Controllability of a 1-D tank containing a fluid modeled by a Boussinesq system

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Joint work with
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The Boussinesq system

J. L. **Bona**, M. **Chen**, J.-C. **Saut** - J. Nonlinear Sci. 12 (2002).

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b \eta_{xxt} = 0 \\ w_t + \eta_x + w w_x + c \eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{1}$$

The model describes the motion of small-amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional.

η is the elevation of the fluid surface from the equilibrium position;

$w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid;

a, b, c, d, are parameters required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c + d = \frac{1}{2}(1 - \theta^2) \geq 0$$

where $\theta \in [0, 1]$.

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then

$$a + b + c + d = \frac{1}{3}.$$

If we assume that

$$a \neq 0, \quad c \neq 0 \text{ and } b = d = 0,$$

due to global well-posedness restrictions

$$a \leq 0 \text{ and } c \leq 0 \quad \text{or} \quad a = c > 0.$$

Since $a + c = \frac{1}{3}$, this leads to

$$a = c = \frac{1}{6}, \qquad \theta = \sqrt{\frac{2}{3}}.$$

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A tank containing a fluid

The tanks is filled with liquid and should be be moved to different steady-state workbenches as fast as possible.

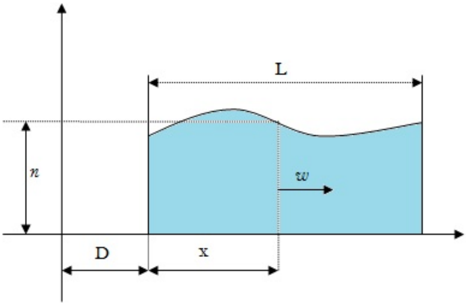

Figure: Fluid in the 1-D tank

- L is the length of the tank and, for simplicity, we assume that $L = \pi$;
- $\eta(t, x)$ is the elevation of the fluid surface at time t and at the position $x \in (0, \pi)$;
- $\omega(t, x)$ is the horizontal fluid velocity (for some parameter $\theta \in [0, 1]$) *in a referential attached to the tank* at time t and at the position $x \in (0, \pi)$;
- $D = D(t)$ is the horizontal displacement of the tank;
- $s = s(t)$ is the horizontal velocity of the tank;
- $u = u(t)$ is the horizontal acceleration of the tank.

$$\frac{dD}{dt} = s, \quad \frac{ds}{dt} = u \quad \text{and} \quad \frac{d^2D}{dt^2} = u.$$

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The full dynamics

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} = 0 \\ \omega_t + \eta_x + c\eta_{xxx} = -u(t) \\ \frac{ds}{dt} = u \\ \frac{dD}{dt} = s, \end{cases}$$

where $0 < x < L$ and $t > 0$, with the boundary conditions

$$\begin{cases} \eta_x(t,0) = \eta_x(t,L) = -u(t) \\ \omega(t,0) = \omega(t,L) = 0 \\ \omega_{xx}(t,0) = \omega_{xx}(t,L) = 0. \end{cases}$$

Since $\ddot{D}(t) = u(t)$, we have the following initial conditions

$$\eta(0,x) = \eta^0(x), \quad \omega(0,x) = \omega^0(x), \quad D(0) = D^0, \quad \dot{D}(t) = D^1.$$

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A control problem

Can we move the tank, continuously, from any steady state to any other steady state?

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Exact controllability

The system is exactly controllable in some appropriate Hilbert space $\mathcal{H}$ when $u(\cdot) \in L^2(0, T)$.

More precisely, *given* $T > 0$, *the initial state* $(\eta^0, \omega^0, D^0, D^1)$ *and the terminal state* $(\eta^T, \omega^T, D^{0,T}, D^{1,T})$ *in* $\mathcal{H}$, *we can find a control* $u \in L^2(0, T)$ *such that the system admits a solution satisfying* $(\eta(T), \omega(T), D(T), \dot{D}(T)) = (\eta^T, \omega^T, D^{0,T}, D^{1,T})$.

