1.4 implies that for any \( \varepsilon > 0 \), for \( P \)-a.e. \( \omega \), \( P_\omega \)-a.s.,

\[
X_n \leq w^{-1}(n^{1+\varepsilon}) \leq (\log n)^{1/\beta} (\log \log n)^{(2+2\varepsilon)/\beta},
\]

for all but finitely many \( n \). This proves the claimed upper bound.

It remains to prove the lower bound in the theorem. For fixed \( \omega \), let \( a_x \) denote the probability that the random walk \( X_n \) hits \( x \) in finite time, given that it starts at \( 2x \). For \( x \geq 1 \) define

\[
M_x := 1 + \sum_{y=1}^\infty \prod_{z=1}^y \frac{p_{x+z}}{q_{x+z}} = 1 + \sum_{y=1}^\infty \exp \sum_{z=1}^y \log \left( \frac{p_{x+z}}{q_{x+z}} \right).
\]  

(6.25)

Standard hitting probability arguments yield \( a_0 = 1 \), and, if \( M_x < \infty \),

\[
 a_x = M_x^{-1} \sum_{y=x}^\infty \prod_{z=1}^y \frac{p_{x+z}}{q_{x+z}} = M_x^{-1} \sum_{y=x}^\infty \exp \sum_{z=1}^y \log \left( \frac{p_{x+z}}{q_{x+z}} \right), \quad x \geq 1.
\]  

(6.26)

Since \( a < 0 \) and \( E \zeta_0 = 0 \), we have from (6.13) with (6.12) that

\[
E \sum_{z=1}^y \log(p_{x+z}/q_{x+z}) \leq -c \sum_{z=1}^y (x+z)^{-\beta} \leq -c' y^{1-\beta},
\]  

(6.27)

for all \( y \geq x \geq x_0 \), for positive constants \( c \), \( c' \), and \( x_0 \). Also, by the Azuma–Hoeffding inequality (Theorem 2.4.14) and an argument similar to Lemma 6.1.13, we have that, for \( P \)-a.e. \( \omega \),

\[
\sum_{z=1}^y (\log(p_{x+z}/q_{x+z}) - E[\log(p_{x+z}/q_{x+z})]) \leq Cy^{1/2}(\log y)^{1/2},
\]

for all \( y \geq x \geq x_0 \).
for all but finitely many \((x, y)\) with \(y \geq x\). Together with (6.27) this shows that for \(\mathbb{P}\)-a.e. \(\omega\), for all but finitely many \((x, y)\) with \(y \geq x\),

\[
\sum_{z=1}^{y} \log(p_{x+z}/q_{x+z}) \leq -Cy^{1-\beta} + Cy^{1/2}(\log y)^{1/2} \leq -(C/2)y^{1-\beta}, \quad (6.28)
\]

since \(\beta \in (0, 1/2)\). Hence, for \(\mathbb{P}\)-a.e. \(\omega\), from (6.25) with (6.12) and (6.28), for \(x \geq 1\) and a constant \(C_\delta\) depending only on \(\delta\),

\[
M_x \leq 1 + (C_\delta)^x + \sum_{y=x}^{\infty} \exp(-Cy^{1-\beta}) < \infty.
\]

Further, since \(M_x \geq 1\) for all \(x\), (6.26) and (6.28) imply, for \(\mathbb{P}\)-a.e. \(\omega\), for all but finitely many \(x\),

\[
a_x \leq \sum_{y=x}^{\infty} \exp(-Cy^{1-\beta}) \leq \exp(-C'x^{1-\beta}),
\]

for some \(C' \in (0, \infty)\). Thus, for \(\mathbb{P}\)-a.e. \(\omega\), \(\sum_x a_x < \infty\).

The Borel–Cantelli lemma then implies that, for \(\mathbb{P}\)-a.e. \(\omega\), \(\mathbb{P}_\omega\)-a.s., for only finitely many sites \(x\) does \(X_n\) return to \(x\) after visiting \(2x\). Denoting by \(\eta_x\) the time of the last visit of \(X_n\) to \(x\), we then have that \(\eta_x \leq \tau_{2x}\), \(\mathbb{P}_\omega\)-a.s. for all but finitely many \(x\). Suppose \(t(x) \leq u(x)\) for all but finitely many \(x\). Then since \(E_\omega \tau_x = t(x)\) we have \(\sum_{x \geq 1} \mathbb{P}_\omega[\tau_x \geq x^2t(x)] \leq \sum_{x \geq 1} x^{-2} < \infty\), so another application of the Borel–Cantelli lemma shows that for \(\mathbb{P}\)-a.e. \(\omega\), \(\mathbb{P}_\omega\)-a.s., for all \(x \geq x_0\) with a finite (random) \(x_0\),

\[
\eta_x \leq \tau_{2x} \leq 4x^2u(2x). \quad (6.29)
\]

Moreover, since, for \(\mathbb{P}\)-a.e. \(\omega\), \(X_n\) is transient, we have that \(X_n \geq x_0\) for all \(n\) sufficiently large. Hence from (6.29), using the fact that \(\eta_{X_n} \geq n\) for all \(n\), we have that for \(\mathbb{P}\)-a.e. \(\omega\), \(\mathbb{P}_\omega\)-a.s., for all but finitely many \(n\),

\[
n \leq \eta_{X_n} \leq 4X_n^2u(2X_n).
\]

Then, with the upper bound in (6.15), we obtain, for any \(\varepsilon > 0\), for \(\mathbb{P}\)-a.e. \(\omega\), \(\mathbb{P}_\omega\)-a.s., \(n < \exp(X_n^{\beta}(\log n)^{1+\varepsilon})\), for all but finitely many \(n\), using the fact that \(X_n \leq n\) to bound the logarithmic term. This gives the claimed lower bound. \(\square\)
6.2 Random strings in random environment

6.2.1 Introduction

Consider a finite alphabet $\mathcal{A} = \{1, \ldots, d\}$. A string is a finite sequence of symbols from $\mathcal{A}$. We write $|s|$ for the length of the string $s$, i.e., if $s = s_1 \ldots s_\ell$, with each $s_i \in \mathcal{A}$, then $|s| = \ell$.

In this section we define a time-homogeneous Markov chain $(X_n, n \geq 0)$ whose state space is the set of all finite strings over $\mathcal{A}$. The transitions of this Markov chain are as follows. If the current state is the string $X_n = s = s_1 \ldots s_\ell$, whose rightmost symbol is $s_\ell = i \in \mathcal{A}$, then a transition to $X_{n+1}$ is one of the following three kinds:

- erase the rightmost symbol of $s$ with probability $r_\ell^i$;
- substitute the rightmost symbol $i$ by $j$ with probability $q_\ell^{ij}$;
- append the symbol $j$ to the right end of the string with probability $p_\ell^{ij}$.

Of course we assume that $r_\ell^i \geq 0$, $q_\ell^{ij} \geq 0$, $p_\ell^{ij} \geq 0$, for all $i, j \in \mathcal{A}$ and all $\ell \in \mathbb{N}$; \hspace{1cm} (6.30)

$$r_\ell^i + \sum_{j \in \mathcal{A}} q_\ell^{ij} + \sum_{j \in \mathcal{A}} p_\ell^{ij} = 1,$$ \hspace{1cm} (6.31)

for all $i \in \mathcal{A}$ and all $\ell \in \mathbb{N}$.

The specification of the Markov chain is completed by describing the possible transitions from the empty string $\emptyset$ (the string with length zero): for definiteness, we suppose that an empty string can transition only to a string of length 1 with probabilities given by $p_0^j$, $j \in \mathcal{A}$.

The asymptotic behaviour of the string will then be determined by the collection of parameters $r_\ell^i$, $q_\ell^{ij}$, $p_\ell^{ij}$, over $\ell \in \mathbb{N}$, $i \in \mathcal{A}$, $j \in \mathcal{A}$. These parameters constitute the environment for the string dynamics.

The environment will itself be random. Let $\mathcal{P}_\mathcal{A}$ denote the set of all admissible $(r_\ell^i, q_\ell^{ij}, p_\ell^{ij})_{i, j}$, i.e., for fixed $\ell$ the set of all parameter choices satisfying (6.30) and (6.31). On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\xi_1, \xi_2, \ldots$ be i.i.d. random elements of $\mathcal{P}_\mathcal{A}$. Then for each $\ell$, $\xi_\ell$ specifies the parameters $r_\ell^i$, $q_\ell^{ij}$, $p_\ell^{ij}$ for all $i, j \in \mathcal{A}$; we will denote the realization of the environment by $\omega = (\xi_1, \xi_2, \ldots)$. This defines the Markov chain describing the evolution of the string, for each fixed environment $\omega$.

Similarly to Section 6.1, we will study this model by choosing a suitable Lyapunov function $f(s) = f(s, \omega)$ of the state $s$ and the environment $\omega$. 
Remark 6.2.1. The strings model includes the random walk in random environment of Section 6.1 as a special case. Indeed, if \( d = 1 \), then the process tracking the length of the string \( |X_n| \) becomes a nearest-neighbour random walk on \( \mathbb{Z}_+ \) with quenched transition probabilities given for \( \ell \in \mathbb{N} \) by

\[
\begin{align*}
P_\omega[|X_{n+1}| = \ell + 1 | |X_n| = \ell] &= p^\ell_{11}; \\
P_\omega[|X_{n+1}| = \ell | |X_n| = \ell] &= q^\ell_{11}; \\
P_\omega[|X_{n+1}| = \ell - 1 | |X_n| = \ell] &= r^\ell_1.
\end{align*}
\]

Note that for general \( d \geq 2 \), the process \( |X_n| \) is not Markov under \( P_\omega \).

6.2.2 Lyapunov exponents and products of random matrices

In this section we review some properties of products of random matrices that we need below. Recall that for a matrix \( A \) we denote its transpose by \( A^\top \) and its largest eigenvalue by \( \lambda_{\max}(A) \), and that for a \( d \times d \) real matrix \( A \), its operator norm is

\[
\|A\|_{\text{op}} = \sup_{u \in S^{d-1}} \|Au\| = (\lambda_{\max}(A^\top A))^{1/2}.
\]

On probability space \((\Omega, \mathcal{F}, \mathbb{P})\), we say that a \( d \times d \) random real matrix \( A \) satisfies condition (N) if:

\[(N) \quad \mathbb{E} \log^+ \|A\|_{\text{op}} < \infty, \text{ where } \log^+ x = \max\{\log x, 0\}.\]

Let \( A, A_1, A_2, \ldots \) be a sequence of i.i.d. matrices satisfying condition (N). Let \( a_1(n) \geq a_2(n) \geq \cdots \geq a_d(n) \geq 0 \) be the square roots of the (random) eigenvalues of \( (A_n \cdots A_1)^\top (A_n \cdots A_1) \). Then there exist constants \( \gamma_j(A) \), \( j \in \{1, \ldots, d\} \), with

\[-\infty \leq \gamma_d(A) \leq \cdots \leq \gamma_1(A) < \infty,
\]

depending on the distribution of \( A \), such that for each \( j \in \{1, \ldots, d\} \),

\[
P\left[ \lim_{n \to \infty} \frac{1}{n} \log a_j(n) = \gamma_j(A) \right] = 1; \quad (6.32)
\]

see Proposition 5.6 of [31] (note that \( A \) does not need to be invertible). The numbers \( \gamma_j(A) \) are called the Lyapunov exponents of the sequence of random matrices \( A_n \). In particular,

\[
\gamma_1(A) = \lim_{n \to \infty} \frac{1}{n} \log a_1(n) = \lim_{n \to \infty} \frac{1}{n} \log \|A_n \cdots A_1\|_{\text{op}}, \quad \mathbb{P}\text{-a.s.}, \quad (6.33)
\]

is the top Lyapunov exponent.
6.2.3 Recurrence classification

To describe our results on recurrence and transience, we need some more notation. With $p_{ij}^\ell$, $q_{ij}^\ell$ and $r_i^\ell$ as defined in Section 6.2.1, we write $P_\ell := (p_{ij}^\ell)_{i,j \in A}$ and $D_\ell := (d_{ij}^\ell)_{i,j \in A}$, where $d_{ij}^\ell = -q_{ij}^\ell$ for $i \neq j$ and

$$d_{ii}^\ell = r_i^\ell + \sum_{j \in A \setminus \{i\}} q_{ij}^\ell.$$ 

Thus $P_1, P_2, \ldots$ and $D_1, D_2, \ldots$ are each i.i.d. sequences of random $d \times d$ matrices on $(\Omega, \mathcal{F}, P)$. We will impose the following assumption.

(D) $E \log(1/r_i^\ell) < \infty$, for all $i \in A$.

We will show shortly (in Proposition 6.2.2) that condition (D) ensures that $D_\ell$ is $P$-a.s. invertible; then we define the sequence of i.i.d. random matrices $A_\ell$, given by

$$A_\ell := D_\ell^{-1} P_\ell,$$  

and we denote the associated Lyapunov exponents by $\gamma_1, \ldots, \gamma_d$. The relevance of condition (D) is displayed by the following result.

Proposition 6.2.2. If (D) holds, then the matrices $D_\ell$ are $P$-a.s. invertible, with $E \log^+ \|D_\ell^{-1}\|_{\text{op}} < \infty$, and the associated matrices $A_\ell$ defined by (6.34) satisfy condition (N).

Proof. For $v = (v_1, \ldots, v_d)^\top$ and $i \in A$, we have

$$(D_\ell v)_i = \sum_j d_{ij}^\ell v_j^\ell = r_i^\ell v_i + \sum_{j \neq i} q_{ij}^\ell (v_i - v_j).$$  

(6.35)

Suppose that (D) holds. Then, $P$-a.s., $r_i^\ell > 0$ for all $i \in A$. Fix $v \neq 0$, and let $v_i$ denote the component of $v$ with greatest absolute value. If $v_i > 0$ then (6.35) shows that $(D_\ell v)_i \geq r_i^\ell v_i > 0$; if $v_i < 0$ then (6.35) shows that $(D_\ell v)_i \leq r_i^\ell v_i < 0$. In particular, any non-zero vector $v \in \mathbb{R}^d$ has a non-zero image $D_\ell v$; thus the kernel of the map $v \mapsto D_\ell v$ is trivial, and hence $D_\ell^{-1}$ exists and the linear map is bijective. Moreover

$$\|D_\ell v\| \geq |(D_\ell v)_i| \geq r_i^\ell |v_i| \geq d^{-1/2} r_i^\ell \|v\|.$$ 

Thus if $u = D_\ell v$ we have $\|D_\ell^{-1} u\| \leq d^{1/2} (1/r_i^\ell) \|u\|$ and hence $\|D_\ell^{-1}\|_{\text{op}} \leq d^{1/2} \max_{i \in A} (1/r_i^\ell)$. Therefore

$$E \log^+ \|D_\ell^{-1}\|_{\text{op}} \leq d^{1/2} \sum_{i \in A} E \log(1/r_i^\ell) < \infty.$$
The final claim in the proposition follows from this, together with the fact that \( \|A\|_{\text{op}} = \|D_{\ell}^{-1}P_{\ell}\|_{\text{op}} \leq \|D_{\ell}^{-1}\|_{\text{op}}\|P_{\ell}\|_{\text{op}} \) and the fact that elements of \( P_{\ell} \) are uniformly bounded.

Now we are ready to formulate our results. Roughly speaking, the top Lyapunov exponent \( \gamma_1 \) determines the recurrence or transience of the process. First we give a sufficient condition for transience. Note that (by Proposition 6.2.2), under the hypotheses of Theorem 6.2.3 both \( D_{\ell} \) and \( P_{\ell} \) are \( \mathbb{P}\)-a.s. invertible, and hence so is \( A_{\ell} \) as defined by (6.34).

**Theorem 6.2.3.** Suppose that (D) holds. Suppose also that \( P_{\ell} \) is \( \mathbb{P}\)-a.s. invertible, and that \( A_{\ell}^{-1} \) satisfies condition (N). If \( \gamma_1 > 0 \), then for \( \mathbb{P}\)-a.e. \( \omega \) the Markov chain \( X_n \) is transient.

Next we give a sufficient condition for positive-recurrence.

**Theorem 6.2.4.** Suppose that (D) holds. If \( \gamma_1 < 0 \), then for \( \mathbb{P}\)-a.e. \( \omega \) the Markov chain \( X_n \) is positive-recurrent.

The final result covers the case \( \gamma_1 = 0 \).

**Theorem 6.2.5.** Suppose that (D) holds. Suppose also that \( A_{\ell} \) is \( \mathbb{P}\)-a.s. invertible, and that no finite union of proper subspaces of \( \mathbb{R}^d \) is a.s. stabilized by \( A_{\ell} \). If \( \gamma_1 = 0 \), then for \( \mathbb{P}\)-a.e. \( \omega \) the Markov chain \( X_n \) is recurrent.

### 6.2.4 Proofs

We will need the following simple lemma, which establishes relations between the Lyapunov exponents of \( A \) and \( A^{-1} \).

**Lemma 6.2.6.** Suppose that \( A \) is \( \mathbb{P}\)-a.s. invertible, and that both \( A \) and \( A^{-1} \) satisfy condition (N). Then for \( j \in \{1, \ldots, d\} \),

\[
\gamma_j(A^{-1}) = -\gamma_{d-j+1}(A).
\]  

(6.36)

**Proof.** Let \( b_1(n) \geq \cdots \geq b_d(n) \) be the square roots of the eigenvalues of \( (A_n^{-1} \cdots A_1^{-1})^\top(A_n^{-1} \cdots A_1^{-1}) = ((A_1 \cdots A_n)(A_1 \cdots A_n)^\top)^{-1} \).

Since \( UV \) has the same eigenvalues as \( VU \), \( (b_1(n))^{-1}, \ldots, (b_d(n))^{-1} \) are the square roots of the eigenvalues of \( (A_1 \cdots A_n)^\top(A_1 \cdots A_n) \). But the product \( A_1 \cdots A_n \) has the same law as the product \( A_n \cdots A_1 \). So for all \( j \in \{1, \ldots, d\} \), we have \( (b_j(n))^{-1} \) has the same law as \( a_{d-j+1}(n) \), and consequently, \( \gamma_j(A^{-1}) = -\gamma_{d-j+1}(A) \). \( \square \)
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We will also need the classical multiplicative ergodic theorem of Oseledets [249].

**Theorem 6.2.7.** Let $A, A_1, A_2, \ldots$ be a sequence of i.i.d. random matrices satisfying condition (N), and let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_d$ be the associated Lyapunov exponents. Let $t := \#\{\gamma_1, \ldots, \gamma_d\}$ denote the number of distinct exponents, define $i_1 := 1$, $i_{t+1} := d+1$, and

$$i_{s+1} := \min\{i > i_s : \gamma_i < \gamma_{i_s}\}, \text{ for } 1 \leq s \leq t-1.$$

Then, $\mathbb{P}$-a.s.:

(i) for every $v \in \mathbb{R}^d$, $\lim_{n \to \infty} n^{-1} \log \|A_n \ldots A_1 v\|$ exists or is $-\infty$;

(ii) for $s \in \{1, 2, \ldots, t, t+1\}$,

$$V(s, \omega) = \left\{ v \in \mathbb{R}^d : \lim_{n \to \infty} n^{-1} \log \|A_n \ldots A_1 v\| \leq \gamma_{i_s} \right\}$$

is a random linear subspace of $\mathbb{R}^d$ with dimension $d - i_s + 1$ (we have $V(1, \omega) = \mathbb{R}^d$ and $V(t+1, \omega) = \{0\}$);

(iii) for $s \in \{1, 2, \ldots, t\}$, $v \in V(s, \omega) \ominus V(s+1, \omega)$ implies that

$$\lim_{n \to \infty} n^{-1} \log \|A_n \ldots A_1 v\| = \gamma_{i_s}.$$

Note that in Theorem 6.2.7 the non-random integers $i_s$ mark the points of decrease of the $\gamma_i$, and $\gamma_1 = \gamma_{i_1} > \gamma_{i_2} > \cdots > \gamma_{i_t} = \gamma_d$.

Now we can give the proof of our transience result.

*Proof of Theorem 6.2.3.* Suppose that (D) holds, that $P_\ell$ is $\mathbb{P}$-a.s. invertible, and that $A_\ell^{-1}$ satisfies condition (N). We construct a Lyapunov function $h(s)$ and apply Corollary 2.5.12 to deduce transience. We show that there exists a sequence of column vectors $v^\ell = (v_1^\ell, \ldots, v_d^\ell)^\top$, depending on the environment $\omega$, such that the function $h(s)$ of the string $s = s_1 \ldots s_{|s|}$ can be defined as

$$h(s) = \sum_{j=1}^{s} v_{s_j}^j. \quad (6.37)$$

Indeed, for $h(s)$ defined by (6.37) we have for $s$ with $|s| = \ell$ and $s_\ell = i \in A$,

$$E_\omega[h(X_{n+1}) - h(X_n) \mid X_n = s] = -r_i^\ell v_i^\ell + \sum_{j \in A} q_{ij}^\ell (-v_i^\ell + v_j^\ell) + \sum_{j \in A} p_{ij}^\ell v_j^{\ell+1}$$
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\[
= (-D_\ell v^\ell + P_\ell v^{\ell+1})_i, \quad (6.38)
\]

where the matrices \(P_\ell\) and \(D_\ell\) are as introduced in Section 6.2.3.

If \(h(s)\) is to satisfy the conditions of Corollary 2.5.12, we require the right-hand side of (6.38) to vanish for all non-empty strings \(s\) and all \(i\). Thus we require the following relation between \(v^\ell\) and \(v^{\ell+1}\):

\[
v^{\ell+1} = D_\ell^{-1} P_\ell v^\ell = A_\ell^{-1} v^\ell. \quad (6.39)
\]

From (6.39) it follows that we may choose

\[
v^{\ell+1} = A_\ell^{-1} \cdots A_1^{-1} v^1, \quad \text{for all } \ell \geq 1,
\]

for some vector \(v^1\).

According to Theorem 6.2.7 (in particular, the \(s = t\) case of part (iii)) applied to the sequence \(A_1^{-1}, A_2^{-1}, \ldots\), for \(\mathbb{P}\)-a.e. environment \(\omega\) there exists a non-zero random vector \(v^1 = v^1(\omega)\) such that

\[
\lim_{\ell \to \infty} \frac{1}{\ell} \log \|v^{\ell+1}\|_1 = \lim_{\ell \to \infty} \frac{1}{\ell} \log \|A_\ell^{-1} \cdots A_1^{-1} v^1\|_1 = \gamma_d(A^{-1}) = -\gamma_1, \quad (6.40)
\]

using Lemma 6.2.6 for the final equality. From (6.40), if \(\gamma_1 > 0\) it follows that there exists a positive constant \(C_1 = C_1(\omega)\) such that \(\|v^\ell\| \leq C_1 \exp\{-\gamma_1 \ell/2\}\) for all \(\ell \geq 1\). Thus from (6.37), we find that

\[
\sup_s |h(s)| \leq \sum_{j=1}^\infty \|v^j\| \leq \frac{C_1 \exp\{-\gamma_1/2\}}{1 - \exp\{-\gamma_1/2\}} < \infty.
\]

So this function satisfies all the conditions of Corollary 2.5.12, which yields transience.

Remark 6.2.8. The string Lyapunov function \(h\) defined at (6.37) bears a close analogy to the nearest-neighbour random walk function of Section 2.2 as defined at (2.7). The string function is a sum of products of the matrices \(A_n^{-1}\), while the walk function is a sum of products of the scalars \(p_n/q_n\).

To prove Theorem 6.2.4, we need one supplementary fact.

Lemma 6.2.9. Suppose that (D) holds. \(\mathbb{P}\)-a.s., all entries of matrix \(D_\ell^{-1}\) are non-negative.

Proof. As shown in Proposition 6.2.2, condition (D) ensures that \(D_\ell^{-1}\) exists and the map \(v \mapsto D_\ell v\) is bijective. It suffices to prove that all components of \(D_\ell^{-1} v\) are non-negative whenever all components of \(v\) are non-negative.
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Thus it suffices to prove that any vector \( v = (v_1, \ldots, v_d)^\top \) with at least one negative component has an image \( D_\ell v \) with at least one negative component. Choose \( i \) such that \( v_i = \min_j v_j \); then, by (6.35), \( (D_\ell v)_i \leq r_\ell^iv_i < 0 \), since \( r_\ell^i > 0 \) under (D).

**Proof of Theorem 6.2.4.** In contrast to the proof of Theorem 6.2.3, where \( h(s) \) was constructed using ingredients provided only implicitly by Theorem 6.2.7, here we explicitly construct a Lyapunov function \( t(s) \), which is an analogue of the random walk Lyapunov function given by (6.3).

Recall that \( I_d \) denotes the \( d \times d \) identity matrix, and denote by \( \mathbf{1} = (1, \ldots, 1)^\top \) the column vector with all \( d \) components equal to 1. Similarly to (6.37), we define

\[
t(s) = \sum_{j=1}^{|s|} v^j,s_j,
\]

but now we define \( v^\ell = (v^\ell_1, \ldots, v^\ell_d)^\top \) by

\[
v^\ell = D_\ell^{-1}\mathbf{1} + \sum_{m \geq \ell} A_\ell \cdots A_m D_{m+1}^{-1}\mathbf{1} = (D_\ell^{-1} + A_\ell D_{\ell+1}^{-1} + A_\ell A_{\ell+1} D_{\ell+2}^{-1} + \cdots)\mathbf{1}.
\]

We need to show that \( v^\ell \) is finite and hence \( t(s) \) is well-defined. First, since for any \( x \in \mathbb{R}_+ \) we have \( x \geq \sum_{m=1}^{\infty} 1\{x \geq m\} \), the assumption \( \mathbb{E} \log^+ \|D_1^{-1}\|_{\text{op}} < \infty \) shows that, for any \( C \in (0, \infty) \),

\[
\sum_{m=1}^{\infty} \mathbb{P} \left( C^{-1} \log \|D_m^{-1}\|_{\text{op}} > m \right) < \infty.
\]

Thus, by the Borel–Cantelli lemma, \( \mathbb{P} \)-a.s., for all but finitely many \( m \), \( \|D_m^{-1}\|_{\text{op}} < \exp(Cm) \). Since \( \gamma_1 < 0 \), we may choose \( C \in (0, -\gamma_1) \). Then we obtain, using (6.33), \( \mathbb{P} \)-a.s.,

\[
\|v^\ell\| \leq (\|D_\ell^{-1}\|_{\text{op}} + \|A_\ell\|_{\text{op}} \|D_{\ell+1}^{-1}\|_{\text{op}} + \|A_\ell A_{\ell+1}\|_{\text{op}} \|D_{\ell+2}^{-1}\|_{\text{op}} + \cdots)\|\mathbf{1}\| < \infty.
\]

In addition, we have that \( v^\ell \) has all components non-negative, by Lemma 6.2.9. Analogously to (6.38), we have for \( s \) with \( |s| = \ell \) and \( s_\ell = i \in A \),

\[
\mathbb{E}_\omega [t(X_{n+1}) - t(X_n) \mid X_n = s] = (-D_\ell v^\ell + P_\ell v^\ell + \cdots)\mathbf{1}.
\]

With the present choice of \( v^\ell \), the vector whose \( i \)th component is on the right-hand side of the last display is equal to

\[
(-I_d - D_\ell A_\ell D_{\ell+1}^{-1} - D_\ell A_\ell A_{\ell+1} D_{\ell+2}^{-1} - \cdots)\mathbf{1}.
\]
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\[ + (P_\ell D_{\ell+1}^{-1} + P_\ell A_{\ell+1} D_{\ell+2}^{-1} + P_\ell A_{\ell+1} A_{\ell+2} D_{\ell+3}^{-1} + \cdots)1 \]

\[ = -I_d 1 - D_\ell (A_{\ell} D_{\ell+1}^{-1} - A_{\ell} A_{\ell+1} D_{\ell+2}^{-1} - \cdots)1 \]

\[ + D_\ell (A_{\ell} D_{\ell+1}^{-1} + A_{\ell} A_{\ell+1} D_{\ell+2}^{-1} + \cdots)1, \]

recalling from (6.34) that \( A_{\ell} = D_{\ell+1}^{-1} P_\ell \). Hence

\[ E_\omega [t(X_{n+1}) - t(X_n) | X_n = s] = -1, \text{ for all } s \neq \emptyset. \]

Therefore the function \( t(s) \) satisfies all the conditions of Theorem 2.6.4 and positive-recurrence follows.

Finally, we complete the proof of Theorem 6.2.5.

**Proof of Theorem 6.2.5.** From a theorem in [30] the assumptions imply

\[ \liminf_{\ell \to \infty} \| A_1 \cdots A_{\ell} \|_{\text{op}} = 0, \mathbb{P}\text{-a.s.,} \]

and therefore

\[ P_1 A_2 \cdots A_\ell 1 \leq D_1 1, \text{ for infinitely many } \ell. \quad (6.41) \]

Define recursively the random times \( \tau_0 = 1, \) and, for \( k \geq 0, \)

\[ \tau_{k+1} = \min \{ n > \tau_k : -D_{\tau_k} 1 + P_{\tau_k} A_{\tau_k+1} \cdots A_n 1 \leq 0 \}. \]

From (6.41) the times \( \tau_k \) are \( \mathbb{P}\)-a.s. finite. Similarly to (6.37), we define

\[ f(s) = \sum_{j=1}^{\lvert s \rvert} \tau^j_{s_j}, \]

but where now \( \psi^{\tau_k} = 1 \) and

\[ \psi^\ell = A_\ell A_{\ell+1} \cdots A_{\tau_k+1} 1, \text{ for } \tau_k < \ell \leq \tau_{k+1}. \quad (6.42) \]

Hence we have \( \psi^\ell = A_\ell \psi^{\ell+1} \) when \( \tau_k < \ell < \tau_{k+1}, \ k \geq 0, \) and

\[ -D_{\tau_k} \psi^{\tau_k} + P_{\tau_k} \psi^{\tau_k+1} = -D_{\tau_k} 1 + P_{\tau_k} A_{\tau_k+1} \cdots A_{\tau_k+1} 1 \leq 0, \]

by definition of \( \tau_{k+1}. \) Similarly to (6.38), this means that \( f(X_n) \) is a super-martingale except at \( s = \emptyset. \) From Lemma 6.2.9 and from the definition (6.42) the vector \( \psi^\ell \) is non-negative, and therefore \( f \) is also non-negative. Finally, since the \( \tau_k \) are a.s. finite and \( \psi^{\tau_k} = 1, f(s) \to \infty \) as \( \lvert s \rvert \to \infty. \) Theorem 2.5.2 applies, and the random string is recurrent. \qed
6.3 Stochastic billiards

6.3.1 Introduction

In this section we consider a model for the dynamics of a particle in a planar domain moving at constant speed and undergoing random reflections on collision with the boundary. The domain is of the form

\[ D := D(\gamma, A) := \{ (x, y) \in \mathbb{R}^2 : x \geq A, |y| \leq x^\gamma \}, \]

where \( \gamma < 1 \) is a fixed parameter and \( A > 0 \) is a constant to be fixed later.

The dynamics are described roughly as follows. A particle moves at unit speed in \( D \); while in the interior \( D \setminus \partial D \), the direction also remains fixed. When the particle hits the boundary \( \partial D \), it is (independently) reflected at a random angle to the inwards-pointing normal vector at the boundary. The law of the process is determined by specifying the distribution of the random reflections. See Figure 6.2 for some simulations of the process.

Suppose that the random variable \( \alpha \) for the angle of reflection satisfies, for some \( \alpha_0 \in (0, \pi/2) \),

\[ \mathbb{P}[0 < |\alpha| < \alpha_0] = 1 \text{ and } \mathbb{E} \tan \alpha = 0. \]

Thus the distribution of \( \alpha \) is bounded strictly away from \( \pm \pi/2 \) and does not have an atom at 0; a sufficient condition for \( \mathbb{E} \tan \alpha = 0 \) is that the distribution of \( \alpha \) be symmetric about 0.

The process that we have described informally above is a continuous-time process on \( D \). It is more convenient to work with a discrete-time process, obtained by recording the locations of the successive hits of the particle on the boundary \( \partial D \). This is a Markov chain with state-space \( \partial D \) that we denote by \( (\xi_n, n \geq 0) \), where we write in coordinates \( \xi_n = (\xi_n^{(1)}, \xi_n^{(2)}) \). Thus when \( \xi_n^{(1)} > A, \xi_n^{(2)} = \pm(\xi_n^{(1)})^\gamma \). We call \( \xi_n \) the collisions process.

To construct \( \xi_n \) formally, suppose that \( \xi_0 \in \partial D \). We perform a step of our process as follows.

- Take an independent draw of an angle \( \alpha \) satisfying (6.44). The realization of \( \alpha \) specifies a ray \( \Gamma \) starting at \( \xi_n \) with angle \( \alpha \) to the interior normal to \( \partial D \) at \( \xi_n \). We adopt the convention that positive values of the angle correspond to the right of the normal; negative values correspond to the left.

- Let \( \xi_{n+1} \) be the first point of intersection of the ray \( \Gamma \) with \( \partial D \setminus \{\xi_n\} \).
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It is not a priori clear that $\xi_n$ is well-defined, as the ray $\Gamma$ might never intersect the boundary; however, this does not occur assuming that the $A$ in (6.43) is chosen sufficiently large, as the following result shows.

**Lemma 6.3.1.** Suppose that $\alpha$ satisfies (6.44) for some $\alpha_0 \in (0, \pi/2)$, and that $\gamma < 1$. Then there exists $A_0 \in \mathbb{R}_+$ (depending on the distribution of $\alpha$ as well as on $\alpha_0$ and $\gamma$) such that for any $A \geq A_0$ the process $\xi_n$ is well-defined on $\partial D$ where $D = D(\gamma, A)$ is given by (6.43). Moreover, $\limsup_{n \to \infty} \xi_n^{(1)} = \infty$, a.s.

Thus our standing assumption for this section will be the following.

**(SB)** Suppose that $\alpha$ satisfies (6.44) for some $\alpha_0 \in (0, \pi/2)$, and that $D$ is given by (6.43) for some $\gamma < 1$ and $A \geq A_0$ as in Lemma 6.3.1.

Our first result covers the recurrence/transience classification for $\xi_n$. It turns out that a key quantity is $E[\tan^2 \alpha]$; note that under (6.44) we have $E[\tan^2 \alpha] \leq E[\tan^2 \alpha_0] < \infty$ and $E[\tan^2 \alpha] > 0$ by the fact that $\alpha$ does not degenerate to 0. Define

$$\gamma_c := \frac{E[\tan^2 \alpha]}{1 + 2E[\tan^2 \alpha]},$$

so that $\gamma_c \in (0, 1/2)$ under (6.44).

**Theorem 6.3.2.** Suppose that (SB) holds. Then:

(i) If $\gamma > \gamma_c$ the process is transient, i.e., $\lim_{n \to \infty} \xi_n^{(1)} = \infty$, a.s.;

(ii) If $\gamma \leq \gamma_c$ the process is recurrent, i.e., $\liminf_{n \to \infty} \xi_n^{(1)} < \infty$, a.s.

### 6.3.2 Reduction to Lamperti’s problem

The key to our analysis of the stochastic billiards process is to consider a rescaled version of the process to obtain an instance of the Lamperti problem. As at (3.1), we write $\mu_k(x) = E[(X_{n+1} - X_n)^k \mid X_n = x]$ for the increment moment functions of $X_n$.

**Theorem 6.3.3.** Suppose that (SB) holds. Set $X_n := (\xi_n^{(1)})^{1-\gamma}$. Then $X_n$ is a time-homogeneous Markov process on $\mathbb{R}_+$ satisfying, for some $B \in \mathbb{R}_+$, $\mathbb{P}[|X_{n+1} - X_n| \leq B]$, and, as $x \to \infty$,

$$\mu_1(x) = \frac{2\gamma(1-\gamma)(1 + E[\tan^2 \alpha])}{x} + O(x^{-2});$$

$$\mu_2(x) = 4(1-\gamma)^2 E[\tan^2 \alpha] + O(x^{-1}).$$

(6.44)
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If a near-critical process on $\mathbb{R}_+$ is transformed to find a scale on which the increments of the process have uniformly bounded second moments, one often ends up with a Lamperti process with asymptotically zero drift.

We defer the proof of Theorem 6.3.3 to Section 6.3.3; here we use the theorem and the results of Chapter 3 to establish our recurrence classification.

**Proof of Theorem 6.3.2.** Consider the process $X_n$ given in Theorem 6.3.3; then $X_n$ is recurrent if and only if $\xi_n^{(1)}$ is recurrent. From (6.46) and (6.47) we have that

$$2x\mu_1(x) - \mu_2(x) = 4(1 - \gamma) \left( \gamma (1 + 2E[tan^2 \alpha]) - E[tan^2 \alpha] + O(x^{-1}) \right)$$

$$= 4(1 - \gamma) \left( (\gamma - \gamma_c)(1 + 2E[tan^2 \alpha]) + O(x^{-1}) \right),$$

where $\gamma_c$ is given by (6.45). Thus if $\gamma > \gamma_c$ we have $\liminf_{x \to \infty} (2x\mu_1(x) - \mu_2(x)) > 0$ which yields transience by Theorem 3.5.1.

Similarly, for $\theta \in (0, 1)$, by (6.46) and (6.47),

$$2x\mu_1(x) - \left(1 + \frac{1 - \theta}{\log x}\right) \mu_2(x)$$

Figure 6.2: Simulations of part of the trajectory of the billiards process with $\alpha$ uniform on $[-\pi/4, \pi/4]$ in domains with $\gamma = 1/10$ (left) and $\gamma = 1/2$ (right). Here $E[tan^2 \alpha] = \frac{4 - \pi}{\pi}$ and $\gamma_c = \frac{4 - \pi}{8 - \pi} \approx 0.177$, so the first case is recurrent and the second transient.
which is strictly negative for all $x$ sufficiently large provided $\gamma \leq \gamma_c$; then Theorem 3.5.2 establishes recurrence.

### 6.3.3 Increment estimates

Write $g(x) := x^\gamma$. Suppose that at time $n = 0$ we have $\xi_0 = (\xi_0^{(1)}, \xi_0^{(2)}) = (x, \pm g(x))$ for $x > A$, and then $\xi_1$ is obtained after reflecting at angle $\alpha$ to the normal. Denote $D(x, \alpha) := \xi_1^{(1)} - \xi_0^{(1)}$, the resulting increment of the horizontal component of the process. Also set $\theta := \arctan g'(x)$, so $\tan \theta = g'(x) = \gamma x^{\gamma - 1}$.

We now proceed to obtain estimates for $D(x, \alpha)$ and its moments. The next lemma gives an upper bound on $D(x, \alpha)$ that follows from the fact that for large enough $x$ our domain is almost flat, while $\alpha$ is bounded strictly away from $\pm \pi/2$.

**Lemma 6.3.4.** Suppose that (SB) holds. Then there exist $x_0 \in \mathbb{R}_+$ and $C \in \mathbb{R}_+$, both depending only on $\alpha_0$ and $\gamma$, such that for all $x \geq x_0$ and all $\alpha \in (-\alpha_0, \alpha_0)$,

$$|D(x, \alpha)| \leq Cx^\gamma.$$

**Proof.** We use a geometrical argument depicted in Figure 6.3. Since $\gamma < 1$, we have $g'(x) \to 0$ and $\theta \to 0$ as $x \to \infty$. Thus we may choose $x \geq x_0$ large enough so that $|\theta| < \min\{\alpha_0, (\pi/2) - \alpha_0\}$ and then $\tan(\alpha_0 + \theta) = c_0$ for $c_0 < \infty$; suppose that $x_0$ is also chosen sufficiently large that $|g'(x)| < 1/c_0$ for all $x \geq x_0$.

By symmetry, it suffices to suppose that we start on the positive half of the curve, i.e., at $\xi_{\alpha} = (x, g(x))$. First suppose that $\gamma \geq 0$. Then $g$ is non-decreasing, so $\theta \geq 0$. Consider $D(x, \alpha)$, $\alpha \geq 0$, as represented in Figure 6.3.

In the case $\alpha \geq 0$, $|D(x, \alpha)| \leq |D(x, \alpha_0)|$. The reflected ray at angle $\alpha_0$ to the normal from $(x, g(x))$ has equation in $(u, v)$ given by

$$u - x = -(v - g(x))\tan(\alpha_0 + \theta), \quad u \geq x.$$

Let $a = 1/c_0$; by choice of $x_0$, we have $|g'(u)| < a$ for all $u \geq x \geq x_0$. Consider the line

$$v + g(x) = -a(u - x), \quad u \geq x.$$

Let $a = 1/c_0$; by choice of $x_0$, we have $|g'(u)| < a$ for all $u \geq x \geq x_0$. Consider the line

$$v + g(x) = -a(u - x), \quad u \geq x.$$
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Figure 6.3: The increment \( D(x, \alpha) = \xi_{n+1}^{(1)} - \xi_n^{(1)} \) when the particle at \( \xi_n = (x, x^\gamma) \) reflects at angle \( \alpha \) to the inwards normal (indicated by ‘\( N \)’ in the diagram). Here \( \tan \theta = \gamma x^{\gamma - 1} \). The quantity \( D' \) is an upper bound for \( D(x, \alpha) \), obtained by extending a line of slope \(-a\) for \( 0 < a < \gamma x^{\gamma - 1} \) from \((x, -x^\gamma)\).

This line intersects \( \partial \mathcal{D} \) at \((x, -g(x))\) and it intersects the reflected ray at angle \( \alpha_0 \) whose equation is (6.48) at \((u, v)\) with

\[
 u - x = \frac{2g(x) \tan(\alpha_0 + \theta)}{1 - a \tan(\alpha_0 + \theta)}. \tag{6.50}
\]

Since \( |g'(u)| < a \) for \( u \geq x \), the curve \( v = -g(u) \) remains above the line (6.49) for all \( u \geq x \). Thus \( |D(x, \alpha_0)| \) is bounded by \( u - x \) as given by (6.50); this is \( D' \) in Figure 6.3.
Thus, for some $C \in \mathbb{R}_+$, $|D(x, \alpha_0)| \leq Cg(x)$, for all $x \geq x_0$ large enough. A similar argument applies when $\alpha < 0$ and we use the bound $|D(x, \alpha)| \leq |\Delta(x, -\alpha_0)|$. With minor modifications, the same argument also works for $\gamma < 0$. \hfill \Box

**Proof of Lemma 6.3.1.** Lemma 6.3.4 shows that we may choose $x_0$ sufficiently large, depending on $\alpha_0$ and $\gamma$, so that for all $x \geq x_0$, $\xi_n \in \partial D$ with $\xi_n = (x, \pm x^\gamma)$ implies that $\xi_{n+1} \in \partial D$, a.s. If $\xi_n^{(1)} = A$, then since $\alpha$ has no atom at 0 it is also not hard to see that $\xi_{n+1} \in \partial D$, a.s. Thus if we take $A > x_0$, we ensure that $\xi_n \in \partial D$ is well defined for all $n \geq 0$.

To show the non-confinement result $\lim \sup_{n \to \infty} \xi_n^{(1)} = \infty$, a.s., we apply Proposition 3.3.4. It is sufficient to show that for any admissible distribution for $\alpha$, for all $y \geq A$, there exists $\varepsilon_y > 0$ for which

$$P[\xi_n^{(1)} - \xi_n^{(1)} > \varepsilon_y | \xi_n^{(1)} = x] > \varepsilon_y, \text{ for all } x \in [A, y]. \quad (6.51)$$

If $x = A$, the result is clear. So suppose $x > A$. Since $\alpha$ is not identically 0 and $E \tan \alpha = 0$, there exists $\nu > 0$ for which $P[\alpha > \nu] > \nu$. Thus with positive probability $D(x, \alpha) \geq D(x, \nu)$. Then (see Figure 6.3), $D(x, \nu) \geq D_1$ where $D_1 = g(x) \tan(\theta + \nu)$. If $\gamma \geq 0$, then $\theta \geq 0$ and (6.51) follows easily, since then for $x \in [A, y]$ we have $D(x, \alpha) \geq g(A) \tan(\nu)$ with probability at least $\nu$. (So in the case $\gamma \geq 0$, $\varepsilon_y$ need not depend on $y$.)

If $\gamma < 0$, we may choose $A$ big enough so that the angle $\theta$ to the normal satisfies $|\theta| < \nu$ for all $x \geq A$, and so $\tan(\theta + \nu)$ is still strictly positive, so that for $x \in [A, y]$ we have $D(x, \alpha) \geq g(y) \tan(\theta + \nu)$ with probability at least $\nu$, and we verify (6.51) in that case too. Note that the choice of $A$ for this last argument depended on $\nu$ and hence on the distribution of $\alpha$, as well as on $\alpha_0$ and $\gamma$. \hfill \Box

The next lemma gives crucial estimates for the first two moments of $D(x,\alpha)$.

**Lemma 6.3.5.** Suppose that (SB) holds. Then as $x \to \infty$

$$E[D(x, \alpha)] = 2\gamma x^{2\gamma-1}(1 + 2E[\tan^2 \alpha]) + O(x^{3\gamma-2}); \quad (6.52)$$

and $E[D(x, \alpha)^2] = 4x^{2\gamma} E[\tan^2 \alpha] + O(x^{3\gamma-1}). \quad (6.53)$

**Proof.** To ease notation we write $D = D(x, \alpha)$. Recall that $x \geq A$ is large enough so that if $\xi_n = (x, x^\gamma)$ is on the positive part of $\partial D$ then the next collision $\xi_{n+1} = (x + D, -(x + D)^\gamma)$ is on the negative part. See Figure 6.3.
6.3. Stochastic billiards

We have, repeatedly using the facts that $|\tan(\alpha + \theta)|$ and $|\tan \alpha|$ are uniformly bounded, and $|D| = O(x^\gamma)$ by Lemma 6.3.4,

\[
D = (g(x) + g(x + D)) \tan(\alpha + \theta)
\]
\[
= x^\gamma(1 + (1 + (D/x))\gamma) \tan(\alpha + \theta)
\]
\[
= x^\gamma(2 + D\gamma x^{-1}) \tan(\alpha + \theta) + O(x^{2\gamma - 2})
\]
\[
= x^\gamma(2 + D\gamma x^{-1}) \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} + O(x^{2\gamma - 2})
\]
\[
= x^\gamma(2 + D\gamma x^{-1}) (\tan \alpha + \gamma x^{-1})(1 + \gamma x^{-1} \tan \alpha) + O(x^{3\gamma - 2}),
\]

where, as always, the implicit constants in the $O$ terms are non-random.

Collecting terms we have

\[
D = 2\gamma x^{2\gamma - 1} + (\tan \alpha) (x^\gamma(2 + D\gamma x^{-1}) + 2\gamma x^{2\gamma - 1})
\]
\[
+ (\tan^2 \alpha) (2\gamma x^{2\gamma - 1}) + O(x^{3\gamma - 2}).
\]

Re-arranging, we obtain

\[
D = 2\gamma x^{2\gamma - 1} + (2\gamma x^\gamma + 2\gamma x^{2\gamma - 1}) \tan \alpha + 2\gamma x^{2\gamma - 1} \tan^2 \alpha + O(x^{3\gamma - 2})
\]
\[
= 2\gamma x^{2\gamma - 1} + (2\gamma x^\gamma + 2\gamma x^{2\gamma - 1}) \tan \alpha + 4\gamma x^{2\gamma - 1} \tan^2 \alpha + O(x^{3\gamma - 2}). \quad (6.54)
\]

Taking expectations in (6.54) and using the fact that $\mathbb{E} \tan \alpha = 0$ we obtain (6.52). Similarly, squaring both sides of (6.54) gives

\[
D^2 = 4x^{2\gamma} \tan^2 \alpha + O(x^{3\gamma - 1}),
\]

which on taking expectations yields (6.53).

Now we can complete the proof of Theorem 6.3.3.

Proof of Theorem 6.3.3. Set $X_n = (\xi_n^{(1)})^{1-\gamma}$. Then, $X_n$ is a Markov chain and, by Lemma 6.3.4,

\[
|X_{n+1} - X_n| \leq \max\{|(x + Cx^\gamma)^{1-\gamma} - x^{1-\gamma}|, |(x - Cx^\gamma)^{1-\gamma} - x^{1-\gamma}|\}.
\]

Here

\[
(x \pm Cx^\gamma)^{1-\gamma} - x^{1-\gamma} = x^{1-\gamma} \left( \left(1 \pm Cx^{\gamma-1}\right)^{1-\gamma} - 1 \right),
\]

which is uniformly bounded. Thus $|X_{n+1} - X_n| \leq B$, a.s., for some $B \in \mathbb{R}_+$. Given $\xi_n^{(1)} = x$ we have $X_n = x^{1-\gamma}$, and, by Taylor’s formula,

\[
X_{n+1} - X_n = (x + D(x, \alpha))^{1-\gamma} - x^{1-\gamma}
\]
\[ = (1 - \gamma) x^{-\gamma} D(x, \alpha) - \frac{\gamma(1 - \gamma)}{2} x^{-\gamma - 1} D(x, \alpha)^2 + O(x^{2\gamma - 2}), \quad (6.55) \]

using Lemma 6.3.4 for the error term. Taking expectations in (6.55) and using (6.52) and (6.53) we obtain

\[ \mathbb{E}[X_{n+1} - X_n \mid X_n = x^{1-\gamma}] = 2(1 - \gamma)\gamma(1 + \mathbb{E}[\tan^2 \alpha])x^{\gamma - 1} + O(x^{2\gamma - 2}), \]

which yields (6.46) after a change of variable. Similarly we obtain (6.47) after squaring both sides of (6.55).

\section{6.4 Exclusion and voter models}

\subsection{6.4.1 Introduction}

In this section, we apply the method of Lyapunov functions to some models of interacting particle systems, in which instead of a single particle on a countable state space, there many particles each of which can influence the others’ dynamics.

The processes that we will consider are Markov processes on configurations of particles on \( \mathbb{Z} \). Thus we take as our state space \( \{0,1\}^\mathbb{Z} \); we call \( s \in \{0,1\}^\mathbb{Z} \) a configuration, and we interpret a coordinate value \( s(x) = 1 \) as the presence of a particle at the site \( x \in \mathbb{Z} \) in the configuration \( s \), and \( s(x) = 0 \) as the absence of a particle at \( x \).

The two processes that we will consider are the exclusion process and the voter model, and we will also study a more general process that is a mixture of the two. The dynamics of these processes may be interpreted in terms of particles that hop on \( \mathbb{Z} \) (the case of the exclusion process) or appear and disappear at the sites of \( \mathbb{Z} \) (the case of the voter model). In either case, the dynamics are driven by the presence of discrepancies 01 or 10 in the configuration. In order to obtain well-defined processes, we consider dynamics on configurations with finitely many discrepancies.

At each time step, the exclusion process selects uniformly at random from among all discrepancies. If the chosen discrepancy is 01, it flips to 10 with probability \( p \) (else there is no change); if the pair is 10, it flips to 01 with probability \( 1 - p \). On the other hand, at each time step the voter model selects uniformly at random from all discrepancies and then flips the chosen pair to either 00 or 11, with equal chance of each. The exclusion-voter process is a hybrid of these two processes, whereby at each time step we determine independently at random whether to perform a voter-type move (with probability \( \beta \)) or an exclusion-type move (probability \( 1 - \beta \)).
6.4. Exclusion and voter models

Individually, the exclusion process and voter model exhibit very different behaviour. For instance, in the exclusion process there is local conservation of 1s: the number of 1s in a bounded interval can change only through the boundary. There is no such conservation in the voter model. In the hybrid process, voter moves and exclusion moves interact in a highly non-trivial way.

Consider the Heaviside configuration defined by \( 1 \{ x \leq 0 \} \), which consists of a single pair 10 abutted by infinite strings of 1s and 0s to the left and right, respectively:

\[
\ldots 11110000 \ldots
\]

If the voter model starts from the Heaviside configuration, then at any future time it is a random translate of the same configuration. Indeed, the position of the rightmost particle performs a symmetric simple random walk. If the voter model starts from a perturbation of the Heaviside configuration, it is natural to study the time it takes to reach a translate of the Heaviside configuration. This example motivates the following notation.

Let \( S' \subset \{0,1\}^{\mathbb{Z}} \) denote the set of configurations with a finite number of 0s to the left of the origin and 1s to the right, i.e., \( s' \in \{0,1\}^{\mathbb{Z}} \) for which there exist \( \ell, r \in \mathbb{Z} \) with \( \ell < r \) such that \( s'(x) = 1 \) for all \( x \leq \ell \) and \( s'(x) = 0 \) for all \( x \geq r \). In other words, \( S' \) contains those configurations of \( \{0,1\}^{\mathbb{Z}} \) in which there is only a finite number of discrepancies, and the number of discrepancies of type 10 minus the number of discrepancies of type 01 is equal to 1; note that \( S' \) is countable.

Let \(~\) denote the equivalence relation on \( S' \) such that for \( s'_1, s'_2 \in S' \), \( s'_1 \sim s'_2 \) if and only if \( s'_1 \) and \( s'_2 \) are translates of each other, i.e., there exists \( y \in \mathbb{Z} \) such that \( s'_1(x) = s'_2(x + y) \) for all \( x \in \mathbb{Z} \). Then set \( S := S' / \sim \).

In other words, \( S \) is the set of configurations of the form infinite string of 1s—finite number of 0s and 1s—infinite string of 0s, modulo translations. For example, one \( s \in S \) is

\[
s = \ldots 11100000000111000010010010000000001111000 \ldots
\]

We now formally define our processes. Fix \( \beta \in [0,1] \) (the mixing parameter) and \( p \in [0,1] \) (the exclusion parameter). The exclusion-voter process \((\xi_n, n \geq 0)\) with parameters \((\beta, p)\) is a time-homogeneous Markov chain on the countable state-space \( S \). The one-step transition probabilities are determined by the following mechanism.