A classical duality approach:

- S. Dolecki, D. L. Russell, A general theory of observation and control, SIAM J. Control Optimization **15** (1977), no. 2, 185–220.
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⇒ Proof of an observability inequality for the solutions of the adjoint system.

- Fourier expansion of the solutions of the adjoint system.
- Splitting into high/small frequencies.
- Ingham's inequality and
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Global well-posedness for the Boussinesq system

We first consider the following system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} = f, \\ \omega_t + \eta_x + c\eta_{xxx} = g, \\ \eta_x(t,0) = \eta_x(t,\pi) = 0, \\ \omega(t,0) = \omega(t,\pi) = \omega_{xx}(t,0) = \omega_{xx}(t,\pi) = 0, \\ \eta(0,x) = \eta^0(x), \quad \omega(0,x) = \omega^0(x), \end{cases} \tag{2}$$

where $0 < x < \pi$ and $t > 0$. At least, formally,

$$(\eta,\omega)(t,x) = \sum_{k\geqslant 1} (\widehat{\eta}_k(t)\cos(kx), \widehat{\omega}_k(t)\sin(kx)),$$

where

$$\begin{cases} (\widehat{\eta}_k)_t + k\widehat{\omega}_k - ak^3\widehat{\omega}_k = \widehat{f}_k, & 0 < t < T, \\ (\widehat{\omega}_k)_t - k\widehat{\eta}_k + ck^3\widehat{\eta}_k = \widehat{g}_k, & 0 < t < T, \\ \widehat{\eta}_k(0) = \widehat{\eta}^0_k, \quad \widehat{\omega}_k(0) = \widehat{\omega}^0_k. \end{cases} \tag{3}$$

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If we set

$$A(k) = \begin{pmatrix} 0 & 1 - ak^2 \\ -1 + ck^2 & 0 \end{pmatrix},$$

it is easy to see that (3) is equivalent to

$$\begin{pmatrix} \widehat{\eta}_k \\ \widehat{\omega}_k \end{pmatrix}_t + kA(k) \begin{pmatrix} \widehat{\eta}_k \\ \widehat{\omega}_k \end{pmatrix} = \begin{pmatrix} \widehat{f}_k \\ \widehat{g}_k \end{pmatrix}, \qquad \begin{pmatrix} \widehat{\eta}_k \\ \widehat{\omega}_k \end{pmatrix}(0) = \begin{pmatrix} \widehat{\eta}_k^0 \\ \widehat{\omega}_k^0 \end{pmatrix}.$$

Note that the eigenvalues of the matrix $A(k)$ are with

$$\sigma(k) = \pm\sqrt{(1 - ak^2)(-1 + ck^2)},$$

and that they are purely imaginary.

We introduce the notations

$$w_1(k) = 1 - ak^2, \quad w_2(k) = 1 - ck^2.$$

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We introduce the space $V^s = H^s_{even}(0,\pi) \times H^s_{odd}(0,\pi)$, where

$$H^s_{odd}(0,\pi) = \left\{ u = \sum_{k\geq 1} \widehat{u}_k \sin(kx);\ \|u\|^2_{H^s_{odd}(0,\pi)} := \sum_{k\geq 1} k^{2s} |\widehat{u}_k|^2 < \infty \right\}$$

$$H^s_{even}(0,\pi) = \left\{ u = \sum_{k\geq 1} \widehat{u}_k \cos(kx);\ \|u\|^2_{H^s_{even}(0,\pi)} := \sum_{k\geq 1} k^{2s} |\widehat{u}_k|^2 < \infty \right\},$$

endowed with the norm

$$\|(\eta,\omega)\|^2_{V^s} := \|\eta\|^2_{H^s_{even}} + \|\mathcal{H}\omega\|^2_{H^s_{odd}}. \tag{4}$$

The operator $\mathcal{H}$ that appears in (4) is defined in the following way:

$$\mathcal{H}\left(\sum_{k\geqslant 1} \widehat{\omega}_k \sin(kx) \right) = \sum_{k\geqslant 1} \sqrt{\frac{w_1(k)}{w_2(k)}}\, \widehat{\omega}_k \sin(kx).$$