- At each time step we decide independently at random whether to perform a voter move or an exclusion move. We choose a voter move with probability \( \beta \) and an exclusion move with probability \( 1 - \beta \).
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- Having decided this, choose a discrepancy (i.e. 01 or 10) uniformly at random from the finite number of available discrepancies.

- Then execute the chosen move. The voter move is such that the chosen pair (01 or 10) flips to 00 or 11 each with probability $1/2$. The exclusion move is such that a chosen pair 01 flips to 00 or 11 each with probability $1/2$. The exclusion move is such that a chosen pair 10 flips to 01 with probability $p := 1 - q$ (otherwise no move) and a chosen pair 10 flips to 01 with probability $q := 1/p$ (otherwise no move).

The case $\beta = 0$ is the (pure) exclusion process with parameter $p$, which we abbreviate as EP($p$), the case $\beta = 1$ is the (pure) voter model, VM, and the general case is the hybrid process, HP($\beta, p$). We use $P^e_p$, $P^v$, and $P^h_{\beta, p}$ to denote probabilities for EP($p$), VM, and HP($\beta, p$) respectively; we use $E^e_p$, $E^v$, and $E^h_{\beta, p}$ for the corresponding expectations.

The following basic result gives some elementary properties of the state-space $S$ under $P^h_{\beta, p}$. In particular, Proposition 6.4.1 says that for $(\beta, p) \in (0, 1)^2$ $\xi_n$ is irreducible and aperiodic. Let $s_H \in S$ denote the equivalence class of the Heaviside configuration $1\{x \leq 0\}$.

**Proposition 6.4.1.** For any $\beta \in [0, 1]$, $s_H$ is an absorbing state under $P^h_{\beta, 1}$. Suppose $\beta \neq 1$ and $(\beta, p) \notin \{(0, 0), (0, 1)\}$. Then all states in $S \setminus \{s_H\}$ communicate under $P^h_{\beta, p}$. Suppose $\beta \neq 1$, $p < 1$, and $(\beta, p) \neq (0, 0)$. Then all states in $S$ communicate under $P^h_{\beta, p}$, and $\xi_n$ is irreducible and aperiodic.

Define the relaxation time for the process $\xi_n$ as

$$\tau := \min\{n \geq 0 : \xi_n = s_H\}.$$  

We introduce some convenient terminology. If $P^h_{\beta, p}[\tau = \infty | \xi_0 = s] > 0$ for $s \in S$, we say that $\xi_n$ is transient started from $s$; if $P^h_{\beta, p}[\tau < \infty | \xi_0 = s] = 1$ for $s \in S$, we say that $\xi$ is recurrent started from $s$. In the latter case, if in addition $E^h_{\beta, p}[\tau | \xi_0 = s] < \infty$ for $s \in S$, we say that $\xi$ is positive-recurrent started from $s$. When $\xi$ is irreducible (see Proposition 6.4.1), this terminology coincides with the standard usage for countable Markov chains.

In the remaining part of this section, we prove a number of results. Namely, we prove

- The exclusion process is positive recurrent for $p > 1/2$ and transient for $p \leq 1/2$ (in particular, there is no null-recurrence regime).

- The voter model is positive recurrent; moreover, all moments for $\tau$ below 3/2 exist while all moment above 3/2 do not exist.
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- The exclusion-voter process is positive recurrent for \( p \geq 1/2 \) and any \( \beta \), or for \( \beta \geq \beta_0 \) and any \( p \), where \( \beta_0 \) is a constant sufficiently close to 1. In particular, note that any proportion of voter moves (\( \beta > 0 \)) added to the transient symmetric exclusion process (\( p = 1/2 \)) makes it positive recurrent.

6.4.2 Configurations and exclusion-voter dynamics

We introduce some more notation and terminology. A 1-block (0-block) is a maximal string of consecutive 1s (0s). Configurations in \( S \) consist of a finite number of such blocks. (From this point onwards we describe elements of \( S \) simply as ‘configurations’ rather than ‘equivalence classes of configurations’.)

For \( s \in S \), let \( N = N(s) \geq 0 \) denote the number of 1-blocks not including the infinite 1-block to the left (this is the same as number of 0-blocks not including the infinite 0-block to the right). Enumerating left to right, let \( n_i = n_i(s) \) denote the size of the \( i \)-th 0-block, and \( m_i = m_i(s) \) the size of the \( i \)-th 1-block: for example, for the configuration from (6.56),

\[
s = \ldots 111 00000000 111 0000 0000 001 001 00000000 1111 000 \ldots
\]

We may thus represent configuration \( s \in S \setminus \{ s_H \} \) by the vector of block sizes \((n_1, m_1, \ldots, n_N, m_N)\). For example, the configuration \( s \) of (6.56), which has \( N(s) = 5 \), has the representation \((8, 3, 4, 1, 2, 1, 2, 1, 8, 4)\).

For \( s \in S \setminus \{ s_H \} \) and \( i \in \{1, \ldots, N\} \) let

\[
R_i := R_i(s) := \sum_{j=1}^{i} n_j, \quad \text{and} \quad T_i := T_i(s) := \sum_{j=i}^{N} m_j;
\]

we adopt the convention \( R_0 = T_{N+1} = 0 \). Set \( |s| := 0 \) and for \( s \in S \setminus \{ s_H \} \) let \( |s| := \sum_{i=1}^{N} (n_i + m_i) = R_N + T_1 \), the length of the string of 0s and 1s between the infinite string of 1s to the left and the infinite string of 0s to the right.

It is convenient to represent a configuration \( s \in S \setminus \{ s_H \} \) diagrammatically as a right-down path in the quarter-lattice \( \mathbb{Z}_2^2 \): starting from \((0, T_1)\), construct a path by reading left-to-right the configuration \( s \) and for each 0 (1) taking a unit step in the right (down) direction. Thus the path starts with a step to the right, and ends at \((R_N, 0)\) after \(|s|\) steps. The lattice squares of \( \mathbb{Z}_2^2 \) bounded by the right-down path determined by \( s \) constitute a polygonal region in the plane that we call the staircase corresponding to \( s \). See Figure 6.4 for a representation of the staircase for \( s \) given by (6.56).
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Figure 6.4: An example staircase configuration \( s \) with \( N(s) = 5 \). The filled square is absent from \( s \) but present in \( s_2^+ \), the configuration obtained after performing the exclusion move \( 10 \rightarrow 01 \) on the rightmost (in this case, only) 1 of the 2nd 1-block.

With this representation of the configuration-space, the exclusion-voter model can be viewed as a growth/depletion process on staircases. For instance, exclusion moves are particularly simple in this context, corresponding to adding or removing a square at a corner.

We now introduce notation for the changes in configuration brought about by voter and exclusion moves. Given the staircase of \( s \), there are \( 2N + 1 \) ‘corners’ representing 10s and 01s alternately, of which \( N + 1 \) are 10s and \( N \) are 01s. In the staircase representation, these corners have coordinates \((R_i, T_{i+1}), i \in \{0, \ldots, N\}\) (for 10s) and \((R_i, T_i), i \in \{1, \ldots, N\}\) (for 01s), where \( R_0 = T_{N+1} = 0 \). Enumerate the 10s left-to-right in the configuration \( S \) by 0, 1, \ldots, \( N \), and similarly the 01s by 1, \ldots, \( N \).

Give a configuration \( s \in S \), we define the configurations to which it can transition under either a voter move or an exclusion move, denoted \( s_k^+ \), \( s_k^- \), \( s_k^{+r} \), \( s_k^{-r} \), \( s_k^{+\ell} \), \( s_k^{-\ell} \), as follows:

- \( s_k^+ \) is the configuration obtained from \( s \) by moving the rightmost 1 of the \( k \)th 1-block by 1 unit to the right, \( k \in \{0, \ldots, N\} \);
- \( s_k^- \) is the configuration obtained from \( S \) by moving the leftmost 1 of the \( k \)th 1-block by 1 unit to the left, \( k \in \{1, \ldots, N\} \);
- \( s_k^{+r} \) is the configuration obtained from \( S \) by adding an extra 1 to the right of the \( k \)th 1-block, \( k \in \{0, \ldots, N\} \);
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- $s_k^{+\ell}$ is the configuration obtained from $S$ by adding an extra 1 to the left of the $k$th 1-block, $k \in \{1, \ldots, N\}$;
- $s_k^{-r}$ is the configuration obtained from $S$ by removing the rightmost 1 from the $k$th 1-block, $k \in \{0, \ldots, N\}$;
- $s_k^{-\ell}$ is the configuration obtained from $S$ by removing the leftmost 1 from the $k$th 1-block, $k \in \{1, \ldots, N\}$.

With this notation, EP($p$) can transform $s$ to $s_k^{+\ell}$ or $s_k^{-r}$, while using VM we can get $s_k^{\pm r}$ or $s_k^{\pm\ell}$. Note that in the above, the 0th 1-block is the infinite block of 1s. See Figure 6.4 for a representation of one possible move as an operation on staircases.

To conclude this section, we sketch the (elementary) proof of Proposition 6.4.1. As well as $s_H$, we introduce special notation for one more configuration. Set 

$$s_D := \ldots 11101000 \ldots,$$

the single-discrepancy configuration with $N(s_1) = 1$ and vector representation $(1, 1)$.

**Proof of Proposition 6.4.1.** It is not hard to see that $s_H$ is an absorbing state for the pure voter model ($\beta = 1$) and for the left-moving totally asymmetric exclusion process ($\beta = 0, p = 1$), and hence also for the mixture model under $\mathbb{P}_{\beta, 1}$ for any $\beta \in [0, 1]$.

To show that all states within $S$ communicate, it suffices to show that $\mathbb{P}_{\beta, p}[^{\xi_n = s_1} | \xi_0 = s_0] > 0$ for some $n \in \mathbb{N}$ for each of the following:

(i) $s_0 = s_H$, $s_1 = s_D$;
(ii) $s_0 = s_D$, $s_1 = s_H$;
(iii) any $s_0$ with $|s_0| \geq 2$ and some $s_1$ with $|s_1| = |s_0| + 1$;
(iv) any $s_0$ with $|s_0| \geq 3$ and some $s_1$ with $|s_1| \leq |s_0| - 1$;
(v) any $s_0$ with $|s_0| \geq 3$ and any $s_1$ where $s_1$ is identical to $s_0$ apart from in a single position $j \in \{2, 3, \ldots, |s_0| - 1\}$.

In other words, given that moves of types (i)–(v) can occur, it is possible (with positive probability) to step, in a finite number of moves, between any two configurations in $S$ by first adjusting the length of the configuration via moves of types (i)–(iv) and then flipping the states in the interior of the configuration via moves of type (v). Similarly, to show that all states in
$\mathcal{S} \setminus \{s_H\}$ communicate, it suffices to show that all moves of types (iii)–(v) have positive probability.

It is not hard to see that voter moves can perform moves of types (ii), (iii) and (iv) in a single step (i.e., with $n = 1$). Similarly exclusion moves with $p < 1$ can perform moves of types (i) and (iii) in one step, while exclusion moves with $p > 0$ can perform moves of types (ii) and (iv), possibly needing multiple steps. We claim that moves of type (v) can be performed provided:

(a) $\beta \in (0, 1)$; or (b) $\beta = 0$ and $p \in (0, 1)$.

In case (a), suppose we need to replace a 0 by a 1 in the interior of a given configuration. If $p < 1$, we may perform a voter move on the first 10 to the left of the position to be changed, and then, if necessary, perform successive 10 $\mapsto$ 01 exclusion moves to ‘step’ the 1 into the desired position. If $p > 0$, an analogous procedure works, starting from the first 01 to the right. On the other hand, if we need to replace a 1 by a 0, a similar argument applies.

In case (b) we cannot use voter moves, but both types of exclusion move are permitted, so we can ‘bring in’ any 0 (1) from outside the disordered region, rearrange as necessary, and ‘take out’ the excess 1 (0) to the other boundary.

It follows that moves of type (ii)–(v) are possible provided $\beta \neq 1$ and $(\beta, p) \notin \{(0, 0), (0, 1)\}$, and all (i)–(v) are possible if we additionally impose $p < 1$.

To complete the proof we need to demonstrate aperiodicity in the case where $\beta \neq 1$, $p < 1$ and $(\beta, p) \neq (0, 0)$, where all states communicate. Since $\beta \neq 1$, exclusion moves may occur. Moreover, every configuration other than $s_H$ contains at least one pair of each type (01 and 10). Hence there is a positive probability that a configuration other than $s_H$ remains unchanged at a given step (when a proposed exclusion move fails to occur). Thus, since all states communicate, we have aperiodicity.

\[ \square \]

### 6.4.3 Lyapunov functions

We define two Lyapunov functions, $f_1$ and $f_2$, from $\mathcal{S}$ to $\mathbb{R}_+$ that will play a crucial role in our arguments. Set $f_1(s_H) = f_2(s_H) = 0$, and for $s \in \mathcal{S} \setminus \{s_H\}$ define

\[
 f_1(s) := \frac{1}{2} \left( \sum_{i=1}^{N} m_i R_i + \sum_{i=1}^{N} n_i T_i \right) = \sum_{i=1}^{N} m_i R_i = \sum_{i=1}^{N} n_i T_i;
\]

and

\[
 f_2(s) := \frac{1}{2} \left( \sum_{i=1}^{N} m_i R_i^2 + \sum_{i=1}^{N} n_i T_i^2 \right) .
\]
Note that $f_1$ is the area under the ‘staircase’ representation of the configuration (see Figure 6.4).

The value $f_1(s)$ is equal exactly to the number of nearest-neighbour transpositions needed to pass from $s$ to $s_{H}$, that is, $f_1(s)$ is in some sense the ‘distance’ from $s$ to the trivial configuration. Unfortunately, as we will see later, the function $f_1$ does not work well for some configurations $s$ (namely, for $s$ such that $N(s)$ is small with respect to $|s|$). The function $f_2$ is the result of our attempts to modify $f_1$ in order to eliminate this disadvantage; we cannot give any intuitive meaning of $f_2(s)$.

The next result gives some basic relations between $|s|$, $f_1(s)$, and $f_2(s)$.

**Lemma 6.4.2.** For any $s \in S$ we have:

(i) $\frac{1}{2}|s| \leq f_1(s) \leq \frac{1}{4}|s|^2$;

(ii) $\frac{1}{2}|s|^2 \leq f_2(s) \leq \frac{1}{8}|s|^3$;

(iii) $f_1(s) \leq (f_2(s))^{3/4}$.

**Proof.** For part (i), we have

$$f_1(s) = \frac{1}{2} \sum_{i=1}^{N} m_i R_i + \frac{1}{2} \sum_{i=1}^{N} n_i T_i \geq \frac{1}{2} (R_N + T_1) = \frac{|s|}{2};$$

and

$$f_1(s) = \sum_{i=1}^{N} m_i R_i \leq R_N \sum_{i=1}^{N} m_i = R_N T_1 \leq \frac{(R_N + T_1)^2}{4} = \frac{|s|^2}{4}.$$

Similarly, for part (ii), we have

$$f_2(s) \geq \frac{1}{2} (R_N^2 + T_1^2) \geq \frac{1}{4} (R_N + T_1)^2 = \frac{|s|^2}{4};$$

and

$$f_2(s) \leq \frac{1}{2} R_N^2 \sum_{i=1}^{N} m_i + \frac{1}{2} T_1^2 \sum_{i=1}^{N} n_i = \frac{1}{2} R_N T_1 (R_N + T_1) \leq \frac{|s|^3}{8}.$$

Finally we prove part (iii). We shall make use of the following simple consequence of Jensen’s inequality: if we have $n$ positive numbers $\gamma_1, \ldots, \gamma_n$ such that $\sum_{i=1}^{n} \gamma_i = 1$, then for any $x_1, \ldots, x_n$,

$$\gamma_1 x_1 + \cdots + \gamma_n x_n \leq \left( \gamma_1 x_1^2 + \cdots + \gamma_n x_n^2 \right)^{1/2}. \quad (6.57)$$
Denote \( \alpha_i = m_i/|s|, \beta_i = n_i/|s|, \) so \( \sum_{i=1}^{N} (\alpha_i + \beta_i) = 1. \) Using (6.57) and part (ii), we get

\[
f_1(s) = \frac{1}{2} \sum_{i=1}^{N} (m_i R_i + n_i T_i) = \frac{|s|}{2} \sum_{i=1}^{N} (\alpha_i R_i + \beta_i T_i)
\]

\[
\leq \frac{|s|}{2} \left( \sum_{i=1}^{N} (\alpha_i R_i^2 + \beta_i T_i^2) \right)^{1/2} = \frac{\sqrt{|s|}}{\sqrt{2}} (f_2(s))^{1/2}
\]

\[
\leq \frac{\sqrt{2}(f_2(s))^{1/4}}{\sqrt{2}} (f_2(s))^{1/2} = (f_2(s))^{3/4},
\]

thus completing the proof of Lemma 6.4.2. \( \square \)

We will need expressions for the expected increments of our Lyapunov functions \( f_1(\xi_n) \) and \( f_2(\xi_n) \). First we deal with \( f_1 \).

**Lemma 6.4.3.** Let \( \beta, p \in [0, 1] \) and \( s \in S \setminus \{s_H\} \). Then

\[
\mathbb{E}_{\beta,p}^h[f_1(\xi_{n+1}) - f_1(\xi_n) \mid \xi_n = s] = (1 - \beta)(1 - p) - \frac{N}{2N + 1}.
\]

**Proof.** Let \( s \in S \setminus \{s_H\} \). As noted in Section 6.4.2, EP can transform a configuration \( s \) only either to \( s^+_k \) or to \( s^-_k \), and we have

\[
f_1(s^+_k) - f_1(s) = -1, \quad \text{and} \quad f_1(s^-_k) - f_1(s) = 1. \tag{6.58}
\]

Thus, summing over the discrepancies where an exclusion move can occur,

\[
\mathbb{E}_{p}^e[f_1(\xi_{n+1}) - f_1(\xi_n) \mid \xi_n = s]
\]

\[
= \frac{p}{2N + 1} \sum_{k=1}^{N} (f_1(s_k^-) - f_1(s)) + \frac{q}{2N + 1} \sum_{k=0}^{N} (f_1(s_k^+) - f_1(s))
\]

\[
= \frac{N(q - p) + q}{2N + 1}. \tag{6.59}
\]

On the other hand, VM can transform \( s \) to one of \( s_k^{\pm \ell} \) or \( s_k^{\pm r} \). Direct computations yield

\[
f_1(s_k^{\pm \ell}) - f_1(s) = R_k - T_k - 1,
\]

\[
f_1(s_k^{\pm r}) - f_1(s) = -R_k + T_k - 1,
\]

\[
f_1(s_k^{\pm r}) - f_1(s) = R_k - T_{k+1},
\]
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\[ f_1(s_k^{+r}) - f_1(s) = -R_k + T_{k+1}. \]

Pairing off the terms \( s_k^\pm \) and the terms \( s_k^\pm \), it follows that

\[ \mathbb{E}^v[f_1(\xi_{n+1}) - f_1(\xi_n) \mid \xi_n = s] = \frac{1}{2N+1} \sum_{k=1}^{N} \frac{1}{2}(-2) + \frac{1}{2N+1} \sum_{k=0}^{N} \frac{1}{2}(0) = -\frac{N}{2N+1}. \quad (6.60) \]

In the hybrid process, a voter transition occurs with probability \( \beta \), and an exclusion transition occurs with probability \( 1 - \beta \); then combining (6.59) and (6.60) we obtain the result after some algebra.

Next we deal with \( f_2 \).

**Lemma 6.4.4.** Let \( \beta, p \in [0, 1] \) and \( s \in \mathcal{S} \setminus \{s_H\} \). Then

\[ \mathbb{E}^{h}_{\beta, p}[f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] = (1-\beta) \left( \frac{N + q}{2N+1} - \frac{2p - 1}{2N+1} \sum_{i=1}^{N} (R_i + T_i) \right). \]

**Proof.** Let \( s \in \mathcal{S} \setminus \{s_H\} \). The possible changes in \( f_2 \) due to the EP are

\[ f_2(s_k^+) - f_2(s) = \frac{1}{2} \left( (R_k + 1)^2 - R_k + (T_{k+1} + 1)^2 - T_{k+1}^2 \right) 
= 1 + R_k + T_{k+1}, \quad (6.61) \]

and

\[ f_2(s_k^-) - f_2(s) = \frac{1}{2} \left( (R_k - 1)^2 - R_k + (T_k - 1)^2 - T_k^2 \right) 
= 1 - R_k - T_k. \quad (6.62) \]

Combining (6.61) and (6.62), we obtain

\[ \mathbb{E}^{c}_{p}[f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] 
= \frac{p}{2N+1} \sum_{k=1}^{N} (f_2(s_k^-) - f_2(s)) + \frac{q}{2N+1} \sum_{k=0}^{N} (f_2(s_k^+) - f_2(s)) 
= \frac{N + q}{2N+1} - \frac{p - q}{2N+1} \sum_{i=1}^{N} (R_i + T_i). \quad (6.63) \]

On the other hand, the possible changes in \( f_2 \) due to the VM are

\[ f_2(s_k^{+r}) - f_2(s) = \frac{1}{2} (R_k + T_{k+1} + R_k^2 - T_{k+1}^2) - \sum_{i=k+1}^{N} m_i R_i + \sum_{i=1}^{k} n_i T_i; \quad (6.64) \]
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\[ f_2(s_k^-) - f_2(s) = \frac{1}{2}(R_k + T_{k+1} - R_k^2 + T_{k+1}^2) + \sum_{i=k+1}^{N} m_i R_i - \sum_{i=1}^{k} n_i T_i; \quad (6.65) \]

for \( k \in \{0, \ldots, N\} \), and, for \( k \in \{1, \ldots, N\} \),

\[ f_2(s_k^+\ell) - f_2(s) = \frac{1}{2}(-R_k - T_k + R_k^2 + T_k^2) - \sum_{i=k}^{N} m_i R_i + \sum_{i=1}^{k} n_i T_i; \quad (6.66) \]

\[ f_2(s_k^-\ell) - f_2(s) = \frac{1}{2}(-R_k - T_k - R_k^2 + T_k^2) + \sum_{i=k}^{N} m_i R_i - \sum_{i=1}^{k} n_i T_i. \quad (6.67) \]

Summing up the contributions from (6.64)–(6.67) we obtain

\[ \mathbb{E}^v [f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] = 0. \quad (6.68) \]

Combining (6.63) and (6.68) gives the result. \( \square \)

6.4.4 Exclusion process

Our first result on the exclusion process is the following.

**Theorem 6.4.5.** If \( p > 1/2 \), then \( EP(p) \) is positive recurrent.

**Proof.** We use the Lyapunov function \( f_2 \). Since \( R_N + T_1 = |s|, R_i \geq i \) and \( T_i \geq N - i + 1 \), it is straightforward to get that

\[ \sum_{i=1}^{N} (R_i + T_i) \geq \max\{|s|, N(N+1)\}. \]

Using this fact, we get from (6.63) that

\[ \mathbb{E}_p^v [f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] \leq \frac{N + 1}{2N + 1} - (2p - 1) \frac{\max\{|s|, N(N+1)\}}{2N + 1}. \]

Let \( p > 1/2 \). If \( (2p - 1)|s| > 2N + 2 \), we have

\[ \mathbb{E}_p^v [f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] \leq \frac{N + 1}{2N + 1} \leq \frac{1}{2}; \]

and for \( (2p - 1)N > 3 \), we have

\[ \mathbb{E}_p^v [f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] \leq 1 - \frac{3(N + 1)}{2N + 1} \leq \frac{1}{2}. \]
The set of $s$ for which both $(2p - 1)N \leq 3$ and $(2p - 1)|s| \leq 2N + 2$ is finite, so we have shown that

$$\mathbb{E}_p^e[f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] \leq -\frac{1}{2},$$

for all but finitely many $s \in \mathcal{S}$, and hence Theorem 2.6.2 shows that $\mathbb{E}(p)$ is positive recurrent when $p > 1/2$.

**Theorem 6.4.6.** If $p \leq 1/2$, then $\mathbb{E}(p)$ is transient.

**Proof.** First we consider the case $p < 1/2$, where we will use the Lyapunov function $f_1$. In this case, we have from (6.58) that $|f_1(\xi_{n+1}) - f_1(\xi_n)| \leq 1$, $\mathbb{P}_p^e$-a.s., and, from (6.59), for some $\varepsilon = \varepsilon(p) > 0$ and all $s \in \mathcal{S} \setminus \{s_H\}$,

$$\mathbb{E}_p^e[f_1(\xi_{n+1}) - f_1(\xi_n) \mid \xi_n = s] \geq \frac{N(1 - 2p)}{2N + 1} \geq \varepsilon.$$

Then by Theorem 2.5.14, the process $\xi_n$ is transient.

We now turn now to the more delicate case where $p = q = 1/2$. In this case we have from (6.59) that

$$\mathbb{E}_{1/2}^e[f_1(\xi_{n+1}) - f_1(\xi_n) \mid \xi_n = s] = \frac{1}{2(2N + 1)}, \quad (6.69)$$

and so, since the right-hand side of (6.69) can be arbitrarily small, we cannot apply Theorem 2.5.14. As observed in Section 2.5, this theorem provides only a sufficient condition for transience, so there is no guarantee that it should be applicable in all situations. Therefore, to deal with the case $p = 1/2$, we seek to apply the more general result, Theorem 2.5.8.

We fix an arbitrary $\alpha > 0$ and define the function $\psi : \mathcal{S} \setminus \{s_H\} \to \mathbb{R}_+$ by

$$\psi(s) := (f_1(s))^{-\alpha}.$$

Note that the definition is correct because $f_1(s) > 0$ for $s \neq s_H$. (Actually, for our proof of Theorem 6.4.6 it is sufficient to take $\alpha = 1$, but, since we will need analogous calculations for the hybrid process in Section 6.4.6, at this point we prefer to do the calculations for arbitrary $\alpha > 0$.)

For any $C > 0$, define the set $A_C \subset \mathcal{S}$ by

$$A_C := \{s \in \mathcal{S} : f_1(s) \leq CN(s)\}.$$

We claim that $A_C$ is finite for any $C > 0$. Indeed, $R_i \geq i$ and $m_i \geq 1$, so $f_1(s) \geq N(N + 1)/2$. Thus, for a configuration to belong to $A_C$, it is necessary that the number of 1-blocks be less than $2C - 1$, so

$$A_C \subseteq \{s \in \mathcal{S} : f_1(s) \leq C(2C - 1)\},$$
which is clearly finite (by e.g. Lemma 6.4.2(i)).

Now we have from (6.58) that
\[
\mathbb{E}_{1/2}^e [(f_1(\xi_{n+1}) - f_1(\xi_n))^2 \mid \xi_n = s] = \frac{1/2}{2N + 1} \left( \sum_{k=1}^{N} (f_1(s_k^{-}) - f_1(s))^2 + \sum_{k=0}^{N} (f_1(s_k^{+}) - f_1(s))^2 \right) = \frac{1}{2}. \tag{6.70}
\]

By elementary calculus, for any \( \alpha > 0 \) there exist positive constants \( C_1 = C_1(\alpha) \) and \( C_2 = C_2(\alpha) \) such that for all \( |x| < C_2 \),
\[
(1 + x)^{-\alpha} - 1 \leq -\alpha x + C_1 x^2.
\]

Hence, using (6.69) and (6.70), we get
\[
\mathbb{E}_{1/2}^e [\psi(\xi_{n+1}) - \psi(\xi_n) \mid \xi_n = s] = (f_1(s))^{-\alpha} \mathbb{E}_{1/2}^e \left[ \left( 1 + \frac{f_1(\xi_{n+1}) - f_1(\xi_n)}{f_1(\xi_n)} \right)^{-\alpha} - 1 \right] \mid \xi_n = s \leq (f_1(s))^{-\alpha} \left( -\frac{\alpha f_1(s)}{f_1(s)} \cdot \frac{1}{2(2N + 1)} + \frac{C_1}{2(f_1(s))^2} \right),
\]
provided \( s \) is such that \( f_1(s) > 1/C_2 \) (where we have used the fact that \( |f_1(\xi_{n+1}) - f_1(\xi_n)| \leq 1 \)). Thus we have
\[
\mathbb{E}_{1/2}^e [\psi(\xi_{n+1}) - \psi(\xi_n) \mid \xi_n = s] = (f_1(s))^{-\alpha - 2} \left( -\frac{\alpha f_1(s)}{2(2N + 1)} + \frac{C_1}{2} \right), \tag{6.71}
\]
which is negative for all \( s \) with \( f_1(s) > 1/C_2 \) and \( f_1(s) > C_1(2N(s) + 1)/\alpha \). The complementary set, with \( f_1(s) \leq 1/C_2 \) or \( f_1(s) \leq C_1(2N(s) + 1)/\alpha \) is the union of two finite sets, using the fact that \( A_C \) is finite. Thus (6.71) holds for all but finitely many \( s \in S \). Theorem 2.5.8 finishes the proof. \( \Box \)

6.4.5 Voter model

Our first result on the voter model in the following.

**Theorem 6.4.7.** The VM is positive recurrent. Moreover, for any \( \varepsilon > 0 \) and any initial configuration \( s \in S \),
\[
\mathbb{E}^e [\tau^{(3/2) - \varepsilon} \mid \xi_0 = s] < \infty. \tag{6.72}
\]
Proof. In the terminology introduced in Section 6.4.1, positive recurrence is the existence of \( \mathbb{E} \tau \); thus we shall turn directly to the proof of (6.72). The idea is to apply Theorem 2.7.1 to the process \((f_2(\xi_n))^\alpha\) for some \(\alpha < 1\).

First, elementary calculus gives us that for \(\alpha\) taking

\[
(1 + x)^\alpha - 1 \leq \alpha x - c_1 x^2.
\]

Hence using (6.73) we obtain

\[
\mathbb{E}^v[(f_2(\xi_{n+1}))^\alpha - (f_2(\xi_n))^\alpha \mid \xi_n = s] = (f_2(s))^\alpha \mathbb{E}^v \left[ \left( 1 + \frac{f_2(\xi_{n+1}) - f_2(\xi_n)}{f_2(\xi_n)} \right)^\alpha - 1 \mid \xi_n = s \right]
\]

\[
\leq -c_1(f_2(s))^{\alpha-2} \mathbb{E}^v[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s],
\]

by the fact from (6.68) that \(f_2(\xi_n)\) is a martingale under \(\mathbb{P}^v\).

To obtain a lower bound for the final expectation in (6.74), observe from (6.64) and (6.65) that

\[
|f_2(s_0^-) - f_2(s)| \geq \frac{T_1^2}{2}, \quad \text{and} \quad |f_2(s_N^+) - f_2(s)| \geq \frac{R_4^2}{2}.
\]

It follows from these two inequalities that

\[
\mathbb{E}^v[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s] \geq \frac{1}{4N+2} (f_2(s_0^-) - f_2(s))^2 + \frac{1}{4N+2} (f_2(s_N^+) - f_2(s))^2
\]

\[
\geq \frac{T_1^4 + R_4^4}{16N+8} \geq \frac{c_s^4}{N},
\]

for all \(s \in \mathcal{S} \setminus \{s_H\}\), where \(c > 0\) is a constant. Now, a very important observation is that the VM does not increase the number of blocks \(N(\xi_n)\).

So we have from (6.75) that

\[
\mathbb{E}^v[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s] \geq c_0 |s|^4, \quad \text{for all } s \in \mathcal{S} \setminus \{s_H\},
\]

with \(c_0 = c_0(\xi_0) = c/N(\xi_0)\). Using (6.76) in (6.74), we obtain

\[
\mathbb{E}^v[(f_2(\xi_{n+1}))^\alpha - (f_2(\xi_n))^\alpha \mid \xi_n = s] \leq -c_0 c_1 (f_2(s))^{\alpha-2} |s|^4.
\]

Applying Lemma 6.4.2(ii) it follows from (6.77) that

\[
\mathbb{E}^v[(f_2(\xi_{n+1}))^\alpha - (f_2(\xi_n))^\alpha \mid \xi_n = s] \leq -8^{4/3} c_0 c_1 (f_2(s))^{\alpha-(2/3)}.
\]

We apply Corollary 2.7.3 to the process \(X_n = (f_2(\xi_n))^\alpha\) taking \(\alpha\) to be close to 1 to finish the proof of (6.72). \(\square\)
The following theorem complements Theorem 6.4.7 by providing a result in the other direction.

**Theorem 6.4.8.** For any $\varepsilon > 0$ and any $s \in S \setminus \{s_H\}$,

$$\mathbb{E}^s[\tau^{3/2} \mid \xi_0 = s] = \infty. \quad (6.78)$$

**Proof.** For any initial configuration $\xi_0 = s \in S \setminus \{s_H\}$, since $N(\xi_{n+1}) - N(\xi_n)$ is either 0 or $-1$ under the VM dynamics, to reach $s_H$ the process must pass through a configuration $s_0$ with $N(s_0) = 1$. Thus it suffices to prove (6.78) for initial configurations of this type.

Representing the process started with $N(\xi_0) = 1$ by the vector of block sizes, we obtain a random walk on $\mathbb{Z}_2^+$, for which $\tau$ is the time to reach the boundary of the quarter-plane. Formally, write $X_n$ for the size of the finite 0-block in $\xi_n$, and $Y_n$ for the size of the finite 1-block, and set $\mathcal{F}_n = \sigma(\xi_0, \ldots, \xi_n)$. Then $(X_n, Y_n)$ is a nearest-neighbour random walk on $\mathbb{Z}_2^+$ with transition probabilities as follows: from the state $(x, y)$ with $xy \neq 0$, the transition can occur to each of the six states $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, $(x, y-1)$, $(x+1, y-1)$ and $(x-1, y+1)$ with probability $1/6$. We stop the random walk if $xy = 0$ (at this point the coupling breaks down, because whereas the random walk can end up at any point on the boundary, the voter model must end up at $(0, 0)$). In any case, the relaxation time is now $\tau = \min\{n \geq 0 : X_nY_n = 0\}$, the first hitting time of the boundary of $\mathbb{Z}_2^+$; we show that $\mathbb{E}[\tau^{3/2}] = \infty$.

It suffices to suppose that $\tau < \infty$, a.s. Let $Z_n = X_n^2Y_n + X_nY_n^2$. Then $Z_n = f_2(\xi_n)$ on $\{n < \tau\}$. In particular, $Z_{n \wedge \tau}$ is a martingale, by (6.68), or as can be verified by a calculation for the random walk.

Our argument is a variation on Example 2.6.6. Consider the Doob decomposition $X_n^3 = M_n + A_n$, where $M_n$ is a martingale with $M_0 = 0$ and

$$A_n = \sum_{m=1}^{n-1} \mathbb{E}[X_{m+1}^3 - X_m^3 \mid \mathcal{F}_m]$$

$$= \mathbb{E}[(X_{m+1} - X_m)^3 \mid \mathcal{F}_m] + 3 \sum_{m=1}^{n-1} X_m \mathbb{E}[(X_{m+1} - X_m)^2 \mid \mathcal{F}_m]$$

$$= 2 \sum_{m=1}^{n-1} X_m, \text{ on } \{n < \tau\}.$$

Hence

$$\mathbb{E} A_{n \wedge \tau} \leq 2\tau \max_{0 \leq m \leq n} X_{m \wedge \tau}.$$
Suppose, for the purpose of deriving a contradiction, that \( \mathbb{E}[\tau^{3/2}] < \infty \). Then, by Hölder’s inequality,

\[
\mathbb{E} A_{n \wedge \tau} \leq 2 \left( \mathbb{E}[\tau^{3/2}] \right)^{2/3} \left( \mathbb{E}\left[ \max_{0 \leq m \leq n} X_{m \wedge \tau}^3 \right] \right)^{1/3}.
\] (6.79)

The quadratic variation associated with the martingale \( X_n \) satisfies

\[
\langle X \rangle_n \leq n - 1 \sum_{m=0}^{n-1} \mathbb{E}[(X_{m+1} - X_m)^2 \mid F_m] = \frac{2}{3} n.
\]

It then follows from Burkholder’s inequality (see e.g. [286, p. 499]) that

\[
\mathbb{E}\left[ \max_{0 \leq m \leq n} X_{m \wedge \tau}^3 \right] \leq C \mathbb{E}\langle X \rangle_{\tau}^{3/2} \leq C \mathbb{E}[\tau^{3/2}],
\] (6.80)

for some constant \( C \in \mathbb{R}_+ \). Thus if \( \mathbb{E}[\tau^{3/2}] < \infty \) we have from (6.79) and (6.80) that \( \mathbb{E} A_{n \wedge \tau} \leq C \) for some \( C \in \mathbb{R}_+ \). Since \( \mathbb{E}[X_{n \wedge \tau}^3] = \mathbb{E} A_{n \wedge \tau} \), it follows that the martingale \( X_{n \wedge \tau} \) is uniformly bounded in \( L^3 \), and hence converges a.s. and in \( L^3 \) to \( X_{\tau} \) (see e.g. Theorem 5.4.5 of [83]). A similar argument applies to \( Y_{n \wedge \tau} \). Hence the martingale \( Z_{n \wedge \tau} \) converges a.s. and in \( L^1 \). In particular,

\[
0 < \mathbb{E} Z_0 = \mathbb{E} Z_{\tau} = 0,
\]

which is the desired contradiction. So \( \mathbb{E}[\tau^{3/2}] = \infty \).

\[\square\]

### 6.4.6 Hybrid process

Finally, we turn to the hybrid process that allows both exclusion and voter moves. The complete recurrence classification here is an open problem (see the bibliographical notes at the end of this chapter). The next result shows that if voter moves are sufficiently prominent, one has positive recurrence: in particular, \( HP(\beta, p) \) is positive recurrent for \( \beta > 2/3 \) and any \( p \).

**Theorem 6.4.9.** If \( \beta, p \in [0, 1] \) are such that \( (1 - p)(1 - \beta) < 1/3 \), then \( HP(\beta, p) \) is positive recurrent.

**Proof.** We have from Lemma 6.4.3 that for all \( s \in \mathcal{S} \setminus \{s_H\} \),

\[
\mathbb{E}_{\beta, p}^h [f_1(\xi_{n+1}) - f_1(\xi_n) \mid \xi_n = s] \leq (1 - \beta)(1 - p) - \frac{1}{3};
\]

provided \( (1 - p)(1 - \beta) < 1/3 \) this last expression is strictly negative, and so applying Theorem 2.6.2 we finish the proof. \[\square\]
The next result is not surprising when $p > 1/2$, since $\text{EP}(p)$ and $\text{VM}$ are (separately) positive recurrent in that case, but recall that $\text{EP}(1/2)$ is transient.

**Theorem 6.4.10.** For any $\beta > 0$ and $p \geq 1/2$ the process $\text{HP}(\beta, 1/2)$ is positive recurrent.

To prove Theorem 6.4.10, we will apply Theorem 2.6.4 with the function $(f_2(s))^\alpha$ for some $\alpha < 1$. This entails a computation similar to that in the proof of Theorem 6.4.7 on the positive recurrence of the VM. In the proof of Theorem 6.4.7 we used the fact that the number of blocks cannot increase under the VM dynamics to obtain from (6.75) the bound (6.76). For the hybrid process, this latter bound is not valid; instead, we have the following.

**Lemma 6.4.11.** Let $\beta > 0$. Then there exists $c > 0$ such that for all $s \in S$,

\[
E_{\beta,p}^h[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s] \geq c|s|^{16/5}. \tag{6.81}
\]

*Proof.* Since $\beta > 0$, there is positive probability of executing a voter move. Similarly to (6.75), writing $\Delta_k = f_2(s_{k+1}^+) - f_2(s_k)$, we thus have

\[
E_{\beta,p}^h[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s] \geq \frac{\beta}{4N+2} \sum_{k=1}^{N} \Delta_k^2. \tag{6.82}
\]

By simple algebraic calculations, one gets from (6.64) that

\[
\Delta_{k+1} - \Delta_k = f_2(s_{k+1}^+) - f_2(s_k^+) \geq m_{k+1}R_{k+1} + n_{k+1}T_{k+1} \geq N, \quad (6.83)
\]

for $k \in \{0, \ldots, N-1\}$. From (6.64) one gets also that $\Delta_0 < 0$ and $\Delta_N > 0$, so denoting $L = \min\{k \geq 0 : \Delta_k \geq 0\}$ we have $1 < L \leq N$, $\Delta_k < 0$ for $k < L$, and $\Delta_k \geq 0$ for $k \geq L$. Then using (6.83), we get

\[
\sum_{k=1}^{N} \Delta_k^2 \geq \sum_{k=0}^{L-1} (N(k-L+1))^2 + \sum_{k=L}^{N} (N(k-L))^2
\]

\[
= N^2 \sum_{k=0}^{L-1} k^2 + N^2 \sum_{k=0}^{N-L} k^2 \geq c_1 N^5,
\]

for some $c_1 > 0$. It then follows from (6.82) that

\[
E_{\beta,p}^h[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s] \geq c_2 N^4,
\]
for some $c_2 > 0$. Combining this with (6.75), we get
\[ \mathbb{E}^h_{\beta, p}[(f_2(\xi_{n+1}) - f_2(\xi_n))^2 \mid \xi_n = s] \geq c_3 \max \left\{ N^4, \frac{|s|^4}{N} \right\} \geq c_3 |s|^{16/5}, \]
thus proving the result.

\[ \square \]

Remark 6.4.12. The exponent $16/5$ in Lemma 6.4.11 is the best possible; to see this, one may take a configuration $s$ with $n_1 = m_N = N^{5/4}$ and $n_2 = \cdots = n_N = m_1 = \cdots = m_{N-1} = 1$ and compute the left-hand side of (6.81).

\textbf{Proof of Theorem 6.4.10.} It suffices to take $\beta \in (0, 1)$, since we already know from Theorem 6.4.7 that VM is positive recurrent. First suppose that $p > 1/2$. Then, by Lemma 6.4.4,
\[ \mathbb{E}^h_{\beta, p}[f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] = (1 - \beta)\mathbb{E}^h_{\beta}[(f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s], \]
which is bounded above by $-(1 - \beta)/2$ for all but finitely many $s \in \mathcal{S}$, as shown in the proof of Theorem 6.4.5. Thus positive recurrence follows from Theorem 2.6.2.

Finally, we consider the more delicate case where $p = 1/2$ and $\beta \in (0, 1)$. This time, by the case $p = q = 1/2$ of Lemma 6.4.4,
\[ \mathbb{E}^h_{\beta, 1/2}[f_2(\xi_{n+1}) - f_2(\xi_n) \mid \xi_n = s] = \frac{1 - \beta}{2}. \quad (6.84) \]

Let $\alpha \in (0, 1)$. Similarly to (6.74) we obtain from (6.73) that
\[ \mathbb{E}^h_{\beta, 1/2}[(f_2(\xi_{n+1}))^{\alpha} - (f_2(\xi_n))^{\alpha} \mid \xi_n = s] \]
\[ = (f_2(s))^{\alpha} \mathbb{E}^h_{\beta, 1/2}\left[\left(1 + \frac{f_2(\xi_{n+1}) - f_2(\xi_n)}{f_2(\xi_n)}\right)^{\alpha} - 1 \mid \xi_n = s\right] \]
\[ \leq \frac{(1 - \beta)}{2} (f_2(s))^{\alpha - 1} - c(f_2(s))^{\alpha - 2} |s|^{16/5}, \]
for some $c > 0$, where we have used (6.84) and (6.81). Since $|s|^{16/5} \geq 2^{16/5}(f_2(s))^{16/15}$ by Lemma 6.4.2(ii), we obtain, for $14/15 < \alpha < 1$,
\[ \mathbb{E}^h_{\beta, 1/2}[(f_2(\xi_{n+1}))^{\alpha} - (f_2(\xi_n))^{\alpha} \mid \xi_n = s] \]
\[ \leq \frac{(1 - \beta)}{2} (f_2(s))^{\alpha - 1} - 2^{16/5}c(f_2(s))^{\alpha - (14/15)} \]
\[ = -(f_2(s))^{\alpha - (14/15)}\left[2^{16/5}c - \frac{(1 - \beta)}{2}(f_2(s))^{-1/15}\right] < -1, \]
for all but finitely many $s \in \mathcal{S}$, as follows from Lemma 6.4.2(ii). Applying Theorem 2.6.2, we finish the proof of Theorem 6.4.10. \[ \square \]
Section 6.1

Random walk in random environment (RWRE) was first considered by M.V. Kozlov [183] (following up on “a hypothesis by A. N. Kolmogorov”) and F. Solomon [291] (a student of F.L. Spitzer); subsequently these prototypical models of random processes in random media have been investigated by many authors, see e.g. [269, 319, 320, 300].

The recurrence classification, Theorem 6.1.1, is due to Solomon [291] in the case of RWRE on $\mathbb{Z}$. Many of the papers on one-dimensional RWRE work on $\mathbb{Z}$ rather than $\mathbb{Z}_+$. Obvious differences include the fact that on $\mathbb{Z}$ one may have $X_n \to -\infty$, while in the i.i.d. case this regime corresponds to the positive-recurrent regime for $X_n$ on $\mathbb{Z}_+$. Taking such differences into account, for many questions of interest the distinction is inessential, and the analysis is more-or-less unchanged.

The null-recurrent case in which $\mathbb{E}[\zeta_0^2] < \infty$ and $\mathbb{E}\zeta_0 = 0$ is known as Sinai’s regime after Sinai’s remarkable result [288] that $(\log n)^{-2}X_n$ converges weakly to a non-degenerate limit under the annealed probability measure $\mathbb{P}^\omega$ given by

$$
\mathbb{P}^\omega[\cdot] = \int_{\Omega} \mathbb{P}_\omega[\cdot] \mathbb{P}[d\omega].
$$

Golosov [119] and Kesten [164] identified the limiting distribution in Sinai’s result.

The almost-sure upper bound in Theorem 6.1.2 is due to Deheuvels and Révész (Theorem 4 of [69]), with a different proof. The (shorter) approach presented here follows [58, 235]. The almost-sure lower bound in Theorem 6.1.6 improves on a result of [235] which in turn improved on Theorem 4 of [69]. The proof of Theorem 6.1.6 given here uses the method of [136], which improved on the method of [235]; the key Lemma 6.1.8 is Lemma 4.3 in [136].

Theorems 6.1.2 and 6.1.6 are not optimal; sharp results on the almost-sure behaviour of the RWRE in Sinai’s regime were provided by Hu and Shi [138], making use of some delicate technical estimates involving the potential associated with the random environment:

$$
\limsup_{n \to \infty} \frac{X_n}{(\log n)^2 \log \log \log n} = \frac{8}{\pi^2 \sigma^2}, \quad \mathbb{P}_\omega\text{-a.s. for } \mathbb{P}\text{-a.e. } \omega.
$$

Thus the lower bound in Theorem 6.1.6 is close to sharp, while the upper bound in Theorem 6.1.2 is rougher. The Lyapunov function methods
presented here are not as sharp as those say of [138], but they are simpler and more robust, and can be employed in more general settings where the random environment is not i.i.d., such as the model of Section 6.1.5.

The model and results of Section 6.1.5 are based on [234, 235]. The recurrence classification for the perturbation of Sinai’s regime, Theorem 6.1.10, is contained in Theorems 6 and 7 of [234], which dealt with more general perturbations as well as a random environment constructed by a random perturbation of symmetric simple random walk. The almost-sure bounds in Theorem 6.1.11 are from Theorem 3 of [235]. In the setting of Theorem 6.1.11, using a more delicate analysis, [113] obtained the precise result that for some constant $c$ depending on $a$, $\beta$, and $E[\zeta_0^2]$,

$$\lim_{n \to \infty} \frac{X_n}{(\log n)^{1/\beta}(\log \log n)^{-1/\beta}} = c, \quad \text{P}_\omega\text{-a.s. for P-a.e. } \omega.$$ 

Thus it is the lower bound in Theorem 6.1.11 that is closest to being sharp.

Different behaviour is observed when $E[\zeta_0^2] = \infty$. The natural model in this heavy-tailed setting takes $\zeta_0$ to be in the domain of attraction of a stable law. Singh [289] gave analogues of the almost-sure results of Hu and Shi [138] in the stable law setting. Analogues of Sinai’s weak convergence result were obtained by Schumacher [279, 280] and Kawazu et al. [154]: now $(\log n)^{-\alpha}X_n$ has an annealed weak limit, where $\alpha \in (0, 2]$ is the index of the stable law. In the heavy-tailed setting, Lyapunov function methods were used to study some perturbations of the i.i.d. environment in [136].

Several techniques are available for studying nearest-neighbour RWRE on $\mathbb{Z}$ or $\mathbb{Z}_+$; since it is reversible, explicit calculations are often possible. In this setting the Lyapunov function method often boils down to performing the ‘classical’ calculations. However, as emphasised elsewhere in this book, the Lyapunov function method is particularly useful in non-reversible situations, such as in the context of random strings in random environment discussed in Section 6.2, or for many-dimensional random walks with rare defects [231].

Section 6.2

The presentation and results in this section follow [58]. Gajrat, Malyshev, Menshikov and Pelikh in [110] studied a version of the model in which the quantities $r^{ij}_\ell$, $q^{ij}_\ell$, and $p^{ij}_\ell$ do not depend on $\ell$, but the maximal jump size for the string length may be greater than one. All these models may be interpreted in terms of LIFO (last in, first out) queueing systems, or as random walks on trees: see [110]. We consider here only the one-sided evolution of
the string; two-sided homogeneous strings were studied by Gajrat, Malyshev and Zamyatin in [111].

In addition to Theorem 6.2.3, [58] gave a second sufficient condition for transience: if (N) holds for $A_n$, and $E \log(1/p_{ii}) < \infty$ for all $i \in A$, then $\gamma_1 > 0$ implies transience.

Section 6.3

Billiards models are dynamical systems describing the motion of a particle in a region with reflection rules at the boundaries. Physical motivation originates with dynamics of an ideal gas in the low-density (Knudsen) regime [179]. The random reflection rule in stochastic billiards models is motivated by the fact that the particle is small and the surface off which it reflects has a complicated (rough) microscopic structure. Stochastic billiards models were studied in [56, 90, 232, 57]; we refer to those papers for further references.

The results of this section on stochastic billiards in unbounded planar domains are based on [232]. In [232], in addition to recurrence and transience, almost-sure bounds on the trajectories of the collisions process and the associated continuous-time trajectory were obtained; in [232] it was assumed that the distribution of $\alpha$ be symmetric around 0, which is stronger than our condition $E \tan \alpha = 0$.

As mentioned in [232], it would be of interest to relax the assumption that $\alpha$ is bounded away from $\pm \pi/2$; there are several natural distributions that are supported on the full interval [56, 90, 179]. We expect that if $\alpha$ has distribution on $(-\pi/2, \pi/2)$ with sufficiently light tails at the endpoints (such that in particular $E[\tan^2 \alpha] < \infty$, say), then the results of Section 6.3 should carry through (technically, one would obtain a Lamperti process whose increments were not uniformly bounded but satisfied a moments condition that should still admit the methods of Chapter 3).

Open problem 6.4.13. How does the stochastic billiards model behave when $E[\tan^2 \alpha] = \infty$? This is the case, for example, when $\alpha$ has the natural Lambertian or cosine law of reflection, with a density proportional to $\cos \alpha$ on $(-\pi/2, \pi/2)$. We conjecture that the process is transient if and only if $\gamma > 1/2$ in this case.

Section 6.4

The results of Section 6.4 are based on [17], while some of the exposition is based on [210]. For background on the exclusion process and the voter model, and interacting particle systems in general, see the books [198, 199].
Liggett [197] described the set of invariant measures for the exclusion process on \( \mathbb{Z} \). If \( p = 1/2 \), the invariant measures are convex combinations of the translation-invariant product measures parametrized by the density of particles. If \( p > 1/2 \), the set of invariant measures contains also measures with support in the set of configurations with a finite number of empty sites to the left of the origin and a finite number of particles to the right of it; this fact can be used to establish positive-recurrence when \( p > 1/2 \) (Theorem 6.4.5). Such measures are sometimes called blocking measures because, due to the exclusion rule and the accumulation of particles to the left of the origin, the flux of particles is null. There are also blocking measures for \( p < 1/2 \); they are obtained by the reflection of those mentioned above. When an asymmetric exclusion process \((p \neq 1/2)\) is considered from a random position determined by a so-called second-class particle, a new set of invariant measures arises. They are called shock measures, they have support on configurations with different asymptotic densities to the left and right of the origin; see [102, 100, 75]. See the review paper [101] and the book [199] for an account of properties of these measures and the asymptotic behaviour of the second class particle.

As mentioned above, Theorem 6.4.5 can be deduced from results of Liggett [197]; for \( p \leq 1/2 \) those results say only that \( \text{EP}(p) \) is not positive-recurrent. The transience result Theorem 6.4.6 is from [17].

In the voter model, only one site changes its value at any given time, so the voter model is an example of a spin-flip model. There are only two invariant measures for the voter model on \( \mathbb{Z} \): one supported on the ‘all-0s’ configuration and the other supported on the ‘all-1s’ configuration. A basic tool to prove such results is duality between the voter model and a dual process obtained when one runs the voter model backwards in time. There are two dual processes for the voter model: coalescing random walks and annihilating random walks. See [82] and Chapter V of [198] for accounts on these and many other properties of the voter model.

Theorem 6.4.7 is from [17]. Theorem 6.4.8 is due to [17], where it was shown that \( \mathbb{E}^v[\tau^{(3/2)+\varepsilon}] = \infty \), and [186], where it was established that the boundary case has \( \mathbb{E}^v[\tau^{3/2}] = \infty \) using a generating function argument. The martingale argument based on Example 2.6.6 used in the proof given here is shorter and more in keeping with the spirit of this book.