Theorem

The family of linear operators $\{S(t)\}_{t\in\mathbb{R}}$ *defined by*

$$S(t)(\eta^0,\omega^0) = \sum_{k\geqslant 1}(\widehat{\eta}_k(t)\cos(kx), \widehat{\omega}_k(t)\sin(kx)), \tag{5}$$

where the Fourier coefficients of $(\eta(t),\omega(t))$ *are obtained from those of* (η^0,ω^0) *by*

$$\begin{cases} \widehat{\eta}_k(t) = \cos(k\lambda(k)t)\widehat{\eta}_k^0 - \sqrt{\dfrac{w_1(k)}{w_2(k)}}\sin(k\lambda(k)t)\widehat{\omega}_k^0, \\ \widehat{\omega}_k(t) = \sqrt{\dfrac{w_2(k)}{w_1(k)}}\sin(k\lambda(k)t)\widehat{\eta}_k^0 + \cos(k\lambda(k)t)\widehat{\omega}_k^0, \end{cases}$$

is a group of isometries in V^s*, for any* $s\in\mathbb{R}$*.*

Theorem

The infinitesimal generator of the group $\{S(t)\}_{t\in\mathbb{R}}$ is the unbounded operator $(D(\mathcal{A}),\mathcal{A})$, where $D(\mathcal{A})=V^{s+3}$ and

$$\mathcal{A}(\eta,\omega)=\left(-\omega_x-a\omega_{xxx},-\eta_x-c\eta_{xxx}\right),\quad \forall(\eta,\omega)\in D(\mathcal{A}).$$

Theorem

Let $T>0$ and $s\in\mathbb{R}$ be given. If $(\eta^0,\omega^0)\in V^s$ and $(f,g)\in C^1([0,T],V^{s-3})$, then the problem admits a unique solution $(\eta,\omega)\in C([0,T],V^s)\cap C^1([0,T],V^{s-3})$. Moreover, there exists a positive constant $C>0$, such that

$$\|(\eta,\omega)\|_{C([0,T],V^s)}+\|(\eta,\omega)\|_{C^1([0,T],V^{s-3})}\leqslant C\left[\|(f,g)\|_{C^1([0,T];V^{s-3})}+\left\|(\eta^0,\omega^0)\right\|_{V^s}\right].$$

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$$\|(\eta,\omega)\|_{C([0,T],V^s)} + \|(\eta,\omega)\|_{C^1([0,T],V^{s-3})} \leqslant C\left[\|(f,g)\|_{C^1([0,T];V^{s-3})} + \left\|(\eta^0,\omega^0)\right\|_{V^s}\right].$$

Global well-posedness for the tank problem

Now consider the following system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} = 0 \\ \omega_t + \eta_x + c\eta_{xxx} = -u(t) \\ \ddot{D}(t) = u(t), \end{cases}$$

where $0 < x < L$ and $t > 0$, with the boundary conditions

$$\begin{cases} \eta_x(t,0) = \eta_x(t,L) = -u(t) \\ \omega(t,0) = \omega(t,L) = 0 \\ \omega_{xx}(t,0) = \omega_{xx}(t,L) = 0, \end{cases}$$

and the following initial conditions

$$\eta(0,x) = \eta^0(x), \quad \omega(0,x) = \omega^0(x), \quad D(0) = D^0, \quad \dot{D}(t) = D^1.$$

The change of functions

$$(\varphi(t,x),\psi(t,x)) = (\eta(t,x),\omega(t,x)) - S(t)(\eta^0,\omega^0) + (u(t)\phi(x),0)$$

transforms the previous system into

$$\begin{cases} \varphi_t + \psi_x + a\psi_{xxx} = f := u'(t)\phi(x), \\ \psi_t + \varphi_x + c\varphi_{xxx} = g := u(t)(-1+\phi'(x)+c\phi'''(x)), \\ \ddot{D}(t) = u(t), \\ \varphi_x(t,0) = \varphi_x(t,\pi) = 0, \\ \psi(t,0) = \psi(t,\pi) = 0, \\ \psi_{xx}(t,0) = \psi_{xx}(t,\pi) = 0, \\ \varphi(0,x) = 0, \quad \psi(0,x) = 0, \quad D(0) = D^0, \quad \dot{D}(0) = D^1, \end{cases}$$

for a convenient $\phi = \phi(x)$ and $u \in C^2([0,T],\mathbb{R})$; $u(0) = 0$.