The hybrid exclusion-voter process was introduced in [17], and Theorems 6.4.9 and 6.4.10 are taken from that paper; some further developments can be found in [144, 297, 210, 186]. The process can also be viewed as a particular case of a model of random grammars [215]. The continuous-time exclusion-voter model can be defined via its infinitesimal generator and con-
structed via a Harris-type graphical construction. The discrete-time process studied in Section 6.4 is naturally embedded in the continuous-time process. We rather study these discrete versions in depth (after all, this book is mainly about discrete-time Markov chains), and then only note that the corresponding results for the continuous-time versions of the processes are either straightforward or may be obtained in an elementary way, as shown in Section 8 of [17].

The voter model has been used to model the spread of opinion through a static population via nearest-neighbour interactions; see, for example [132]. The hybrid model studied here is a natural extension of this model whereby individuals do not have to remain static, but may move by switching places. Alternative motivations, such as from the point of view of competition of species (see e.g. [48]) can also be adapted to the hybrid process. Models consisting of a mixture of a spin-flip dynamics and exclusion dynamics come under the title of reaction-diffusion processes; see e.g. [17] for more discussion on such processes.

The main open problem for the exclusion-voter process is:

Open problem 6.4.14. Give the complete recurrence classification for $\text{HP}(\beta, p)$ for $\beta, p \in [0, 1]$.

It is known from Theorems 6.4.9, 6.4.10 and Theorem 4 of [210] (which is obtained using another Lyapunov function) that the process is recurrent if one of the following holds:

- $\beta > 0$ and $p \geq 1/2$;
- $(1 - p)(1 - \beta) < 1/3$;
- $p < 1/2$ and $\beta \geq 4/7$.

On the other hand, transience is established only for $\beta = 0$ and $p \leq 1/2$ (Theorem 6.4.6). See [210] for some other open problems related to this process.

Further remarks

To conclude this chapter we give bibliographic details for some other applications of Lyapunov function ideas that did not make it into this book.
Queueing theory. The process that tracks the lengths of a system of $d$ queues often gives rise to a random walk in a positive orthant $\mathbb{Z}^d_+$. Depending on the details of the queueing model, the random walk may possess varying degrees of spatial homogeneity. In the book [96] basic applications of the Lyapunov function method to non-critical queueing systems were given. Subsequently, near-critical cases have been treated; for example, polling systems are studied in [225, 208]. Polynomial rates of convergence to stationarity are treated in [308, 309]. There are also applications to some more exotic queueing models, such as the supermarket model [207] or polling systems with randomized service-regime regenerations [209, 206].

Branching random walks. A particle performs a random walk while at the same time producing offspring as in a classical branching process. The natural asymptotic question regards local survival and extinction, i.e., is the origin visited infinitely often by some member of the population? We refer to [233, 203, 227] for approaches to this question via Lyapunov functions. The model can be extended to consider a branching random walk in random environment, and this too has received attention [52, 71, 204, 205, 54, 55].

Self-interacting random walks. There are several models currently popular concerning random walks whose dynamics depends on their past trajectory in some way. For example, reinforced random walks or excited random walks where transition probabilities are functions of nearby occupation times for the process (see e.g. [255]); in this context ideas related to Lamperti’s problem were recently applied in [184]. Other possibilities are non-local interactions with the occupation time distribution, such as interaction mediated by the centre of mass of the previous trajectory [53]; analysis of this model leads to a time-inhomogeneous version of Lamperti’s problem on $\mathbb{R}_+$ in which the mean drift at $x$ at time $n$ is of order $x^{-\beta} - (x/n)$.

Growth and deposition models. Various stochastic growth models have been studied via Lyapunov function and related methods. We mention classes of state-dependent or controlled branching processes [159, 155, 120], cooperative growth processes on graphs [282], and models associated with the growth of crystals [5] or biomolecules [134].

Urn models and related random walks. Urn models are prototypical reinforced processes [255]. In [60] an urn model that can be phrased as a non-homogeneous random walk model on $\mathbb{Z}^2$ was studied by making use of
the connection to Lamperti’s problem. In [223] generalizations of Pólya’s urn model give rise to another time-inhomogeneous version of Lamperti’s problem on $\mathbb{R}_+$ in which the mean drift at $x$ at time $n$ is of order $n^\alpha x^{-\beta}$. 
Chapter 7

Markov chains in continuous time

7.1 Introduction and notation

So far, this book has been devoted to discrete-time stochastic processes. For this final chapter, we switch to continuous time, and present Lyapunov function methods for studying the asymptotic behaviour of continuous-time Markov chains on countable state spaces. We establish general theorems for the recurrence classification of continuous-time Markov chains, and for quantifying recurrence by studying which moments of the passage time exist, analogously to the corresponding results for discrete-time processes from Chapter 2.

In keeping with the theme of this book, we are particularly interested in the near-critical case, when even the discrete-time jump chain embedded in the continuous-time chain has non-trivial behaviour. Continuous-time Markov chains have the additional feature, compared to discrete-time ones, that on each visit to a state they spend a random (exponentially distributed) holding time; in the case where the jumps rates are spatially inhomogeneous and not bounded away from 0 and $\infty$, the continuous-time chain can exhibit the phenomena of explosion (whereby a transient chain makes an infinite number of steps in finite time) or implosion (see Definition 7.1.3 below). These phenomena can also be explored with the help of Lyapunov functions. In particular, we present locally verifiable conditions for explosion or non-explosion.

We adapt the notational conventions introduced in Chapter 2 as follows. Now $\left(\xi_t, t \in \mathbb{R}_+\right)$ will be a time-homogeneous continuous-time Markov chain...
taking values in a countably infinite state-space \( \Sigma \), equipped with its full \( \sigma \)-algebra \( \mathcal{E} = 2^\Sigma \).

Denote by \( \Gamma = (\Gamma_{xy}, x, y \in \Sigma) \) the infinitesimal generator of the Markov chain, that is, the matrix satisfying \( \Gamma_{xy} \geq 0 \) if \( y \neq x \) and \( \Gamma_{xx} = -\gamma_x \), where 
\[
\gamma_x := \sum_{y \in \Sigma \setminus \{x\}} \Gamma_{xy}.
\]
Assume that
\[
0 < \gamma_x < \infty \text{ for all } x \in \Sigma.
\]
(7.1)
(The assumption that \( \sum_y \Gamma_{xy} = 0 \) and \( \gamma_x < \infty \) for all \( x \) is to say that the chain is conservative.) We construct a stochastic matrix \( P = (P_{xy}, x, y \in \Sigma) \) out of \( \Gamma \) by defining
\[
P_{xy} := \begin{cases} \frac{\Gamma_{xy}}{\gamma_x} & \text{if } \gamma_x \neq 0; \\ 0 & \text{if } \gamma_x = 0; \end{cases}
\]
and \( P_{xx} := 0 \). Then \( P \) is the transition matrix of a discrete-time Markov chain \( \tilde{\xi} = (\tilde{\xi}_n, n \in \mathbb{Z}_+) \) on \( \Sigma \) called the jump chain. Throughout this chapter, we assume the following:

\((\text{CT})\) Suppose that (7.1) holds, and that the jump chain is irreducible.

Define a sequence \((\sigma_n, n \in \mathbb{N})\) of random holding times distributed, given \( \tilde{\xi} \), according to an exponential law with parameter \( \gamma_{\tilde{\xi}_{n-1}} \) : for \( n \geq 1 \),
\[
\mathbb{P}[\sigma_n > s \mid \tilde{\xi}] = \exp(-s\gamma_{\tilde{\xi}_{n-1}}), \text{ for } s \geq 0,
\]
so that \( \mathbb{E}[\sigma_n \mid \tilde{\xi}] = \frac{1}{\gamma_{\tilde{\xi}_{n-1}}} \). The sequence \((J_n, n \in \mathbb{Z}_+)\) of random jump times is defined accordingly by \( J_0 := 0 \) and for \( n \geq 1 \) by \( J_n := \sum_{k=1}^{n} \sigma_k \). The life time of the process is
\[
\zeta := \lim_{n \to \infty} J_n = \sum_{k=1}^{\infty} \sigma_k,
\]
and we say that explosion occurs if \( \{\zeta < \infty\} \).

Remark 7.1.1. The parameter \( \gamma_x \) must be interpreted as the proper frequency of the internal clock of the Markov chain multiplicatively modulating the local speed of the chain. If \( \sup_{x \in \Sigma} \gamma_x < \infty \) then it is not hard to see that \( \mathbb{P}[\zeta = \infty] = 1 \) so that the chain is a.s. non-explosive (also called regular). Interesting phenomena are observed if

- \( \sup_x \gamma_x = \infty \): the internal clock ticks unboundedly fast (leading to an unbounded local speed of the chain), or
7.1. Introduction and notation

• \( \inf_x \gamma_x = 0 \): the internal clock ticks arbitrarily slowly (leading to a local speed that can be arbitrarily close to 0).

To have a unified description of both explosive and non-explosive processes, we extend the state space to \( \hat{\Sigma} = \Sigma \cup \{ \partial \} \) by adjoining a special absorbing state \( \partial \). The continuous-time Markov chain is then the process \( (\xi_t, t \in \mathbb{R}_+) \) defined by

\[
\xi_t = \begin{cases} 
\sum_{n \in \mathbb{Z}_+} \tilde{\xi}_n 1 \{ t \in [J_n, J_{n+1}) \} & \text{for } 0 < t < \zeta, \\
\partial & \text{for } t \geq \zeta.
\end{cases}
\]

Note that although no algebraic structure is imposed on the set \( \Sigma \), the above sum is well-defined since for every fixed \( t \) only one term survives.

Under condition (CT), we say that the continuous-time Markov chain \( \xi_t \) is irreducible.

If \( A \in \mathcal{E} \), we denote by \( \tau_A := \inf \{ t \geq 0 : \xi_t \in A \} \) the first hitting time of \( A \). We use the notation \( \mathbb{P}_x[\cdot] = \mathbb{P}_x[\cdot | \xi_0 = x] \) for probabilities conditional on starting the chain at \( x \in \Sigma \); we use \( \mathbb{E}_x \) for the corresponding expectation.

Analogously to the discrete-time case, we define the first passage time of a set \( A \subset \Sigma \) by

\[
\tau^+_A := \inf \{ t \geq J_1 : \xi_t \in A \},
\]

and for \( x \in \Sigma \) we set \( \tau^+_x := \tau^+_{\{x\}} \).

**Definition 7.1.2.** Suppose that (CT) holds. The irreducible Markov chain \( \xi_t \) is called

• **recurrent** if \( \mathbb{P}_x[\tau^+_x < \infty] = 1 \) for all \( x \in \Sigma \);

• **transient** if \( \mathbb{P}_x[\tau^+_x < \infty] < 1 \) for all \( x \in \Sigma \).

A recurrent Markov chain is classified further as

• **positive recurrent** if \( \mathbb{E}_x \tau^+_x < \infty \) for all \( x \in \Sigma \);

• **null recurrent** if \( \mathbb{E}_x \tau^+_x = \infty \) for all \( x \in \Sigma \).

A dual notion to explosion is that of implosion.

**Definition 7.1.3.** Let \( (\xi_t, t \in \mathbb{R}_+) \) be a continuous-time Markov chain on \( \Sigma \) and let \( A \subset \Sigma \) be a proper subset of \( \Sigma \). We say that the Markov chain imploids towards \( A \) if there exists \( K > 0 \) such that \( \mathbb{E}_x \tau_A \leq K \) for all \( x \in \Sigma \setminus A \).
Remark 7.1.4. It will be shown in Proposition 7.4.11 that if $A$ is finite and the chain is irreducible, implosion towards $A$ is equivalent to implosion towards any state. In this situation, we speak about *implosion of the chain*.

We denote the measurable functions on $(\Sigma, \mathcal{E})$ by

$$m\mathcal{E} := \{ f : \Sigma \rightarrow \mathbb{R}, \ f \text{ is } \mathcal{E}\text{-measurable} \},$$

and similarly we use $b\mathcal{E}$ to denote bounded measurable functions, $m\mathcal{E}_+$ to denote non-negative measurable functions, etc. For $f \in m\mathcal{E}_+$ and $\alpha > 0$, we denote by $S_\alpha(f)$ the *sublevel set* of $f$ of height $\alpha$ defined by

$$S_\alpha(f) := \{ x \in \Sigma : f(x) \leq \alpha \}.$$

We recall that a function $f \in m\mathcal{E}_+$ is *unbounded* if $\sup_{x \in \Sigma} f(x) = +\infty$, while $f \rightarrow \infty$ means that for every $n \in \mathbb{N}$ the sublevel set $S_n(f)$ is finite. Measurable functions $f$ defined on $\Sigma$ can be extended to functions $\hat{f}$, defined on $\hat{\Sigma}$, by $\hat{f}(\partial) := 0$ (with obvious extension of the $\sigma$-algebra).

We denote the domain of the generator by

$$\text{Dom}(\Gamma) := \{ f \in m\mathcal{E} : \sum_{y \in \Sigma \setminus \{x\}} \Gamma_{xy}|f(y)| < \infty, \ \text{for all } x \in \Sigma \}, \quad (7.2)$$

and by $\text{Dom}_+(\Gamma)$ we denote the set of non-negative functions in the domain. The action of the generator $\Gamma$ on $f \in \text{Dom}(\Gamma)$ then reads

$$\Gamma f(x) := \sum_{y \in \Sigma} \Gamma_{xy} f(y). \quad (7.3)$$

### 7.2 Recurrence and transience

We have the following criteria for recurrence and transience.

**Theorem 7.2.1.** Suppose that (CT) holds. The following are equivalent.

(a) The chain is recurrent.

(b) There exist a finite non-empty $F \subset \Sigma$ and $f \in \text{Dom}_+(\Gamma)$ satisfying $\Gamma f(x) \leq 0$ for all $x \notin F$, and $f \rightarrow \infty$.

**Theorem 7.2.2.** Suppose that (CT) holds. The following are equivalent.
7.2. Recurrence and transience

(a) The chain is transient.

(b) There exist a (finite or infinite) non-empty \( F \subset \Sigma \) and \( f \in \text{Dom}_+(\Gamma) \) satisfying \( \Gamma f(x) \leq 0 \) for all \( x \not\in F \), and \( f(y) < \inf_{x \in F} f(x) \) for some \( y \not\in F \).

After introducing convenient notation, we will see that these theorems are immediate translations of the discrete-time Foster–Lyapunov criteria of Section 2.5. For \( p > 0 \), we denote the \( p \)-integrable functions by

\[
\ell^p(\Gamma) = \left\{ f \in m\mathcal{E} : \sum_{y \in \Sigma} \Gamma_{xy} |f(y) - f(x)|^p < \infty, \text{ for all } x \in \Sigma \right\};
\]

by \( \ell^p_+(\Gamma) \) we denote the positive \( p \)-integrable functions. Note that \( \ell^1(\Gamma) = \text{Dom}(\Gamma) \) as defined at (7.2). For \( f \in \ell^1(\Gamma) \), we denote the \( f \)-increment of the jump chain by

\[
\Delta_{n+1}^f := \Delta f(\tilde{\xi}_{n+1}) := f(\tilde{\xi}_{n+1}) - f(\tilde{\xi}_n),
\]

the local mean \( f \)-drift by

\[
m_f(x) := \mathbb{E}[\Delta_{n+1}^f \mid \tilde{\xi}_n = x] = \sum_{y \in \Sigma} P_{xy} (f(y) - f(x)) = \mathbb{E}_x \Delta_1^f,
\]

and for \( p \geq 1 \) and \( f \in \ell^p(\Gamma) \) the \( p \)-th moment of the \( f \)-increment by

\[
v_p^f(x) := \mathbb{E}[|\Delta_{n+1}^f|^p \mid \tilde{\xi}_n = x] = \sum_{y \in \Sigma} P_{xy} |f(y) - f(x)|^p = \mathbb{E}_x [||\Delta_1^f||^p].
\]

In terms of the local mean \( f \)-drift, the action of the generator \( \Gamma \) on \( f \in \ell^1(\Gamma) \) given by (7.3) reads

\[
\Gamma f(x) = \Gamma_{xx} f(x) + \sum_{y \in \Sigma \setminus \{x\}} \Gamma_{xy} f(y)
\]

\[
= -\gamma_x f(x) + \sum_{y \in \Sigma \setminus \{x\}} \gamma_x P_{xy} f(y)
\]

\[
= \gamma_x \sum_{y \in \Sigma} P_{xy} (f(y) - f(x)) = \gamma_x m_f(x).
\]

Proof of Theorems 7.2.1 and 7.2.2. Armed with (7.4), we see that the two theorems for the continuous-time Markov chain \( \xi_t \) follow from Theorems 2.5.2 and 2.5.8, respectively, applied to the jump chain \( \tilde{\xi}_n \). \( \Box \)
7.3 Existence and non-existence of moments of passage times

The next result is the following Lyapunov function condition for existence of moments of passage times.

**Theorem 7.3.1.** Suppose that (CT) holds. Let $f \in \text{Dom}_+(\Gamma)$ be such that $f \to \infty$. Suppose that there exist constants $a > 0$, $c > 0$ and $p > 0$ such that $f^p \in \text{Dom}_+(\Gamma)$ and

$$\Gamma f^p(x) \leq -cf^{p-2}(x), \text{ for all } x \not\in S_a(f).$$

Then $E_x[\tau^{q}_{S_a(f)}] < \infty$ for all $q < p/2$ and all $x \in \Sigma$. Moreover, if $p \leq 2$, then $E_x[\tau^{p/2}_{S_a(f)}] < \infty$ as well.

We also have the following result in the other direction.

**Theorem 7.3.2.** Suppose that (CT) holds. Let $f \in \text{Dom}_+(\Gamma)$ be such that $f \to \infty$. Suppose that there exist

(a) constants $a > 0$ and $c_1 > 0$ such that $\Gamma f(x) \geq -c_1$ for $x \not\in S_a(f)$;

(b) constants $c_2 > 0$ and $r > 1$ such that $f^r \in \text{Dom}(\Gamma)$ and $\Gamma f^r(x) \leq c_2 f^{r-1}(x)$ for $x \not\in S_a(f)$; and

(c) a constant $p > 0$ such that $f^p \in \text{Dom}(\Gamma)$ and $\Gamma f^p \geq 0$ for $x \not\in S_a(f)$.

Then $E_x[\tau^{q}_{S_a(f)}] = +\infty$ for all $q > p$ and all $x \not\in S_a(f)$.

**Remark 7.3.3.** The conditions in Theorem 7.3.1 guarantee recurrence of the chain, as well as providing existence of polynomial moments of the hitting time. When $\tau_A$ is integrable, the chain is positive recurrent. In the null recurrent case, $\tau_A$ is almost surely finite but not integrable; nevertheless, some fractional moments $E[\tau^{q}_{A}]$ with $q < 1$ can exist. Similarly, in the positive recurrent case, some higher moments $E[\tau^{q}_{A}]$ with $q > 1$ may fail to exist.

When $p = 2$, Theorem 7.3.1 says that if $\Gamma f(x) \leq -\varepsilon$ for some $\varepsilon > 0$ and for $x$ outside a finite set $F$, then $E_x \tau_F < \infty$ for all $x \in \Sigma$. In this situation, we have the following stronger result, which is the analogue of Foster’s criterion Theorem 2.6.4.

**Theorem 7.3.4.** Suppose that (CT) holds. Suppose that the chain is non-explosive, i.e., $\mathbb{P}[\zeta < \infty] = 0$. The following are equivalent.
7.3. Existence and non-existence of moments of passage times

(a) The chain is positive recurrent.

(b) There exist a triple \((\varepsilon, F, f)\), with \(\varepsilon > 0\), \(F\) a finite non-empty subset of \(\Sigma\) and \(f\) a function in \(\text{Dom}_+ (\Gamma)\) satisfying \(\Gamma f(x) \leq -\varepsilon\) for all \(x \notin F\).

Remarks 7.3.5. (a) To check that the chain is non-explosive, it is sufficient to show that the chain is recurrent, for which, by Theorem 7.2.1, it is sufficient that the \(f\) in part (b) satisfies \(f \rightarrow \infty\).

(b) It is clear that the triple \((\varepsilon, F, f)\) in Theorem 7.3.4 is not uniquely determined. Mostly, it will be possible to choose a function \(f \rightarrow \infty\) and \(F\) as the sublevel set of \(f\) at height \(a\), for some \(a > 0\). Sometimes it will be possible to choose the function \(f\) uniformly bounded; this case will be further considered in Theorem 7.4.12 and leads to implosion. It is also immediate that if \(f\) satisfies the condition \(\Gamma f(x) \leq -\varepsilon\) for \(x \notin F\) then the modified function \(f + c\), where \(c\) is an arbitrary positive constant, also satisfies the same condition. Further, if a function \(f\) satisfies this condition, the function \(g(x) = f(x)1\{x \notin F\}\) satisfies a fortiori the same condition.

The rest of this section is devoted to the proofs of the preceding results.

First we introduce some more notation that we use throughout the rest of this chapter. The natural right-continuous filtration \((\mathcal{F}_t, t \in \mathbb{R}_+)\) is defined by \(\mathcal{F}_t := \sigma(\xi_s, s \leq t)\); similarly \(\mathcal{F}_s := \sigma(\xi_s, s < t)\), and \(\mathcal{F}_n := \sigma(\xi_m, m \leq n)\) for \(n \in \mathbb{N}\). A (possibly infinite) \(\tau \in \mathbb{Z}_+\) is a stopping time relative to \((\mathcal{F}_t, t \in \mathbb{R}_+)\) if \(\{\tau \leq t\} \in \mathcal{F}_t\) for all \(t \in \mathbb{R}_+\). Given a stopping time \(\tau\), we define \(\mathcal{F}_\tau\) to be the set of all \(A \in \mathcal{F}_\infty\) such that \(A \cap \{\tau \leq t\} \in \mathcal{F}_t\) for all \(t \in \mathbb{R}_+\). Similarly,

\[
\mathcal{F}_{\tau -} := \sigma\left(\mathcal{F}_0 \cup \bigcup_{t \geq 0}\{A \cap \{t < \tau\} : A \in \mathcal{F}_t\}\right).
\]

Since it is not hard to show that \(\tau\) is \(\mathcal{F}_{\tau -}\)-measurable, the only information contained in \(\mathcal{F}_{J_{n+1}}\) but not in \(\mathcal{F}_{J_{n+1} -}\) is conveyed by the random variable \(\tilde{\xi}_{n+1}\), i.e., the position to which the chain jumps at the moment \(J_{n+1}\).

Given \(f \in \text{Dom}(\Gamma)\), the process \((X_t, t \in \mathbb{R}_+)\) defined by \(X_t = f(\xi_t)\) is specified by, for \(t < \zeta\),

\[
X_t = f(\xi_t) = \sum_{n=0}^{\infty} f(\tilde{\xi}_n)1\{t \in [J_n, J_{n+1})\} = X_0 + \sum_{n=0}^{\infty} \Delta^{f}_{n+1}1\{J_n \in (0, t]\}.
\]

If there is no explosion, the process \(X_t\) is an \(\mathcal{F}_t\)-semimartingale admitting the decomposition \(X_t = X_0 + M_t + A_t\) (see e.g. [174]), where \(M_t\) is a martingale.
vanishing at 0 and \( A_t \) is the predictable compensator given by

\[
A_t = \int_{[0,t]} \Gamma f(\xi_s)ds = \int_{[0,t]} \Gamma f(\xi_s)ds.
\]

Note that, although not explicitly marked, \( X_t, M_t, \) and \( A_t \) all depend on \( f \).

We use in the sequel also the infinitesimal form of the above decomposition. For any admissible \( f \) we have

\[
dX_t = dM_t + dA_t = dM_t + \Gamma f(\xi_t)dt;
\]

in particular, since \( M_t \) is an \( \mathcal{F}_t \)-martingale,

\[
E[dX_t | \mathcal{F}_t^-] = E[dA_t | \mathcal{F}_t^-] = \Gamma f(\xi_t)dt \tag{7.5}
\]

represents the conditional increment of \( X_t \) as an ordinary differential multiplied by a previsible random factor.

We also recall Itô’s formula, which for a real-valued semimartingale \((S_t, t \in \mathbb{R}_+)\) and any twice continuously differentiable \( g : \mathbb{R} \to \mathbb{R} \), says

\[
dg(S_t) = g'(S_t-)dS^c_t + g(S_t) - g(S_t-), \tag{7.6}
\]

where \( S^c_t \) denotes the continuous part of \( S_t \).

The next result is a continuous-time analogue of Theorem 2.6.2.

**Lemma 7.3.6.** Let \((Y_t, t \in \mathbb{R}_+)\) be an \( \mathbb{R}_+ \)-valued process adapted to a filtration \((\mathcal{G}_t, t \in \mathbb{R}_+)\), and let \( T \) be a stopping time. Suppose that there exists \( \varepsilon > 0 \) such that

\[
E[dY_t | \mathcal{G}_t^-] \leq -\varepsilon dt, \text{ on } \{t \leq T\}.
\]

Then \( E[T | \mathcal{G}_0] \leq \varepsilon^{-1}Y_0 \).

**Proof.** Since \( \{s \leq T\} \in \mathcal{G}_{s-} \), the hypothesis of the lemma can be written as

\[
E[dY_{s\wedge T} | \mathcal{G}_{s-}] \leq -\varepsilon 1\{T \geq s\}ds.
\]

Taking expectations conditional on \( \mathcal{G}_0 \) and integrating over \( s \in [0,t] \) yields

\[
0 \leq E[Y_{t\wedge T} | \mathcal{G}_0] \leq Y_0 - \varepsilon \int_0^t P[T \geq s | \mathcal{G}_0]ds, \text{ for all } t \in \mathbb{R}_+.
\]

Taking \( t \to \infty \) it follows from monotone convergence that

\[
0 \leq Y_0 - \varepsilon \int_0^\infty P[T \geq s | \mathcal{G}_0]ds = Y_0 - \varepsilon E[T | \mathcal{G}_0],
\]

which gives the result. \( \square \)
We can now present the proof of Theorem 7.3.4.

**Proof of Theorem 7.3.4.** We assume that $\mathbb{P}[\zeta = \infty] = 1$, i.e., there is no explosion. First, we prove that (b) implies (a). Without loss of generality, using Remark 7.3.5, possibly modifying $f$, we can always assume that $a_0 := \inf_{x \notin F} f(x) > 0$ and $f(x) = 0$ for all $x \in F$. Choose then an arbitrary $c \in (0, a_0)$. Let $X_t = f(\xi_t)$ and recall that $\tau_F = \inf\{t \geq 0 : \xi_t \in F\}$. Then using (7.5) condition (b) can be expressed as

$$
\mathbb{E}[dX_t | F_{t-}] = \Gamma_f(\xi_{t-}) dt \leq -\epsilon dt, \text{ on } \{t \leq \tau_F\}.
$$

Hence we may apply Lemma 7.3.6 with $Y_t = X_t$, $G_t = F_t$ and $T = \tau_F$ to get $\mathbb{E}_x \tau_F \leq \epsilon^{-1} f(x) < \infty$ for every $x \notin F$; clearly $\mathbb{E}_x \tau_F$ is also finite for $x \in F$.

It follows that $\xi_t$ is positive-recurrent by the continuous-time analogue of Lemma 2.6.1 (we omit the details).

Now, we prove that (a) implies (b). Let $F = \{z\}$ for some fixed $z \in \Sigma$; positive recurrence of the chain implies that $\mathbb{E}_x \tau_F < \infty$ for all $x \in \Sigma$. Define

$$
f(x) = \begin{cases} 
\mathbb{E}_x \tau_F & \text{if } x \notin F, \\
0 & \text{if } x \in F.
\end{cases}
$$

Then, for all $x \notin F$,

$$
m_f(x) = \sum_{y \neq z} P_{xy} \mathbb{E}_y \tau_F - \mathbb{E}_x \tau_F = \mathbb{E}_x[\tau_F - \sigma_1] - \mathbb{E}_x \tau_F = -\mathbb{E}_x \sigma_1 = -\frac{1}{\gamma_x}.
$$

It follows that $\Gamma f(x) = \gamma_x m_f(x) = -1$ for $x \notin F$. By adding the constant 1 to the function $f$ determined above (see Remark 7.3.5), we see that $f$ meets all the requirements.

Next we work towards the proof of Theorem 7.3.1. We proceed via two technical results that deal separately with the cases $p \geq 2$ and $p < 2$, analogously to the proof of Theorem 2.7.1.

**Lemma 7.3.7.** Let $f \in \text{Dom}_+(\Gamma)$ with $f \to \infty$, $p \geq 2$, and $a > 0$. Denote $X_t = f(\xi_t)$ and assume further that $f^p \in \text{Dom}_+(\Gamma)$. Suppose that there exists $c > 0$ such that

$$
\Gamma f^p(x) \leq -cf^{p-2}(x), \text{ for all } x \notin S_a(f).
$$

Then there exists $C \in \mathbb{R}_+$ such that for all $q \in [0, p/2]$,

$$
\mathbb{E}_x[\tau_{S_a(f)}^q] \leq C f(x)^{2q}, \text{ for all } x \in \Sigma.
$$
Proof. Use the abbreviation $A := S_{a_i}(f)$. We first show that under the hypotheses of the lemma, the process $(Z_t, t \in \mathbb{R}_+)$ defined by

$$Z_t = \left( X_{t \wedge \tau_A}^2 + \frac{c}{p/2} (t \wedge \tau_A) \right)^{p/2}$$

is a non-negative supermartingale. Introducing the predictable decomposition

$$1 = 1[t > \tau_A] + 1[t \leq \tau_A],$$

we get

$$E[dZ_t | \mathcal{F}_{t-}] = E[d\left( X_{t-}^2 + \frac{c}{p/2} t \right)^{p/2} | \mathcal{F}_{t-}] 1[t \leq \tau_A].$$

Itô’s formula (7.6) applied to $S_t = X_t^2 + \frac{c}{p/2} t$ with $g(x) = x^{p/2}$ yields

$$d\left( X_t^2 + \frac{c}{p/2} t \right)^{p/2} = c\left( X_{t-}^2 + \frac{c}{p/2} t \right)^{(p/2)-1} dt \left( X_{t-}^2 + \frac{c}{p/2} t \right)^{p/2} - \left( X_{t-}^2 + \frac{c}{p/2} t \right)^{p/2}.$$

Writing the semimartingale decomposition for the process $X_t^p$, similarly to (7.5), we note that the hypothesis of the lemma implies that

$$E[dX_t^p | \mathcal{F}_{t-}] = \Gamma f(\xi_{t-})^p dt \leq -cX_{t-}^{p-2} dt, \text{ on } \{t \leq \tau_A\}. \quad (7.8)$$

Since $p \geq 2$, we may apply the conditional Minkowski inequality to obtain

$$E\left[ \left( X_t^2 + \frac{c}{p/2} t \right)^{p/2} \right] \leq \left( E[X_t^p | \mathcal{F}_{t-}] \right)^{2/p} \left( E[\left( X_{t-}^2 + \frac{c}{p/2} t \right)^{p/2}] \right)^{2/p} \leq \left( (X_{t-}^p - cX_{t-}^{p-2} dt)^{2/p} + \frac{c}{p/2} t \right)^{p/2},$$

on $\{t \leq \tau_A\}$, by (7.8). Hence, on $\{t \leq \tau_A\}$,

$$E\left[ \left( X_t^2 + \frac{c}{p/2} t \right)^{p/2} \right] \leq \left( X_{t-}^2 \left( 1 - \frac{c}{p/2} X_{t-}^2 dt \right)^{2/p} + \frac{c}{p/2} t \right)^{p/2} \leq \left( X_{t-}^2 \left( 1 - \frac{c}{p/2} \frac{1}{X_{t-}^2} dt \right) + \frac{c}{p/2} t \right)^{p/2},$$

using the inequality $(1 - y)^r \leq 1 - ry$ for $r \in [0, 1]$ and all $y \geq 0$. Hence

$$E\left[ \left( X_t^2 + \frac{c}{p/2} t \right)^{p/2} \right] \leq \left( X_{t-}^2 - \frac{c}{p/2} dt + \frac{c}{p/2} t \right)^{p/2}$$
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\[
\left( X_t^2 + \frac{c}{p/2} t \right)^{p/2} - c \left( X_t^2 + \frac{c}{p/2} t \right)^{(p/2)-1} dt,
\]

by Taylor’s formula. Thus taking expectations in (7.7), we obtain \( \mathbb{E}[dZ_t \mid \mathcal{F}_{t-}] \leq 0 \). Hence \( Z_t \) is a supermartingale as claimed.

To prove the statement in the lemma, it suffices to suppose that \( x \notin A \).

Since the function \( x \mapsto x^{2q/p} \) is concave for \( q \in [0, p/2] \), we have that the supermartingale property holds for \( Z_t^{2q/p} \) as well as for \( Z_t \). Since \( X_t \) is non-negative,

\[
\frac{c}{p/2} \mathbb{E}_x[(t \wedge \tau_A)^q] \leq \mathbb{E}_x[Z_t^{2q/p}] \leq \mathbb{E}_x[Z_0^{2q/p}] = f(x)^{2q}.
\]

Fatou’s lemma now completes the proof. \( \square \)

**Lemma 7.3.8.** Let \( f \in \text{Dom}_+(\Gamma) \) with \( f \to \infty \), \( 0 < p \leq 2 \), and \( a > 0 \). Denote \( X_t = f(\xi_t) \) and assume further that \( f^p \in \text{Dom}_+(\Gamma) \). Suppose that there exists \( c > 0 \) such that

\[
\Gamma f^p(x) \leq -cf^{p-2}(x), \text{ for all } x \notin S_a(f).
\]

Then for any \( q \in [0, p/2) \) there is a constant \( C \in \mathbb{R}_+ \) such that \( \mathbb{E}_x[\tau_{S_a(f)}^q] \leq Cf(x)^p \) for all \( x \in \Sigma \).

**Proof.** Let \( q \in [0, p/2) \). Again use the abbreviation \( A := S_a(f) \). We first show that under the hypotheses of the lemma, the process \( (Z_t, t \in \mathbb{R}_+) \) defined by

\[
Z_t = X_t^p + \frac{c}{q}(1 + (t \wedge \tau_A))^q
\]

satisfies for some \( C \in \mathbb{R}_+ \),

\[
\mathbb{E}_x Z_t \leq Cf(x)^p, \text{ for all } x \in \Sigma. \tag{7.9}
\]

Once more, the hypothesis of the lemma implies that (7.8) holds. Then since \( d(X_t^p + \frac{c}{q}(1 + t)^q) = dX_t^p + c(1 + t)^{q-1}dt \), we have

\[
\mathbb{E}[dZ_t \mid \mathcal{F}_{t-}] \leq c(-X_t^{p-2} + (1 + t)^{q-1})dt, \text{ on } \{ t \leq \tau_A \}.
\]

Since \( t \leq \tau_A \) implies \( X_t \geq a \), we have that, on \( \{ t \leq \tau_A \} \),

\[
\mathbb{E}[dZ_t \mid \mathcal{F}_{t-}] \leq c(-X_t^{p-2} + (1 + t)^{q-1})1\{X_t \geq a, (1 + t)^{1/2}\}dt
\]

\[
+ c(-X_t^{p-2} + (1 + t)^{q-1})1\{(1 + t)^{1/2}, \infty\})dt.
\]
Here we note that if $X_{t-} \leq (1 + t)^{1/2}$,

$$-X_{t-}^{p-2} + (1 + t)^{q-1} \leq -(1 + t)^{(p/2) - 1} + (1 + t)^{q-1} \leq 0,$$

for all $t \geq 0$, since $0 < q < p/2 \leq 1$. Hence

$$\mathbb{E}_x [dZ_t] \leq c(1 + t)^{q-1} \mathbb{P}_x [X_{\tau_A \wedge t-} \geq (1 + t)^{1/2}] dt$$

$$\leq c(1 + t)^{q-1-(p/2)} \mathbb{E}_x [X_{\tau_A \wedge t-}^p],$$

by Markov’s inequality. A simple consequence of the hypothesis of the lemma is that $\mathbb{E}[dX_{\tau_A \wedge t}] \leq 0$, so that $\mathbb{E}_x [X_{\tau_A \wedge t-}^p] \leq \mathbb{E}_x [X_0^p] = f(x)^p$. Hence

$$\mathbb{E}_x [dZ_t] \leq cf(x)^p (1 + t)^{q-1-(p/2)}.$$

Integrating this differential inequality yields

$$\mathbb{E}_x Z_t \leq cf(x)^p \int_0^\infty (1 + t)^{q-1-(p/2)} dt;$$

the condition $q < p/2$ ensures the finiteness of the last integral, which establishes the claim (7.9). Since $X_t$ is non-negative, we obtain from (7.9) that

$$\frac{c}{q} \mathbb{E}_x [(t \wedge \tau_A)^q] \leq \mathbb{E}_x Z_t \leq Cf(x)^p;$$

Fatou’s lemma completes the proof. \(\square\)

Theorem 7.3.1 is now immediate from the preceding two lemmas.

**Proof of Theorem 7.3.1.** The statement is a combination of Lemmas 7.3.7 and 7.3.8. \(\square\)

Now we turn to the proof of Theorem 7.3.2 on conditions for some passage-time moments to be infinite. We first need the following crucial estimate, which is a continuous-time analogue of Lemma 2.7.5. Lemma 7.3.9 serves twice in this chapter: once on the road to Theorem 7.3.2 and once more in a different context, namely for finding conditions for non-implosion (Theorem 7.4.13).

**Lemma 7.3.9.** Let $(Y_t, t \in \mathbb{R}_+)$ be an $\mathbb{R}_+$-valued process adapted to a filtration $(\mathcal{G}_t, t \in \mathbb{R}_+)$. Let $a > 0$ and $T_a := \inf \{t \geq 0 : Y_t \leq a\}$. Suppose that there exist constants $c_1 > 0$, $c_2 > 0$, and $r > 1$ such that

(a) $\mathbb{E}[dY_t \mid \mathcal{G}_{t-}] \geq -c_1 dt$ on $\{t \leq T_a\}$; and
(b) $E[dY_t^r \mid G_{t-}] \leq c_2 Y_t^{r-1} dt$ on $\{t \leq T_\alpha\}$. 

Then, for all $\alpha \in (0,1)$, there exist $\varepsilon > 0$ and $\delta > 0$ such that, for all $t \geq 0$,

$$P[T_\alpha > t + \varepsilon Y_t \wedge T_\alpha \mid G_t] \geq 1 - \alpha,$$

on $\{T_\alpha > t, Y_t > a(1 + \delta)\}$.

Proof. Fix $t \in \mathbb{R}_+$ and let $\sigma = (T_\alpha - t)^+$; then for all $s > 0$ we have $\{\sigma > s\} = \{T_\alpha > t + s\} \in G_{t+s}$. To prove the lemma, it is enough to establish $P[\sigma > \varepsilon Y_t \mid G_t] \geq 1 - \alpha$ on $\{\sigma > 0, Y_t > a(1 + \delta)\}$. Observe that

$$P[\sigma > \varepsilon Y_t \mid G_t] = P[Y_{t+(\varepsilon Y_t)} > a \mid G_t], \text{ on } \{\sigma > 0, Y_t > a(1 + \delta)\}.$$ 

Write $U_t := Y_{t+(\varepsilon Y_t)}$. Let $r > 1$ be as given in condition (b) of the lemma. Then, by Hölder’s inequality,

$$E[U_t \mid G_t] = E[U_t 1\{U_t \leq a\} \mid G_t] + E[U_t 1\{U_t > a\} \mid G_t] \leq a + (E[U_t^r \mid G_t])^{1/r} (P[U_t > a \mid G_t])^{1-(1/r)},$$ 

therefore

$$P[U_t > a \mid G_t] \geq \left(\frac{(E[U_t \mid G_t] - a)^+}{(E[U_t^r \mid G_t])^{1/r}}\right)^{r/(r-1)}. \tag{7.10}$$

For a lower bound on the numerator on the right-hand side of (7.10),

$$E[U_t \mid G_t] = E[Y_{t+(\varepsilon Y_t)\wedge \sigma} - Y_t \mid G_t] + Y_t$$

$$= \int_t^{t+(\varepsilon Y_t)\wedge \sigma} E[dY_s \mid G_t] + Y_t \geq -c_1 \varepsilon Y_t + Y_t,$$

by hypothesis (a). Obtaining an upper bound on the denominator on the right-hand side of (7.10) requires more work. We obtain an upper bound for $E[Y_{t+s\wedge \sigma}^r \mid G_t]$ for arbitrary $s > 0$. Let $\tau \in (0, \infty)$ be a $G_t$-measurable random variable. For $c_3 = c_2/r$ and any $s \in (0, \tau]$, define

$$F_r(s) = E[(Y_{t+s\wedge \sigma} + c_3 \tau - c_3(s \wedge \sigma))^r \mid G_t].$$

We shall show that $F_r(s) \leq F_r(s-)$ for all $s \in (0, \tau]$. It is enough to show this inequality on $\{s \leq \sigma\}$ since otherwise $F_r(s) = F_r(s-)$ and there is nothing to prove. To show that $F_r$ is decreasing in $(0, \tau]$, it is enough to show that $E[dZ_s \mid G_{t+s-}] \leq 0$ for all $s \in (0, \tau]$, where $Z_s = (Y_{t+s\wedge \sigma} + c_3 \tau - c_3(s \wedge \sigma))^r$. 

Now, on $\{s \leq \sigma\}$, Itô’s formula (7.6) yields

$$dZ_s = -rc_3(Y_{t+s-} + c_3 \tau - c_3 s)^{r-1} ds + (Y_{t+s} + c_3 \tau - c_3 s)^r - (Y_{t+s-} + c_3 \tau - c_3 s)^r.$$
Moreover, using Minkowski’s inequality, we get
\[
E[(Y_{t+s} + c_3 \tau - c_3 s)^r \mid \mathcal{G}_{t+s-}] \leq \left( E[Y_{t+s}^r \mid \mathcal{G}_{t+s-}]^{1/r} + c_3 \tau - c_3 s \right)^r,
\]
using the fact that \( \tau \) is \( \mathcal{G}_t \)-measurable. Also, by hypothesis (b),
\[
E[Y_{t+s}^r \mid \mathcal{G}_{t+s-}] \leq Y_{t+s-}^r + c_2 Y_{t+s-}^{r-1} 1\{s \leq \sigma\} ds
\]
\[
= Y_{t+s-}^r \left( 1 + \frac{c_2}{Y_{t+s-}} 1\{s \leq \sigma\} ds \right)
\]
\[
\leq (Y_{t+s-} + c_3 1\{s \leq \sigma\} ds)^r,
\]
using the elementary inequality \( 1 + y \leq (1 + r^{-1}y)^r \) for all \( r > 1 \) and all \( y > 0 \). Therefore,
\[
E[(Y_{t+s} + c_3 \tau - c_3 s)^r \mid \mathcal{G}_{t+s-}] \leq (Y_{t+s-} + c_3 ds + c_3 \tau - c_3 s)^r
\]
\[
= (Y_{t+s-} + c_3 \tau - c_3 s)^r + rc_3 (Y_{t+s-} + c_3 \tau - c_3 s)^{r-1} ds.
\]
It follows that, for all \( s \in (0, \tau] \), \( E[dZ_s \mid \mathcal{G}_{t+s-}] \leq 0 \). So, for all \( \tau > 0 \),
\[
E[Y_{t+\tau}^r \mid \mathcal{G}_t] = F_\tau(\tau) \leq \lim_{s \to 0^+} F_\tau(s) = (Y_t + c_3 \tau)^r, \text{ on } \{\tau \leq \sigma\}.
\]
Choosing \( \tau = \varepsilon Y_t \), we get, on \( \{\varepsilon Y_t \leq \sigma\} \),
\[
(E[Y_{t+(\varepsilon Y_t) \wedge \sigma}^r \mid \mathcal{G}_t])^{1/r} \leq Y_t + c_3 \varepsilon Y_t.
\]
On the other hand, on \( \{\varepsilon Y_t > \sigma\} \cap \{Y_t > a(1 + \delta)\} \),
\[
(E[Y_{t+(\varepsilon Y_t) \wedge \sigma}^r \mid \mathcal{G}_t])^{1/r} \leq a \leq Y_t + c_3 \varepsilon Y_t.
\]
Thus it follows from (7.10) that, on \( \{Y_t > a(1 + \delta)\} \),
\[
P[U_t > a \mid \mathcal{G}_t] \geq \left( \frac{(1 - c_1 \varepsilon - Y_t^{-1}a)^+}{1 - c_3 \varepsilon} \right)^{r/(r-1)}
\]
\[
\geq \left( \frac{(1 - c_1 \varepsilon - (1 + \delta)^{-1} a)^+}{1 - c_3 \varepsilon} \right)^{r/(r-1)},
\]
which exceeds any \( 1 - \alpha, \alpha \in (0, 1) \), for suitable \( \varepsilon > 0 \) and \( \delta > 0 \).

**Lemma 7.3.10.** Let \( (Y_t, t \in \mathbb{R}_+) \) be an \( \mathbb{R}_+ \)-valued process adapted to a filtration \( (\mathcal{G}_t, t \in \mathbb{R}_+) \). Let \( a > 0 \) and \( T_a := \inf\{t \geq 0: Y_t \leq a\} \). Suppose that there exist positive constants \( a, c_1, c_2, r, p \) such that
7.4 Explosion and implosion

(a) $Y_0 = y > a$;
(b) $\mathbb{E}[dY_t \mid \mathcal{G}_t^-] \geq -c_1 dt$ on $\{t \leq T_a\}$;
(c) $\mathbb{E}[dY^p_{t} \mid \mathcal{G}_t^-] \leq c_2 Y^p_{t-1} dt$ on $\{t \leq T_a\}$;
(d) $(Y^p_{t \wedge T_a})$ is a submartingale.

Then $\mathbb{E}[T^q_a] = \infty$ for all $q > p$.

Proof. It suffices to suppose that $\mathbb{P}[T_a < \infty] = 1$. Let $q > p$. Suppose, for the purposes of deriving a contradiction, that $\mathbb{E}[T^q_a] < \infty$. Under the conditions of the lemma, Lemma 7.3.9 applies, and taking $\alpha = 1/2$ there shows that there exist positive constants $\varepsilon$ and $\delta$ such that $\mathbb{P}[T_a > t + \varepsilon Y_{t \wedge T_a} \mid \mathcal{G}_t] \geq 1/2$, on $\{Y_{t \wedge T_a} > a(1 + \delta)\}$, for all $t \in \mathbb{R}_+$. Hence

$$
\mathbb{E} T^q_a \geq \mathbb{E}\left[\mathbb{E}[T^q_a1\{T_a > t + \varepsilon Y_{t \wedge T_a}\} \mid \mathcal{G}_t]1\{Y_{t \wedge T_a} > a(1 + \delta)\}\right]
\geq \frac{1}{2} \mathbb{E}\left[(\varepsilon Y_{t \wedge T_a})^q 1\{Y_{t \wedge T_a} > a(1 + \delta)\}\right]
\geq \frac{\varepsilon^q}{2} \mathbb{E}[Y^q_{t \wedge T_a}] - \frac{\varepsilon^q}{2} a^q (1 + \delta)^q.
$$

Thus under the assumption $\mathbb{E}[T^q_a] < \infty$, re-arranging the preceding inequality shows that there exists a finite constant $K_1$ such that $\mathbb{E}[Y^q_{t \wedge T_a}] \leq K_1$ for all $t \in \mathbb{R}_+$. Hence $Y^q_{t \wedge T_a}$ is uniformly integrable; by hypothesis (d) it is a submartingale. Hence $\lim_{t \to \infty} \mathbb{E}[Y^q_{t \wedge T_a}] = \mathbb{E}[Y^q_{T_a}] \leq a^q$. On the other hand, the submartingale property shows that $\mathbb{E}[Y^q_{t \wedge T_a}] \geq \mathbb{E}[Y^q_0] = y^q$. Choosing $Y_0 = y > a$ leads to the desired contradiction.

Proof of Theorem 7.3.2. On identifying $Y_t$ in Lemma 7.3.10 with $f(\xi_t)$, we see that the conditions of the theorem imply the hypotheses of the lemma. The non-existence of moments immediately follows.

7.4 Explosion and implosion

The classical criterion for explosion due to Chung [45] is in terms of the jump chain and states

$$
\mathbb{P}[\zeta < \infty] = 1 \text{ if and only if } \sum_{n=0}^{\infty} \gamma_{\xi_n}^{-1} < \infty. \quad (7.11)
$$
Chapter 7. Markov chains in continuous time

This condition is usually difficult to check since it is global, i.e., requires the knowledge of the entire trajectory of the embedded Markov chain. The purpose of this section is to give some conditions whose validity can be verified by local estimates. The first result gives a Lyapunov function criterion for the explosion time to be integrable; it is worth noting that although explosion can only occur in the transient case, the result is strongly reminiscent of Foster’s criterion in discrete time (Theorem 2.6.4).

Theorem 7.4.1. Suppose that (CT) holds. The following are equivalent.

(a) The explosion time \( \zeta \) satisfies \( \mathbb{E}_x \zeta < \infty \) for all \( x \in \Sigma \).

(b) There exist \( f \in \text{Dom}_+ (\Gamma) \) strictly positive, a finite set \( A \subseteq \Sigma \), and \( \varepsilon > 0 \) such that \( \Gamma f(x) \leq -\varepsilon \) for all \( x \notin A \), and either (i) \( A = \emptyset \); or (ii) the chain is transient.

Remarks 7.4.2. (a) If \( A = \emptyset \), then the function \( f \) in condition (b) cannot have \( f \to \infty \), or else Theorem 7.4.9 below would imply non-explosion; note however, that \( f \) need not be uniformly bounded.

(b) By Theorem 7.2.2, we may replace the transience condition in (b)(ii) by the condition \( f(y) < \inf_{x \in A} f(x) \) for some \( y \notin A \).

The following result will be shown to follow from Theorem 7.4.1.

Proposition 7.4.3. Suppose that (CT) holds. Let \( f \in \text{Dom}_+ (\Gamma) \) be bounded and strictly positive, and denote \( b = \sup_{x \in \Sigma} f(x) \). Suppose that there exist a finite set \( A \subseteq \Sigma \) and a non-decreasing function \( g : \mathbb{R}_+ \to (0, \infty) \) such that \( \int_0^b \frac{du}{g(u)} < \infty \), and \( \Gamma f(x) \leq -g(f(x)) \) for all \( x \notin A \). Then \( \mathbb{E}_x \zeta < \infty \) for all \( x \in \Sigma \).

Remark 7.4.4. The interest of Proposition 7.4.3 is in the case \( \inf_{x \in \Sigma} g(f(x)) = 0 \), because then the hypothesis is weaker than the requirement of Theorem 7.4.1(b) that \( \Gamma f(x) \leq -\varepsilon \).

The following result permits an infinite exceptional set, and, roughly speaking, gives a condition for explosion to occur in a direction in which the Lyapunov function \( f \) tends to zero.

Theorem 7.4.5. Suppose that (CT) holds. Suppose that there exist an increasing sequence \( E_1 \subseteq E_2 \subseteq \cdots \) of (finite or infinite) sets in \( \mathcal{E} \) with \( \cup_n E_n = \Sigma \), and a function \( f \in \text{Dom}_+ (\Gamma) \), such that

(a) for each \( n \in \mathbb{N} \), \( \sigma_n := \inf \{ t \geq 0 : \xi_t \notin E_n \} \) is finite a.s.;

(b) \( \sup_{x \in \Sigma} f(x) < \infty \), \( \inf_{x \in E_n} f(x) > 0 \), and \( \lim_{n \to \infty} \sup_{x \notin E_n} f(x) = 0 \);
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(c) for some $\varepsilon > 0$, $\Gamma f(x) \leq -\varepsilon$ for all $x \notin E_1$.

Then $\mathbb{P}_x[\zeta < \infty] = 1$ for all $x \in \Sigma$.

If $\mathbb{P}_x[\zeta < \infty] > 0$ for some $x \in \Sigma$, then irreducibility of the chain implies that $\mathbb{P}_x[\zeta < \infty] > 0$ for all $x$, i.e., explosion can occur from any starting state. However, we may have (see Example 7.4.8) $0 < \mathbb{P}_x[\zeta < \infty] < 1$. Additionally, the results in this section establish conditions that guarantee $\mathbb{E}_x \zeta < \infty$, implying explosion, but we may have (see Example 7.4.10) $\mathbb{P}_x[\zeta < \infty] = 1$ but $\mathbb{E}_x \zeta = \infty$. The following conditional explosion result is useful in such circumstances.

**Theorem 7.4.6.** Suppose that (CT) holds. Suppose that there exists a triple $(\varepsilon, A, f)$ with $\varepsilon > 0$, $A$ a proper (finite or infinite) subset of $\Sigma$ such that $\Sigma \setminus A$ is infinite, and $f \in \text{Dom}_+(\Gamma)$, such that

(a) there exists $x_0 \notin A$ with $f(x_0) < \inf_{x \in A} f(x)$;

(b) $\Gamma f(x) \leq -\varepsilon$ for all $x \notin A$.

Then $\mathbb{P}_{x_0}[\tau_A = \infty] > 0$ and $\mathbb{E}_{x_0}[\zeta \mid \tau_A = \infty] < \infty$; in particular, $\mathbb{P}_{x_0}[\zeta < \infty] > 0$, so explosion occurs with positive probability.

**Remark 7.4.7.** Note that if the conditions of Theorem 7.4.6 hold, then the chain is necessarily transient (cf. Theorem 7.2.2). If the set $A$ is finite, then the conditions of Theorem 7.4.6 imply that $\mathbb{E} \zeta < \infty$, by Theorem 7.4.1.