Exact controllability

For each s, we introduce the spaces

$$\widehat{H}^s_{odd}(0,\pi) = \{u \in H^s_{odd}(0,\pi); \sum_{n\geq 1} |c_n|^2 n^{2s} < \infty \text{ e } c_n = 0 \text{ for } n \in 2\mathbb{Z}\};$$

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$$\mathcal{H} = \widehat{H}^1_{even} \times \widehat{H}^1_{odd} \times \mathbb{R} \times \mathbb{R} \text{ and } \mathcal{H}' = \widehat{H}^{-1}_{even} \times \widehat{H}^{-1}_{odd} \times \mathbb{R} \times \mathbb{R}.$$

Theorem

Let $T > 0$. Then, for any $(\eta^0, \omega^0, D^0, D^1) \in \mathcal{H}'$ and any $(\eta^T, \omega^T, D^{0,T}, D^{1,T}) \in \mathcal{H}'$, there exists a control input $u \in L^2(0,T)$ such that the solution (η, ω, D) of the system satisfies $(\eta(T), \omega(T), D(T), \dot{D}(T)) = (\eta^T, \omega^T, D^{0,T}, D^{1,T})$.

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The adjoint system

We consider (p, q, E), solution of

$$\begin{cases} p_t + q_x + cq_{xxx} = 0, \\ q_t + p_x + ap_{xxx} = 0, \\ \ddot{E}(t) = 0, \end{cases}$$

satisfying the boundary conditions

$$\begin{cases} p_x(t,0) = p_x(t,\pi) = 0, \\ q(t,0) = q(t,\pi) = 0, \quad q_{xx}(t,0) = q_{xx}(t,\pi) = 0, \end{cases}$$

and initial conditions

$$p(T,x) = p^T(x), \quad q(T,x) = q^T(x), \quad E(T) = E^{0,T}, \quad \dot{E}(T) = E^{1,T},$$

where $0 < x < \pi$ and $t > 0$.

Observe that $E(t) = \beta t + \alpha$, where $\alpha = E^0$, $\beta = E^1$.

The adjoint system

We consider (p, q, E), solution of

$$\begin{cases} p_t + q_x + cq_{xxx} = 0, \\ q_t + p_x + ap_{xxx} = 0, \\ \ddot{E}(t) = 0, \end{cases}$$

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$$\begin{cases} p_x(t,0) = p_x(t,\pi) = 0, \\ q(t,0) = q(t,\pi) = 0, \quad q_{xx}(t,0) = q_{xx}(t,\pi) = 0, \end{cases}$$

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where $0 < x < \pi$ and $t > 0$.

Observe that $E(t) = \beta t + \alpha$, where $\alpha = E^0$, $\beta = E^1$.

Definition of the solution by transposition

Multiply the first equation of the system by p, the second one by q and the third one by $E(t)$. Integrating by parts over $(0,T)\times(0,\pi)$ and assuming that the functions (η,ω,D) and (p,q,E) are sufficiently regular, we obtain

$$-\int_0^T\int_0^\pi \eta(p_t+q_x+cq_{xxx})dxdt-\int_0^T\int_0^\pi \omega(q_t+p_x+ap_{xxx})dxdt$$
$$+\int_0^\pi \{\cdots\cdots\}_0^T\,dx+\int_0^T \{\cdots\cdots\}_0^\pi\,dt=-\int_0^T\int_0^\pi u(t)qdxdt,$$
$$\left[\dot D(t)E(t)\right]_0^T-\left[D(t)\dot E(t)\right]_0^T+\int_0^T D(t)\ddot E(t)dt=\int_0^T u(t)E(t)dt.$$

If (p,q,E) is a solution of (6), then we obtain

$$\int_0^\pi [\eta p]_0^T + \int_0^\pi [\omega q]_0^T - c\int_0^T [\eta_x q_x]_0^\pi = -\int_0^T\int_0^\pi u(t) q dx dt$$
$$\left[\dot D(t)E(t)\right]_0^T - \left[D(t)\dot E(t)\right]_0^T = \int_0^T u(t)E(t)dt.$$

Definition

A function $(\eta, \omega, D) \in \mathcal{H}' := \widehat{H}^{-1}_{even} \times \widehat{H}^{-1}_{odd} \times \mathbb{R} \times \mathbb{R}$, such that

$$\left\langle (\eta(t), \omega(t), -\dot D(t), D(t)), (p(t), q(t), E(t), \dot E(t)) \right\rangle_{\mathcal{H}',\mathcal{H}}$$
$$= -\int_0^t u(\tau)\left\{\int_0^\pi (q + cq_{xx})dx + E(\tau)\right\} d\tau \qquad (6)$$
$$+ \left\langle (\eta^0, \omega^0, -D^1, D^0), (p^0, q^0, E(0), \dot E(0)) \right\rangle_{\mathcal{H}',\mathcal{H}}$$

solution by transposition of the tank model.

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A function $(\eta, \omega, D) \in \mathcal{H}' := \widehat{H}^{-1}_{even} \times \widehat{H}^{-1}_{odd} \times \mathbb{R} \times \mathbb{R}$, such that

$$\begin{aligned} &\left\langle (\eta(t), \omega(t), -\dot D(t), D(t)), (p(t), q(t), E(t), \dot E(t)) \right\rangle_{\mathcal{H}',\mathcal{H}} \\ &= -\int_0^t u(\tau)\left\{ \int_0^\pi (q + c q_{xx}) dx + E(\tau) \right\} d\tau \\ &+ \left\langle (\eta^0, \omega^0, -D^1, D^0), (p^0, q^0, E(0), \dot E(0)) \right\rangle_{\mathcal{H}',\mathcal{H}} \end{aligned} \tag{6}$$

solution by transposition of the tank model.

- Identity (6) defines $(\eta(t), \omega(t), -\dot{D}(t), D(t)) \in \mathcal{H}'$ in a unique way and $(\eta, \omega, -\dot{D}, D) \in C([0, T]; \mathcal{H}')$.
- We can assume that $D^0 = D^1 = 0$ and $\eta^0 = \omega^0 = 0$. Then, the following equivalent condition for the controllability holds:

$$-\dot{D}(T)E^{0,T} + D(T)E^{1,T} + \left\langle \eta(T), p^T \right\rangle_{\widehat{H}^{-1}_{even}, \widehat{H}^1_{even}} + \left\langle \omega(T), q^T \right\rangle_{\widehat{H}^{-1}_{odd}, \widehat{H}^1_{odd}} + \int_0^T u(t) \left\{ \int_0^\pi (q + cq_{xx})dx + E(t) \right\} dt = 0.$$

Observability Inequality: For some $C > 0$,

$$\left|E^{0,T}\right|^2 + \left|E^{1,T}\right|^2 + \left\|p^T\right\|_1^2 + \left\|q^T\right\|_1^2 \leqslant C \int_0^T \left| \int_0^\pi (q + cq_{xx})dx + E(t) \right|^2 dt.$$

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The change of variables $t \to T - t$ and $x \to L - x$ give us the following "initial conditions"

$$p^0(x) = p^T(\pi - x),\ q^0(x) = q^T(\pi - x),\ E^0, = E^{0,T},\ E^1 = -E^{1,T}.$$

Therefore, the above observability inequality is equivalent to the following one:

$$\left|E^0\right|^2 + \left|E^1\right|^2 + \left\|p^0\right\|_1^2 + \left\|q^0\right\|_1^2 \leqslant C\int_0^T \left|\int_0^\pi (q + cq_{xx})dx + E(t)\right|^2 dt,$$

for some constant $C > 0$ and any $(p^0, q^0, E^0, E^1) \in \mathcal{H}$ corresponding solution (p, q, E) of the "new" adjoint system.