**Example 7.4.8.** Consider a nearest-neighbour random walk on $\mathbb{Z}$ with jump rates given for $x \leq 0$ by $\Gamma_{x-1} = 2$ and $\Gamma_{x+1} = 1$, and for $x > 0$ by $\Gamma_{xx} = \alpha x$ and $\Gamma_{x+1} = 2\alpha x$.

Let $A = \{0, -1, -2, \ldots\}$ and define $f : \mathbb{Z} \to \mathbb{R}_+$ by $f(x) = 1$, $x \leq 0$ and $f(x) = x^{-\alpha}$, $x \geq 1$, where $\alpha > 0$. Then for $x \geq 1$,

$$
\Gamma f(x) = 2x^2((x+1)^{-\alpha} - x^{-\alpha}) + x^2((x-1)^{-\alpha} - x^{-\alpha})
= 2x^2x^{-\alpha}(-\alpha x^{-1} + O(x^{-2})) + x^2x^{-\alpha}(\alpha x^{-1} + O(x^{-2}))
= -\alpha x^{1-\alpha} + O(x^{-\alpha}).
$$

Thus for $x \notin A$ we have $\Gamma f(x) \leq -\varepsilon$ provided $\alpha \in (0, 1]$. Thus an application of Theorem 7.4.6 shows that, for any $x \geq 1$, $\mathbb{E}_x[\zeta \mid \tau_A = \infty] < \infty$ and $\mathbb{P}_x[\zeta < \infty] > 0$, so explosion occurs with positive probability.

On the other hand, it may be shown that $\mathbb{P}_x[\zeta < \infty] < 1$; we sketch the argument. Indeed, we have $\mathbb{P}_x[\zeta_n \to -\infty] > 0$ for any $x$, and if the process is transient to $-\infty$ it will not explode, as shown by Chung’s criterion (7.11) and the fact that the rates in that direction are uniformly bounded. △
A sufficient condition for non-explosion is the next result.

**Theorem 7.4.9.** Suppose that (CT) holds. Let \( f \in \text{Dom}_+(\Gamma) \) with \( f \to \infty \). Suppose also that

(a) there exists a non-decreasing function \( g: \mathbb{R}_+ \to \mathbb{R}_+ \) for which \( G(z) := \int_0^z \frac{dy}{g(y)} < \infty \) for all \( z \in \mathbb{R}_+ \) but \( \lim_{z \to \infty} G(z) = \infty \); and

(b) \( \Gamma f(x) \leq g(f(x)) \) for all \( x \in \Sigma \).

Then \( \mathbb{P}_x[\zeta = \infty] = 1 \) for all \( x \in \Sigma \).

**Example 7.4.10.** Consider a nearest-neighbour random walk on \( \mathbb{Z} \) with jump rates given for \( x \leq 0 \) by \( \Gamma_{x-1} = \Gamma_{x+1} = 1 \), and for \( x > 0 \) by \( \Gamma_{x-1} = x^\beta \) and \( \Gamma_{x+1} = 2x^\beta \), where \( \beta \in \mathbb{R} \) is a fixed parameter.

First we show that explosion does not occur if \( \beta \leq 1 \). Define \( f: \mathbb{Z} \to \mathbb{R}_+ \) by \( f(x) = 1 + |x| \). Then, \( \Gamma f(0) = 1 + 1 = 2 \leq 2f(0) \), while, for \( x \leq -1 \),

\[
\Gamma f(x) = (|x| + 1 - |x|) + (|x| - 1 - |x|) = 0 \leq 2f(x),
\]

and, for \( x \geq 1 \),

\[
\Gamma f(x) = 2x^\beta(|x| + 1 - |x|) + x^\beta(|x| - 1 - |x|) = |x|^\beta \leq 2f(x),
\]

provided \( \beta \leq 1 \). Thus an application of Theorem 7.4.9 shows that, if \( \beta \leq 1 \), \( \mathbb{P}_x[\zeta = \infty] = 1 \) for all \( x \in \mathbb{Z} \).

Next, we show that explosion is certain if \( \beta > 1 \). Let \( E_n = (-\infty, n] \cap \mathbb{Z} \). Then \( \sigma_n = \inf\{t \geq 0 : \zeta_t \notin E_n\} < \infty \) for each \( n \). Consider the Lyapunov function \( f: \mathbb{Z} \to (0, 1] \) given by \( f(x) = 1 \), \( x \leq 0 \) and \( f(x) = x^{-\alpha} \), \( x \geq 1 \), where \( \alpha > 0 \). Then for \( x \geq 1 \),

\[
\Gamma f(x) = 2x^\beta((x+1)^{-\alpha} - x^{-\alpha}) + x^\beta((x-1)^{-\alpha} - x^{-\alpha})
\]

\[
= 2x^\beta x^{-\alpha}(\alpha x^{-1} + O(x^{-2})) + x^\beta x^{-\alpha}(\alpha x^{-1} + O(x^{-2}))
\]

\[
= -\alpha x^{\beta-\alpha-1} + O(x^{\beta-\alpha-2}).
\]

Thus for \( x \notin A \) we have \( \Gamma f(x) \leq -\varepsilon \) provided \( \alpha \in (0, \beta - 1] \), which requires \( \beta > 1 \). Thus an application of Theorem 7.4.5 shows that, if \( \beta > 1 \), \( \mathbb{P}_x[\zeta < \infty] = 1 \) for all \( x \in \mathbb{Z} \). Moreover, an application of Theorem 7.4.6 shows that for \( A = \{0, -1, -2, \ldots\} \) and any \( x \geq 1 \), \( \mathbb{E}_x[\zeta | \tau_A = \infty] < \infty \).

It may be shown that in this example \( \mathbb{E}_x \zeta = \infty \). We do not have any suitable Lyapunov function result here, so we just sketch the argument. (In principle Theorem 7.4.1 gives a condition for \( \mathbb{E}_x \zeta = \infty \), but in practice one
cannot show that no appropriate function exists.) The idea is that there is positive probability that before the process explodes it takes a sojourn on the negative integers, and in so doing the expected time to return to the positive integers is infinite.

Finally, we state results about implosion. The first result shows that implosion implies the existence of all moments and even of exponential moments.

**Proposition 7.4.11.** Suppose that (CT) holds. Suppose that there exists a finite set \( A \in \mathcal{E} \) and a constant \( C_A \in \mathbb{R}_+ \) such that \( \sup_{x \in \Sigma} E_x \tau_A \leq C_A \) holds. Then the chain implodes towards any state \( z \in \Sigma \), and, moreover, for each \( z \in \Sigma \) there exists \( \alpha > 0 \) such that \( \sup_{x \in \Sigma} E_x e^{\alpha \tau_z} < \infty \).

Next we give a criterion for implosion; as before, we remark that to check that the chain is non-explosive it is sufficient to show that it is recurrent.

**Theorem 7.4.12.** Suppose that (CT) holds. Suppose that the chain is non-explosive, i.e., \( \mathbb{P}[\zeta < \infty] = 0 \). The following are equivalent.

(a) There exists a triple \((\epsilon, F, f)\) with \( \epsilon > 0 \), \( F \) a finite set and \( f \in \text{Dom}_+(\Gamma) \) such that \( \sup_{x \in \Sigma} f(x) < \infty \) and \( \Gamma f(x) \leq -\epsilon \) for all \( x \notin F \).

(b) For every finite \( A \in \mathcal{E} \), there exists a constant \( C_A \in \mathbb{R}_+ \) such that \( E_x \tau_A \leq C_A \) for all \( x \in \Sigma \), i.e., there is implosion towards \( A \).

Next we give a sufficient condition for non-implosion.

**Theorem 7.4.13.** Suppose that (CT) holds. Let \( f \in \text{Dom}_+(\Gamma) \) be such that \( f \to \infty \) and suppose that there exist constants \( a > 0 \), \( c > 0 \), \( \epsilon > 0 \), and \( r > 1 \) such that \( f^r \in \text{Dom}_+(\Gamma) \). Suppose also that

(a) \( \Gamma f(x) \geq -\epsilon, \text{ for all } x \notin S_a(f) \); and

(b) \( \Gamma f^r(x) \leq cf^{r-1}(x), \text{ for all } x \notin S_a(f) \).

Then the chain does not implode towards \( S_a(f) \).

In some applications, it is quite difficult to guess the form of the function \( f \) satisfying the uniform condition \( \Gamma f(x) \leq -\epsilon \) required to apply Theorem 7.4.12. It is sometimes more convenient to check merely that \( \Gamma f(x) \leq -g(f(x)) \) for some function \( g \) vanishing at 0 in some controlled way. Thus the following result provides us with a convenient alternative condition to be checked.
Proposition 7.4.14. Suppose that (CT) holds. Suppose that the chain is non-explosive, i.e., \( \mathbb{P}[\zeta < \infty] = 0 \). Let \( f \in \text{Dom}_+(\Gamma) \) be strictly positive and such that \( \sup_{x \in \Sigma} f(x) = b < \infty \); assume further that for any \( a \in (0, b) \) the sublevel set \( S_a(f) \) is finite. Let \( g : [0, b] \to \mathbb{R}_+ \) be an increasing function such that \( B := \int_0^b \frac{du}{g(u)} < \infty \). If \( \Gamma f(x) \leq -g(f(x)) \) for all \( x \notin S_a(f) \) then \( \mathbb{E}_x \tau_{S_a(f)} \leq B \) for all \( x \notin S_a(f) \), i.e., the chain implodes towards \( S_a(f) \).

The remainder of this section is devoted to the proofs of the results on explosion and implosion stated above.

Since we wish to treat explosion, we work with the Markov chain \( \xi_t \) evolving on the augmented state space \( \hat{\Sigma} = \Sigma \cup \{\partial\} \) with the generator \( \hat{\Gamma} \); we work with augmented functions \( \hat{f} \in \text{Dom}(\hat{\Gamma}) \) obtained from \( f \in \text{Dom}(\Gamma) \) by setting \( \hat{f}(\partial) = 0 \) as explained earlier.

First we treat our results on conditional explosion. We will need the following continuous-time analogue of Lemma 2.5.10.

Lemma 7.4.15. Suppose that there exist \( f \in \text{Dom}_+(\Gamma) \) and \( A \subset \Sigma \) such that \( \Gamma f(x) \leq 0 \) for all \( x \notin A \). Then
\[
\mathbb{P}_x[\tau_A = \infty] \geq 1 - \frac{f(x)}{\inf_{y \in A} f(y)}.
\]

Proof. Let \( Y_t = f(\xi_t \wedge \tau_A) \). Then since \( \Gamma f(x) \leq 0 \) for \( x \notin A \), we have that \( Y_t \) is a non-negative supermartingale, and hence converges a.s. to some \( Y_\infty \). For every \( x \notin A \), Fatou’s lemma implies that \( \mathbb{E}_x Y_\infty \leq \lim_{t \to \infty} \mathbb{E}_x Y_t \leq f(x) \).

Hence
\[
f(x) \geq \mathbb{E}_x Y_\infty \geq \mathbb{E}_x [Y_\infty 1\{\tau_A < \infty\}] = \mathbb{E}_x [f(\xi_{\tau_A}) 1\{\tau_A < \infty\}] \quad \geq \inf_{y \in A} f(y) \mathbb{P}_x[\tau_A < \infty].
\]

This gives the result. \( \square \)

Proof of Theorem 7.4.6. First of all, Lemma 7.4.15 shows that
\[
\mathbb{P}_{x_0}[\tau_A = \infty] \geq 1 - \frac{f(x_0)}{\inf_{y \in A} f(y)} > 0.
\]

An application of Lemma 7.3.6 with \( Y_t = f(\xi_t \wedge \tau_A \wedge \zeta) \) and \( T = \tau_A \wedge \zeta \) shows that \( \mathbb{E}_{x_0}[\tau_A \wedge \zeta] \leq \varepsilon^{-1} f(x_0) \). Hence
\[
\frac{f(x_0)}{\varepsilon} \geq \mathbb{E}_{x_0}[\tau_A \wedge \zeta] \geq \mathbb{E}_{x_0}[\zeta \mid \tau_A = \infty] \mathbb{P}_{x_0}[\tau_A = \infty].
\]

Re-arranging and using the fact that \( \mathbb{P}_{x_0}[\tau_A = \infty] > 0 \) gives the result. \( \square \)
Proof of Theorem 7.4.5. First we show that for any $\delta > 0$ we can choose $n \in \mathbb{N}$ so that

$$\mathbb{P}_x[\tau_{E_1} = \infty] \geq 1 - \delta, \text{ for all } x \notin E_n. \quad (7.12)$$

Indeed, we have from Lemma 7.4.15 that

$$\inf_{x \notin E_n} \mathbb{P}_x[\tau_{E_1} = \infty] \geq 1 - \frac{\sup_{x \notin E_n} f(x)}{\inf_{y \in E_1} f(y)},$$

then since $\sup_{x \notin E_n} f(x) \to 0$ and $\inf_{x \in E_1} f(x) > 0$ we can choose $n$ so that (7.12) holds. Then we can apply Theorem 7.4.6 with $A = E_1$ to show that,

$$\mathbb{P}_x[\zeta < \infty] \geq \mathbb{P}_x[\zeta < \infty | \tau_A = \infty] \mathbb{P}_x[\tau_A = \infty] = \mathbb{P}_x[\tau_A = \infty] \geq 1 - \delta.$$

Let $\sigma_n = \inf \{t \geq 0 : \xi_t \notin E_n \}$; by hypothesis, $\sigma_n < \infty$ a.s., and the strong Markov property shows that for any $x \in \Sigma$,

$$\mathbb{P}_x[\zeta < \infty] \geq \mathbb{E}_x[\mathbb{P}[\zeta < \infty | \mathcal{F}_{\sigma_n}] \geq 1 - \delta.\]$$

Since $\delta > 0$ was arbitrary, the result follows.

Now we turn to the criterion for integrability of the explosion time, Theorem 7.4.1.

Proof of Theorem 7.4.1. Suppose that (b) holds. First suppose that $A = \emptyset$. Let $Y_t = \hat{f}(\xi_t)$ for $t \geq 0$. Then $(Y_t, t \in \mathbb{R}_+)$ is an $\mathbb{R}_+$-valued process adapted to $(\mathcal{F}_t, t \in \mathbb{R}_+)$. We write $\tau_\partial := \inf \{t \geq 0 : \xi_t = \partial \}$ for the hitting time of $\partial$; note

$$\tau_\partial = \zeta = \inf \{t \geq 0 : Y_t = 0 \}.$$

The condition $\Gamma f(x) \leq -\varepsilon$ for all $x \in \Sigma$ shows that $\mathbb{E}[dY_{t \wedge \tau_\partial} | \mathcal{F}_{t-}] \leq -\varepsilon 1\{t \leq \tau_\partial\} dt$. Then (a) follows from an application of Lemma 7.3.6.

Next suppose that $A \neq \emptyset$ is finite. First let us show that we can modify the function $f$ so that (b) holds with $A$ a singleton. So suppose $A$ is a set with at least two elements, and choose $x, y$ distinct elements of $A$. By irreducibility, there exists a path of distinct elements $z_0, z_1, \ldots, z_m \in \Sigma$ with $z_0 = x$ and $z_m = y$, such that $\Gamma_{z_k z_{k+1}} \geq \delta > 0$ for $k = 0, 1, \ldots, m - 1$. Define a new function $f_1 : \Sigma \to \mathbb{R}$ by taking $f_1(z) = f(z)$ for $z \neq z_1$ and $f_1(z_1) = f(z_1) - M_1$ for some constant $M_1 \in \mathbb{R}_+$ chosen so that $\Gamma f_1(z_0) \leq -\varepsilon$; then $\Gamma f_1(z) \leq -\varepsilon$ for all $z \in (\Sigma \cup \{z_0\}) \setminus (A \cup \{z_1\})$. Iteratively, define $f_k (k = 2, \ldots, m)$ by setting $f_k(z) = f_{k-1}(z)$ for $z \neq z_k$ and $f_k(z_k) = f_{k-1}(z_k) - M_k$ for $M_k \in \mathbb{R}_+$ chosen so that $\Gamma f_k(z_k) \leq -\varepsilon$; then $\Gamma f_k(z) \leq -\varepsilon$ for all
z \in (\Sigma \cup \{z_0, z_1, \ldots, z_{k-1}\}) \setminus (A \cup \{z_k\}). At the last step of this procedure, we obtain \( f_m : \Sigma \to \mathbb{R} \) with inf\(_{x \in \Sigma} f_m(x) \geq -\sum_{k=1}^{m} M_k \) and \( f_m(z) \leq -\varepsilon \) for all \( z \in \Sigma \setminus (A \setminus \{x\}) \). So taking \( f'(z) = f_m(z) + \sum_{k=1}^{m} M_k \) we have replaced \( f \) by a function \( f' \) that satisfies the conditions in (b) but with the size of the exceptional set \( A \) reduced by 1. Iterating this procedure we can ultimately reduce the exceptional set to a singleton. Thus it suffices to suppose that \( A = \{x\} \).

Define \( N := \#\{t \in (0, \zeta) : \xi_t \neq y, \xi_t = y\} \), the total number of returns to \( y \). Then set \( T_0 := 0 \) and for \( 1 \leq k < N+2 \) define recursively the quantities

\[
\begin{align*}
H_k &:= \inf\{t \geq 0 : \xi_{T_{k-1}+t} \neq y\}; \\
\tau_k &:= (\zeta - T_{k-1}) \wedge \inf\{t > H_k : \xi_{T_{k-1}+t} = y\}; \\
T_k &:= \tau_1 + \cdots + \tau_k.
\end{align*}
\]

In words, \( \tau_k \) are the successive times between returns to \( y \), \( H_k \) are the associated holding times, and \( T_k \) is the total time elapsed on the \( k \)th return to \( y \). For \( k \geq N + 2 \), set \( T_k = \zeta \) and \( \tau_k = 0 \). Let \( A_k := \{T_{k-1} + \tau_k < \zeta\} = \{k \leq N\} \), the event that the process returns to \( y \) for the \( k \)th time. Note that \( T_{k-1} + \tau_k = \zeta \) for \( k \geq N + 1 \).

With this notation in place, we may write

\[
\zeta = \tau_1 + \tau_2 + \cdots + \tau_{N+1} = \sum_{k=1}^{\infty} \tau_k \prod_{j=1}^{k-1} 1(A_j).
\]

By the strong Markov property, we have that on \( A_{k-1} \) the distribution of \( \tau_k \) is the same as the distribution of \( \tau_1 \), so taking expectations in (7.13) gives

\[
\mathbb{E}_x \zeta = \sum_{k=1}^{\infty} \mathbb{E}_x[\tau_1] \mathbb{P}_x[A_1 \cap \cdots \cap A_{k-1}].
\]

Using the strong Markov property and the fact that, by (b)(ii), the Markov chain is transient, we have that there exists \( \delta > 0 \) such that \( \mathbb{P}_x[A_1 \cap \cdots \cap A_{k-1}] \leq (1 - \delta)^{k-1} \). Thus we get \( \mathbb{E}_x \zeta \leq \delta^{1-1} \mathbb{E}_x \tau_1 \). Now let \( Y_t = f(\xi_t) \); then if \( \xi_0 = y \neq x \), on \( \{t \leq \tau_1\} \) we have \( \mathbb{E}[dY_t | \mathcal{F}_{t-}] \leq -\varepsilon dt \), and we may apply Lemma 7.3.6 to deduce that \( \mathbb{E}_y \tau_1 \leq \varepsilon^{-1} f(y) \). Then

\[
\mathbb{E}_x \tau_1 \leq \gamma x^{-1} + \varepsilon^{-1} \sum_y P_{xy} f(y) < \infty
\]

since \( f \in \text{Dom}_+(\Gamma) \). So we conclude that \( \mathbb{E}_x \zeta < \infty \), which establishes (a).
7.4. Explosion and implosion

On the other hand, suppose that (a) holds. Define \( \hat{f}(x) = \varepsilon E_x \tau_\partial \) for \( x \in \Sigma \). Obviously \( \hat{f}(\partial) = 0 \) while \( 0 < \hat{f}(x) < \infty \) for all \( x \in \Sigma \). Conditioning on the first move of the embedded Markov chain \( \tilde{x} \) we get

\[
m_f(x) = E[f(\tilde{x}_1) - f(x) \mid \tilde{x}_0 = x] = E[f(\xi_{n+1}) - f(\xi_0) \mid \xi_0 = x]
\]

\[
= \varepsilon \sum_{y \in \Sigma \setminus \{x\}} P_{xy} E_y \tau_\partial - \varepsilon E_x \tau_\partial = \varepsilon (E_x \tau_\partial - E_x \sigma_1) - \varepsilon E_x \tau_\partial
\]

\[
= -\varepsilon E_x \sigma_1 = -\frac{\varepsilon}{\gamma_x}.
\]

Hence \( \Gamma f(x) \leq -\varepsilon \) for all \( x \in \Sigma \), and we have the \( A = \emptyset \) case of (b).

Proof of Proposition 7.4.3. Let \( G(z) = \int_0^z \frac{dy}{g(y)} \). Then \( G \) is differentiable, with \( G'(z) = \frac{1}{g(z)} > 0 \) hence an increasing function of \( z \in [0, b] \). Since \( g \) is increasing, \( G' \) is decreasing and hence \( G \) is concave satisfying \( \lim_{z \to 0} G(z) = 0 \) and \( \lim_{z \to \infty} G(z) < \infty \). Additionally, boundedness of \( G \) implies that \( G \circ f \in \ell^1(\Gamma) \). Due to differentiability and concavity of \( G \), we have:

\[
\Gamma(G \circ f)(x) = \gamma_x E[G(f(\xi_n) + \Delta_{n+1}^f) - G(f(\xi_n)) \mid \xi_n = x]
\]

\[
\leq \gamma_x G'(f(x)) E[\Delta_{n+1}^f \mid \xi_n = x]
\]

\[
= \gamma_x \frac{1}{g(f(x))} m_f(x) = \frac{\Gamma f(x)}{g(f(x))} \leq -1,
\]

for all \( x \not\in A \); we conclude by Theorem 7.4.1 because \( G \circ f \) is strictly positive and bounded.

Next we give the proof of the condition for non-explosion, Theorem 7.4.9.

Proof of Theorem 7.4.9. Let \( G(z) = \int_0^z \frac{dy}{g(y)} \). Then \( G \) is differentiable, with \( G'(z) = \frac{1}{g(z)} > 0 \), hence increasing. Since \( g \) is increasing, \( G' \) is decreasing, and hence \( G \) is concave. Concavity and differentiability of \( G \) imply that \( G(y + d) - G(y) \leq dG'(y) \) for all \( y \) and \( d \) with \( y \geq 0 \) and \( y + d \geq 0 \). Thus

\[
0 \leq E[G(f(\tilde{x}_n) + \Delta_{n+1}^f) \mid \tilde{x}_n = x] \leq G(f(x)) + \frac{m_f(x)}{g(f(x))} \leq \infty,
\]

so that, setting \( U_n := G(f(\tilde{x}_n)) \), we obtain

\[
E[U_{n+1} - U_n \mid \tilde{x}_n = x] \leq \frac{m_f(x)}{g(f(x))} \leq \gamma_x^{-1},
\]
Chapter 7. Markov chains in continuous time

by hypothesis (b). Now let $V_n := U_n - \sum_{k=0}^{n-1} \gamma_{\tilde{\xi}_k}^{-1}$. Then $V_n$ is a supermartingale adapted to $\tilde{\mathcal{F}}_n$. Let $y \in \mathbb{R}_+$ and define

$$\sigma := \min\left\{n \geq 0 : \sum_{k=0}^{n} \gamma_{\tilde{\xi}_k}^{-1} \geq y\right\}.$$ 

Then $V_n \wedge \sigma$ is a supermartingale bounded below by $-y$, hence $V_n \wedge \sigma$ converges to some $V_\infty$ with $V_\infty < \infty$ a.s. On the event $\{\sigma = \infty\}$ we have $\sum_{k=0}^{\infty} \gamma_{\tilde{\xi}_k}^{-1} < y$ and so $\gamma_{\tilde{\xi}_n} \to \infty$. This can only happen if $\tilde{\xi}_n$ visits every state only finitely often, and so $f(\tilde{\xi}_n)$ and $G(f(\tilde{\xi}_n))$ both tend to $\infty$. But on $\{\sigma = \infty\}$,

$$V_n \wedge \sigma \geq G(f(\tilde{\xi}_n)) - y \to \infty.$$ 

But since $V_\infty < \infty$ a.s., we must have $P[\sigma = \infty] = 0$. In other words, we have shown that

$$\sum_{k=0}^{\infty} \gamma_{\tilde{\xi}_k}^{-1} \geq y, \text{ a.s.}$$

Since $y$ was arbitrary, the result follows by Chung’s criterion (7.11). \qed

Now we turn to the results on implosion.

Proof of Proposition 7.4.11. Note that, for $x \not\in A$, $\sigma_0 \leq \tau_A$, where $\sigma_0$ is the holding time at the initial state, so that $\gamma_x^{-1} = E_x \sigma_0 \leq E_x \tau_A \leq C$ for all $x \not\in A$. Hence, since $A$ is finite, $\tilde{\gamma} := \inf_{x \in \Sigma} \gamma_x > 0$.

Fix $z \in \Sigma$. Let $\lambda = \min\{n \in \mathbb{Z}_+ : \tilde{\xi}_n = z\}$. Irreducibility and finiteness of $A$ means that there exist $k \in \mathbb{N}$ and $\delta > 0$ such that

$$\min_{a \in A} \mathbb{P}[\lambda \leq k \mid \tilde{\xi}_0 = a] \geq \delta.$$ 

By Markov’s inequality,

$$\mathbb{P}[\sigma_n \geq t] \leq t^{1} \mathbb{E} \sigma_n \leq t^{-1} \tilde{\gamma}^{-1} \leq \frac{\delta}{2k},$$

provided we choose $t = t_0 := \frac{2k}{\tilde{\gamma}}$. Then

$$\mathbb{P}_a\left[\lambda \leq k, \max_{0 \leq i \leq k-1} \sigma_i \leq t_0\right] \geq \delta/2,$$

which means that $\mathbb{P}_a[\tau_x \leq kt_0] \geq \delta/2$ for all $a \in A$. Moreover, Markov’s inequality also shows that $\mathbb{P}_x[\tau_A \leq 2C] \geq 1/2$, so, by the Markov property, with $K = 2C + kt_0$ and $\varepsilon = \delta/4$, we have

$$\mathbb{P}_x[\tau_x \leq K] \geq \varepsilon, \text{ for all } x \in \Sigma.$$
Proof of Theorem 7.4.12. Let us prove that (a) implies (b). The condition \( \Gamma f(x) \leq -\varepsilon \) for \( x \notin F \) shows, by an application of Lemma 7.3.6, that \( E_x \tau_F \leq e^{-1} f(x) \) so that \( \sup_{x \in \Sigma} E_x \tau_F \leq e^{-1} \sup_{x \in \Sigma} f(x) < \infty. \)

Now, we show that (b) implies (a). Suppose that for a finite \( A \in \mathcal{E} \), there exists a constant \( C \) such that \( E_x \tau_A \leq C \) for all \( x \in \Sigma \). We use the same observation as at the start of the proof of Proposition 7.4.11, that \( \gamma_x^{-1} \leq E_x \tau_A \leq C \), and so \( \gamma = \inf_{x \in \Sigma} \gamma_x > 0. \) Define

\[
f(x) = \begin{cases} 
0 & \text{if } x \in A, \\
E_x \tau_A & \text{if } x \notin A.
\end{cases}
\]

Then it is immediate to show that for \( x \notin A \), we have \( \Gamma f(x) \leq -1. \)

Proof of Theorem 7.4.13. Under the conditions of the theorem, we may apply Lemma 7.3.9 with \( Y_t = f(\xi_t) \) and \( G_t = F_t \), so that \( T_a := \inf \{ t \geq 0 : Y_t \leq a \} = \tau_{S_a(f)}. \) We conclude that there exists \( \varepsilon > 0 \) such that, for all \( x_0 \) sufficiently large,

\[
\mathbb{P}_{x_0} [ \tau_{S_a(f)} > t + \varepsilon f(x_0) ] \geq \frac{1}{2}.
\]

Therefore \( E_{x_0} \tau_{S_a(f)} \geq \frac{1}{2} \varepsilon f(x_0) \); since \( f \to \infty \), this expectation cannot be bounded uniformly in \( x_0 \), so implosion is excluded.

Proof of Proposition 7.4.14. Let \( G(z) = \int_0^z \frac{du}{g(u)} \). Since \( G'(z) = \frac{1}{g(z)} > 0 \) the function \( G \) is increasing with \( G(0) = 0 \) and \( G(b) = B \). Since \( g \) is increasing \( G' = \frac{1}{g} \) is decreasing, hence the function \( G \) is concave. Then concavity gives the bound

\[
m_{Gof}(x) \leq G'(f(x))m_f(x) = \frac{m_f(x)}{g(f(x))}.
\]

The condition imposed in the statement of the proposition implies that \( \Gamma G \circ f(x) \leq -1 \). We conclude from Lemma 7.3.6.

7.5 Applications

7.5.1 Lamperti processes in continuous time

Let \((\xi_t, t \in \mathbb{R}_+)\) be an irreducible time-homogeneous continuous-time Markov chain on \( \Sigma \subset \mathbb{R}_+ \) assumed to be locally finite with \( \inf \Sigma = 0 \) and \( \sup \Sigma = +\infty \); then transience can occur only to \( +\infty \). We suppose that the associated
jump chain $\tilde{\xi}_n$ is of Lamperti-type in the manner of Section 3.2; specifically, writing $\Delta_n := \tilde{\xi}_{n+1} - \tilde{\xi}_n$ for the increments of the jump chain, we suppose that (M1) holds, and denoting the increment moment functions by

$$\mu_k(x) := \mathbb{E}[(\tilde{\xi}_{n+1} - \tilde{\xi}_n)^k \mid \tilde{\xi}_n = x], \quad k \in \mathbb{N}, \; x \in \Sigma,$$

we also suppose that (M2) holds. For the jump rates of the continuous-time chain, we suppose that $\gamma_x = x^{2-\kappa}$ for some $\kappa \in \mathbb{R}$.

First we give a result on passage-time moments in the recurrent case (compare Theorem 3.2.6).

**Proposition 7.5.1.** Suppose that (M1) and (M2) hold, and that $\gamma_x = x^{2-\kappa}$ for some $\kappa \in \mathbb{R}$. Suppose that $2a < b$. Define $s_0 := b - 2a b \kappa$.

(i) If $s < s_0$ and $s < \frac{p}{\kappa}$, then $\mathbb{E}_x[\tau_A^s] < \infty$.

(ii) If $s_0 < s < \frac{p}{\kappa}$, then $\mathbb{E}_x[\tau_A^s] = \infty$.

**Proof.** We use the Lyapunov function $g(x) = (1 + x)^{\alpha}$, where $\alpha \in (0, p)$ will be specified later. Similarly to Lemma 3.4.1, a Taylor’s formula calculation shows that

$$m_g(x) = \mathbb{E}[(1 + x + \Delta_n)^\alpha - (1 + x)^\alpha \mid \tilde{\xi}_n = x]$$

$$= \frac{\alpha}{2} (2x \mu_1(x) + (\alpha - 1) \mu_2(x)) x^{\alpha-2} + o(x^{\alpha-2}),$$

so assuming (M2) we have from (7.4) that

$$\Gamma g(x) = \frac{\alpha}{2} (2a + (\alpha - 1)b + o(1)) x^{\alpha-\kappa}. \quad (7.15)$$

Take $f(x) = (1 + x)^{\alpha/2}$ and set $\alpha = s\kappa/2$; then provided $\alpha < \frac{b-2a}{b}$, we have from (7.15) that for some $c > 0$ and all $x$ sufficiently large

$$\Gamma f^s(x) = \Gamma g(x) \leq -c(1 + x)^{\alpha-\kappa} = -cf^{s-2}(x).$$

Hence Theorem 7.3.1 applies with $p = s$ for any $s < \frac{2p}{\kappa} \wedge \frac{2(b-2a)}{b\kappa}$, proving (i).

For part (ii), we again use the function $g(x) = (1 + x)^{\alpha}$, $\alpha \in (0, p)$. Take $f(x) = (1 + x)^{\kappa}$ and $\alpha = s\kappa$. Then from (7.15) we see that if $\alpha > \frac{b-2a}{b}$, then $\Gamma f^s(x) = \Gamma g(x) \geq 0$ for all $x$ sufficiently large. Equation (7.15) (applied with $\alpha = \kappa$ and again with $\alpha = \kappa r$) also shows that $\Gamma f(x) \geq -c_1$ and, for any $r \in (1, p/\alpha)$, that $\Gamma f^r(x) \leq c_2 f^{r-1}(x)$ for all $x$ sufficiently large. Thus Theorem 7.3.2 applies whenever $\frac{b-2a}{b\kappa} < s < \frac{p}{\kappa}$. □
Next we turn to explosion in the transient case.

**Proposition 7.5.2.** Suppose that (M1) and (M2) hold, and that $\gamma_x = x^{2-\kappa}$ for some $\kappa \in \mathbb{R}$. Suppose that $2a > b$.

(i) If $\kappa < 0$, then $\mathbb{E}_x \zeta < \infty$ for all $x \in \Sigma$.

(ii) If $\kappa \geq 0$, then $\mathbb{P}_x[\zeta = \infty] = 1$ for all $x \in \Sigma$.

**Proof.** We use the Lyapunov function $f(x) = (1 + x)\alpha$ for $\alpha \in (0, p-2)$. Similarly to Lemma 3.4.1, a Taylor’s formula calculation shows that

$$m_f(x) = \mathbb{E}[(1 + x + \Delta_n)^{-\alpha} - (1 + x)^{-\alpha} | \xi_n = x]$$

$$= -\frac{\alpha}{2} (2x\mu_1(x) - (1 + \alpha)\mu_2(x)) x^{-\alpha-2} + o(x^{-\alpha-2}),$$

so assuming (M2) we have from (7.4) that

$$\Gamma f(x) = -\frac{\alpha}{2} (2a - (1 + \alpha)b + o(1)) x^{-\alpha-\kappa}.$$ 

Then for any $\kappa < 0$ we may choose $\alpha \in (0, \frac{2a-b}{b})$ with $\alpha + \kappa \leq 0$, and then we may apply Theorem 7.4.1 to establish $\mathbb{E}_x \zeta < \infty$. This proves part (i).

For part (ii) we use the Lyapunov function $f(x) = \log(1 + x)$. Then

$$m_f(x) = \mathbb{E}\left[\log\left(1 + \frac{\Delta_n}{1 + x}\right) - 1 | \xi_n = x\right]$$

$$= \frac{1}{2} (2x\mu_1(x) - \mu_2(x)) x^{-2} + o(x^{-2}),$$

so assuming (M2) we have from (7.4) that

$$\Gamma f(x) = \frac{1}{2} (2a - b + o(1)) x^{-\kappa}. \quad (7.16)$$

Thus for $\kappa \geq 0$ we may apply Theorem 7.4.9 with $g(x) = x$ to establish non-explosion. \qed

Finally, we give a result on implosion.

**Proposition 7.5.3.** Suppose that (M1) and (M2) hold, and that $\gamma_x = x^{2-\kappa}$ for some $\kappa \in \mathbb{R}_+$. Suppose that $2a < b$.

(i) If $\kappa < 0$, then the chain implodes.

(ii) If $\kappa \geq 0$, then the chain does not implode.
Proof. For part (i), we use the Lyapunov function \( f(x) = 1 - (1 + x)^{-\alpha} \), where \( \alpha \in (0, p - 2) \). Then a Taylor’s formula calculation shows that
\[
m_f(x) = \mathbb{E}[(1 + x)^{-\alpha} - (1 + x + \Delta_n)^{-\alpha} | \tilde{\xi}_n = x]
= \frac{\alpha}{2} (2x\mu_1(x) - (1 + \alpha)\mu_2(x)) x^{-\alpha - 2} + o(x^{-\alpha - 2}),
\]
so assuming (M2) we have from (7.4) that
\[
\Gamma f(x) = \frac{\alpha}{2} (2a - (1 + \alpha)b + o(1)) x^{-\alpha - \kappa}.
\]
Then for any \( \kappa < 0 \) we may choose \( \alpha > 0 \) with \( \alpha + \kappa \leq 0 \), and then we may apply Theorem 7.4.12 to deduce implosion.

For part (ii) we use the Lyapunov function \( f(x) = \log^\alpha(1 + x) \), \( \alpha \in (0, 1) \). Then a Taylor’s formula calculation shows that
\[
m_f(x) = \frac{\alpha}{2} x^{-2} \log^{-\alpha - 1} x (2a - b + o(1)),
\]
so that
\[
\Gamma f(x) = \frac{\alpha}{2} x^{-\kappa} \log^{-\alpha - 1} x (2a - b + o(1)).
\]
For \( \kappa \geq 0 \), it follows that \( \Gamma f(x) \geq -\epsilon \) for all \( x \) sufficiently large, and for \( r > 1 \) such that \( \alpha r < 1 \) we have \( \Gamma f^r(x) \leq cf^{r-1}(x) \) for all \( x \) sufficiently large. We conclude by Theorem 7.4.13 that implosion does not occur. \( \square \)

### 7.5.2 Simple symmetric random walk

Take as state space \( \Sigma = \mathbb{Z}^d \), \( d \in \mathbb{N} \), and suppose that the jump chain is simple symmetric random walk on \( \Sigma \).

First we study explosion for the transient case, \( d \geq 3 \).

**Proposition 7.5.4.** Suppose that \( d \geq 3 \) and \( \gamma_x = \|x\|^{2-\kappa} \) for \( \kappa \in \mathbb{R} \). Then the random walk \( \tilde{\xi}_t \) is explosive if \( \kappa < 0 \) and non-explosive if \( \kappa \geq 0 \).

**Proof.** To prove explosion, we use the Lyapunov function \( f(x) = (1 + \|x\|)^{-\alpha} \). Here a Taylor’s formula calculation similar to that in Example 2.5.9 shows that
\[
m_f(x) = \frac{\alpha}{d} \|x\|^{-2-\alpha}(\alpha + 2 - d + o(1)),
\]
so that
\[
\Gamma f(x) = \frac{\alpha}{d} \|x\|^{-\alpha - \kappa}(\alpha + 2 - d + o(1)).
\]
Thus for $\kappa < 0$ and $d \geq 3$ we may choose $\alpha > 0$ such that $\alpha + \kappa \leq 0$ and $\Gamma f(x) \leq -\varepsilon$ for all $x$ outside a finite set. Then we may apply Theorem 7.4.5 with $E_n = B(0; n) \cap \Sigma$ to establish explosion.

To prove non-explosion, we use the Lyapunov function $f(x) = \log(1 + \|x\|)$. Then a Taylor’s formula calculation gives

$$m_f(x) = \frac{1}{2} \left(1 - \frac{2}{d} + o(1)\right) \|x\|^{-2},$$

so that

$$\Gamma f(x) = \frac{1}{2} \left(1 - \frac{2}{d} + o(1)\right) \|x\|^{-\kappa}.$$

Thus for $\kappa \geq 0$ we have $\Gamma f(x) \leq cf(x)$ and we may apply Theorem 7.4.9 with $g(x) = x$ to establish non-explosion.

Finally we study implosion in the recurrent case $d \in \{1, 2\}$.

**Proposition 7.5.5.** Suppose that $d \in \{1, 2\}$ and $\gamma_x = \|x\|^{2-\kappa}$ for $\kappa \in \mathbb{R}$. Then the random walk $\xi_t$ implodes if $\kappa < 0$ but does not implode if $\kappa \geq 0$.

**Proof.** The proof is similar to our previous calculations: for implosion we use the function $f(x) = 1 - (1 + \|x\|)^{-\alpha}$ with Theorem 7.4.12, while for non-implosion we use $f(x) = \log^\alpha(1 + \|x\|)$ with Theorem 7.4.13.

**Bibliographical notes**

**Section 7.1**

The results of this chapter are largely based on [220]. For definitions of continuous-time Markov chains, and the exponential holding-time and jump chain constructions, we refer to standard texts such as [45, 4, 243].

**Section 7.2**

As stated in the text, Theorems 7.2.1 and 7.2.2 are direct consequence of the corresponding discrete-time results, and so can be traced back to Foster [108] (see the bibliographical notes to Chapter 2). The sufficient condition for recurrence in Theorem 7.2.1 can be found for example for the case of $F$ a singleton in an unpublished technical report by Miller [240], and for finite $F$ in [303]; the if and only if statement is given in Theorem 4 of [305]. Miller [240] also gives a more restrictive version of the condition for transience in Theorem 7.2.2.
Section 7.3

Theorems 7.3.1 and 7.3.2 on existence and non-existence of passage-time moments are contained in Theorem 1.5 of [220]. The proofs mirror their discrete-time analogues from [10] (see also Section 2.7) and also parallel those for diffusion processes from [224].

If $\gamma_x$ is bounded away from 0 and $\infty$, then since the Markov chain can be stochastically controlled between two Markov chains with constant jump rates, the asymptotics for the continuous-time process are identical to those of the jump chain and the discrete-time results of [10]; thus Theorems 7.3.1 and 7.3.2 are of most interest when $\sup_{x \in \Sigma} \gamma_x = \infty$ or $\inf_{x \in \Sigma} \gamma_x = 0$. As in discrete time, existence and non-existence of moments of passage times leads to estimates on rates of convergence to stationarity: see e.g. [285].

The continuous-time analogue of Foster’s criterion for positive recurrence, Theorem 7.3.4, is essentially Theorem 3(i) of Tweedie [305]; compare also Theorem 1.7 in [220]. The fact that the drift condition (b) implies positive recurrence (a) was shown by Reuter [268, p. 426] in the case where $F$ is a singleton, and, in the case of finite $F$, in Lemma 1 of Kingman [171]; see also Theorem 2.3(i) of [303]. Both directions of the statement were given for the case of $F$ a singleton in Theorem 2 of Miller [240]. The key integrability condition for stopping times, Lemma 7.3.6, is an improvement of Lemma 2.3 of [220].

Criteria for a continuous-time Markov chain to be not positive-recurrent analogous to the discrete-time Corollary 2.6.11 are given in [38], and extensions based on a continuous-time version of Kaplan’s condition are given in [169]; these results are typically not as sharp as Theorem 7.3.2.

Section 7.4

In the case where $A = \emptyset$, Theorem 7.4.1 is Theorem 1.9 of [220], while the fact that the uniform global drift condition (b) implies $P_x(\zeta < \infty) = 1$ in this case is contained in Theorem 4.3.6 of [296]; similar sufficient conditions of explosion are established in [313] for Markov chains on locally compact separable metric spaces. The case where $A \neq e\emptyset$ of Theorem 7.4.1 is new. Permitting a non-empty exceptional set is very useful in applications, where often Taylor’s theorem is used.

Theorem 7.4.5 is new. The conditional explosion result, Theorem 7.4.6, is Theorem 1.12 in [220]. Results for establishing $P(\zeta < \infty | \xi_t \to \infty)$ are given in [161, 162].

Theorem 7.4.9 on non-explosion is Theorem 1.14 of [220]; earlier versions
of this result were established by Khas’minskii [168], in Theorem (1.11) of Chen [37] (both with $g(x) = x$), and in Theorem 1 of Kersting and Klebaner [161] (in the case of processes on $\mathbb{R}_+$, with $f$ the identity); see also Theorem 1 of [162] and Theorem 1 of [125], which has $g(x) = x$. The argument of [161] is the basis for the one given here. A similar idea was used in [85] to prove non-explosion for Markov chains on separable metric spaces, while Lyapunov functions are used in [39] for the study of explosion for Markov chains on $\mathbb{Z}$ and time-dependent holding times.

The notion of implosion was introduced in [220]; it is reminiscent of Doeblin’s condition for general Markov chains [76]. Theorems 7.4.12 and 7.4.13 improve slightly on results contained in Theorem 1.15 of [220].
Glossary of named assumptions

Section 2.8: Growth bounds on trajectories

(A0) Suppose that for some \( a \in \mathbb{Z}^+ \) the function \( a : [n_a, \infty) \to (0, \infty) \) is such that (i) \( x \mapsto a(x) \) is increasing on \( x \geq n_a \); (ii) \( \lim_{x \to \infty} a(x) = \infty \); and (iii) \( \sum_{n \geq n_a} \frac{1}{a(n)} < \infty \).

(A1) Suppose that for some \( n_a \in \mathbb{Z}^+ \) the function \( a : [n_a, \infty) \to (0, \infty) \) is such that (i) \( x \mapsto a(x) \) is increasing on \( x \geq n_a \); (ii) \( \lim_{x \to \infty} a(x) = \infty \); and (iii) \( \sum_{n \geq n_a} \frac{1}{na(n)} < \infty \).

Section 3.2: Makovian Lamperti problem

(M0) Let \( X_n \) be an irreducible, time-homogeneous Markov chain on \( \Sigma \), a locally finite, unbounded subset of \( \mathbb{R}_+ \), with \( 0 \in \Sigma \).

(M1) Suppose that for some \( p > 2 \),
\[
\sup_{x \in \Sigma} \mathbb{E}[|\Delta_n|^p | X_n = x] < \infty.
\]

(M2) Suppose that there exist \( a \in \mathbb{R} \) and \( b \in (0, \infty) \) such that
\[
\lim_{x \to \infty} \mu_2(x) = b, \quad \text{and} \quad \lim_{x \to \infty} x \mu_1(x) = a.
\]

Section 3.3: General Lamperti problem

(L0) Let \( X_n \) be a stochastic process adapted to a filtration \( \mathcal{F}_n \) and taking values in \( S \subseteq \mathbb{R}_+ \) with \( \inf S = 0 \) and \( \sup S = +\infty \).

(L1) Suppose that for some \( p > 2 \), \( \delta \in (0, p - 2] \), and \( C \in \mathbb{R}_+ \),
\[
\mathbb{E}[|\Delta_n|^p | \mathcal{F}_n] \leq C(1 + X_n)^{p-2-\delta}, \text{ a.s., for all } n \geq 0.
\]
(L2) Suppose that \( \limsup_{n \to \infty} X_n = +\infty \), a.s.

(L3) Suppose that for each \( x \in \mathbb{R}_+ \) there exist \( r_x \in \mathbb{N} \) and \( \delta_x > 0 \) such that, for all \( n \geq 0 \),

\[
P \left[ \max_{n \leq m \leq n + r_x} X_m \geq x \mid \mathcal{F}_n \right] \geq \delta_x, \quad \text{on} \{X_n \leq x\}.
\]

### Section 3.6: Irreducibility and regeneration

(I) Suppose that for each \( x, y \in S \) there exist constants \( m(x, y) \in \mathbb{N} \) and \( \varphi(x, y) > 0 \), and a collection of \( \mathcal{F}_n \)-measurable random variables \( M_n(y) \in \mathbb{Z}_+ \), such that, for all \( n \geq 0 \), \( M_n(y) \leq m(X_n, y) \), a.s., and

\[
P[X_n + M_n(y) = y \mid \mathcal{F}_n] \geq \varphi(X_n, y), \quad \text{a.s.}
\]

(I') Suppose that for each \( x, y \in S \) there exist constants \( m(x, y) \in \mathbb{N} \) and \( \varphi(x, y) > 0 \) such that, for all \( n \geq 0 \),

\[
P \left[ \bigcup_{k=0}^{m(X_n, y)} \{X_{n+k} = y\} \right] \geq \varphi(X_n, y), \quad \text{a.s.}
\]

(Ra) Suppose that, for all \( k \in \mathbb{N} \), the distribution of \( E_k \) on \( \{\nu_{k-1} < \infty\} \) is the same as the distribution of \( E_1 \).

(Rb) Suppose that, on \( \{N = \infty\} \), \( (E_k)_{k \in \mathbb{N}} \) is an i.i.d. sequence.

(R) Suppose that \( X_0 = 0 \) and both (Ra) and (Rb) hold.

### Section 3.12: Supercritical Lamperti problem

(L5) Suppose that for some \( C \in \mathbb{R}_+ \), \( \mathbb{E}[|\Delta_n|^p \mid \mathcal{F}_n] \leq C \), a.s.

### Section 4.1: Many-dimensional random walks

(MD0) Let \( d \geq 1 \). Suppose that \( (\xi_n, n \geq 0) \) is a discrete-time, time-homogeneous Markov process on an unbounded subset \( \Sigma \) of \( \mathbb{R}^d \), with \( 0 \in \Sigma \).

(MD1) Suppose that \( \mu(x) = 0 \) for all \( x \in \Sigma \).

(MD2) There exists \( v > 0 \) such that \( \text{tr} M(x) \geq v \) for all \( x \in \Sigma \).

(MD3) Suppose that \( \limsup_{n \to \infty} \|\xi_n\| = +\infty \), a.s.

(MD4) Suppose that there exists \( v_0 > 0 \) such that,

\[
\inf_{e \in \mathbb{S}^d} (e^\top M(x)e) \geq v_0, \quad \text{for all} \ x \in \Sigma.
\]
Section 4.2: Elliptic random walk

(E1) Suppose that there exists a positive-definite matrix function \( \sigma^2 \) with domain \( S^{d-1} \) such that, as \( r \to \infty \),
\[
\varepsilon(r) := \sup_{x \in \Sigma : ||x|| \geq r} \| M(x) - \sigma^2(\hat{x}) \|_{op} \to 0.
\]

(E2) Suppose that there exist constants \( U \) and \( V \) with \( 0 < U \leq V < \infty \) such that, for all \( u \in S^{d-1} \),
\[
\langle u, u \rangle_u = U, \quad \text{and} \quad \text{tr} \sigma^2(u) = V.
\]

Section 4.4: Centrally biased random walk

(CB1) Suppose that for some \( \beta \in [0, 1), \rho \in \mathbb{R}_+, \) and \( p > 1 + \beta \), \( \sup_{x \in \Sigma} \mathbb{E}_x[||\theta_0||^p] < \infty \), and\[
\mu(x) = \rho ||x||^{-\beta} \hat{x} + O(||x||^{-\beta} \log^{-2} ||x||),
\]

(CB2) Suppose that, for some \( p > 2 \), \( \sup_{x \in \Sigma} \mathbb{E}_x[||\theta_0||^p] < \infty \), and that there exists \( C < \infty \) such that\[
||\mu(x)|| \leq C(1 + ||x||)^{-1}, \quad \text{for all} \quad x \in \Sigma.
\]

(CB3) Suppose that there exist \( \rho \in \mathbb{R} \) and \( \sigma^2 \in (0, \infty) \) for which, as \( ||x|| \to \infty \),\[
\mu(x) = \rho \hat{x} ||x||^{-1} + o(||x||^{-1} \log^{-1} ||x||); \quad \text{and} \quad ||M(x) - \sigma^2 I||_{op} = o(\log^{-1} ||x||).
\]

Section 5.2: Directional transience

(H0) Let \( X_n \) be a time-homogeneous Markov chain on \( \Sigma \subseteq \mathbb{R} \) with \( 0 \in \Sigma \), \( \inf \Sigma = -\infty \) and \( \sup \Sigma = +\infty \). Suppose that \( X_0 = 0 \).

(H1) There exist \( \alpha > 0, c > 0, \) and \( y_0 < \infty \) for which, for all \( x \in \Sigma \) and all \( y \geq y_0 \), \( \mathbb{P}[\Delta_n^+ > y \mid X_n = x] \geq cy^{-\alpha} \).

(H2) There exist \( \alpha \in (0, 1), c > 0, \) and \( y_0 < \infty \) for which, for all \( x \in \Sigma \) and all \( y \geq y_0 \), \( \mathbb{E}[\Delta_n^+ \mathbf{1}\{\Delta_n^+ \leq y\} \mid X_n = x] \geq cy^{1-\alpha} \).

(H3) There exist \( \beta > 0 \) and \( C < \infty \) for which \( \mathbb{E}[(\Delta_n^-)^\beta \mid X_n = x] \leq C \) for all \( x \in \Sigma \).
Section 5.3: Oscillating random walk

(Os1) Let \( v_{\alpha} \in \mathcal{D}_{\alpha} \) and \( v_{\beta} \in \mathcal{D}_{\beta} \), for some \( \alpha, \beta > 0 \). For \( x, y \in \mathbb{R} \), let

\[
w_{x}(y) := \begin{cases} v_{\alpha}(-y) & \text{if } x \geq 0, \\ v_{\beta}(y) & \text{if } x < 0. \end{cases}
\]

(Os2) Let \( P = (p(i,j), i,j \in S) \) be an irreducible stochastic matrix, and let \( (\mu_k, k \in S) \) denote the corresponding invariant distribution.

(Os3) Suppose that, for each \( k \in S \), we have an exponent \( \alpha_k \in (0, \infty) \) and a density function \( v_k \in \mathcal{D}_{\alpha_k} \). Then suppose that, for all \( y \in \mathbb{R} \), \( w_k \) is given by \( w_k(y) = v_k(-y) \).

Section 6.2: Random strings in random environment

(N) \( \mathbb{E} \log^+ \|A\|_{\text{op}} < \infty \), where \( \log^+ x = \max\{\log x, 0\} \).

(D) \( \mathbb{E} \log(1/r_i^f) < \infty \), for all \( i \in A \).

Section 6.3: Stochastic billiards

(SB) Suppose that for some \( \alpha_0 \in (0, \pi/2) \),

\[
P[0 < |\alpha| < \alpha_0] = 1 \quad \text{and} \quad \mathbb{E} \tan \alpha = 0,
\]

and that for some \( \gamma < 1 \) and \( A \geq A_0 \) as in Lemma 6.3.1,

\[
\mathcal{D} := \mathcal{D}(\gamma, A) := \{(x,y) \in \mathbb{R}^2 : x \geq A, |y| \leq x^{\gamma}\}.
\]

Section 7.1: Markov chains in continuous time

(CT) Suppose that \( 0 < \gamma_x < \infty \) for all \( x \in \Sigma \), and that the jump chain is irreducible.
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