Proof of the observability inequality

- First case: $E \equiv 0$:

There exists $C > 0$, such that

$$\|p^0\|_1^2 + \|q^0\|_1^2 \leqslant C \int_0^T \left| \int_0^\pi (q + cq_{xx}) dx \right|^2 dt.$$

Observe that

$$(p, q) = \sum_{k \geqslant 1} (\widehat{p}_k(t) \cos(kx), \widehat{q}_k(t) \sin(kx)),$$

where

$$\widehat{p}_k = \cos[k\lambda(k)t]\widehat{p}_k^0 - \sqrt{\frac{\tilde{w}_1}{\tilde{w}_2}} \sin[k\lambda(k)t]\widehat{q}_k^0$$

$$\widehat{q}_k = \sqrt{\frac{\tilde{w}_2}{\tilde{w}_1}} \sin[k\lambda(k)t]\widehat{p}_k^0 + \cos[k\lambda(k)t]\widehat{q}_k^0$$

with $\tilde{w}_1(k) = 1 - ck^2$ and $\tilde{w}_2(k) = 1 - ak^2$.

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We have that

$$\int_0^T \left| \int_0^\pi (q + cq_{xx}) dx \right|^2 dt = \int_0^T \left| \sum_{\substack{k \in \mathbb{Z} \\ |k| \text{ odd}}} a_k e^{i\mu_k t} \right|^2 dt,$$

and

$$\left\| p^0 \right\|_1^2 + \left\| q^0 \right\|_1^2 \leqslant \sum_{\substack{k \in \mathbb{Z} \\ |k| \text{ odd}}} |a_k|^2.$$

where a_k and μ_k can be computed explicitly.

From Ingham's inequality,

$$\sum_{k \in \mathbb{Z}} |a_k|^2 \leqslant C^T \int_0^T \left| \sum_{k \in \mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \leqslant D^T \sum_{k \in \mathbb{Z}} |a_k|^2,$$

for some positive constants C^T and D^T.

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$$
\begin{aligned}
\left\|(p^0, q^0)\right\|_{V^1}^2 &\leqslant C \sum_{\substack{k\in\mathbb{N} \\ k \text{ odd}}} \frac{(1-ck^2)^2}{k^2} \left(\frac{\tilde{w}_2}{\tilde{w}_1} \left|\widehat{p}_k^0\right|^2 + \left|\widehat{q}_k^0\right|^2 \right) \\
&\leqslant \sum_{\substack{k\in\mathbb{Z} \\ |k| \text{ odd}}} |a_k|^2 \\
&\leqslant CC^T \int_0^T \left| \sum_{\substack{k\in\mathbb{Z} \\ |k| \text{ odd}}} a_k e^{i\mu_k t} \right|^2 dt \\
&\leqslant CC^T \int_0^T \left| \int_0^\pi (q + cq_{xx}) dx \right|^2 dt.
\end{aligned}
$$

- Seconde case: $\ddot{E}(t) = 0$

$$E(t) = \beta t + \alpha, \text{ where } \alpha = E^0 \text{ and } \beta = E^1$$

Observability inequality: for some $C > 0$,

$$\left|E^0\right|^2 + \left|E^1\right|^2 + \left\|p^0\right\|_1^2 + \left\|q^0\right\|_1^2 \leqslant C \int_0^T \left| \int_0^\pi (q + cq_{xx})dx + E(t) \right|^2 dt.$$

Set $f(t) = \int_0^\pi (q + cq_{xx})dx$. Then,

$$\int_0^T |f(t) + E(t)|^2 \, dt = \int_0^T |f(t)|^2 \, dt + 2\int_0^T f(t)E(t)dt + \int_0^T |E(t)|^2 \, dt.$$

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Proof of the observability inequality

If the statement is false, then there exists a sequence

$$(p_n^0, q_n^0, E_n^0, E_n^1)_{n\geq 0} \text{ in } \mathcal{H} := \widehat{H}^1_{even} \times \widehat{H}^1_{odd} \times \mathbb{R} \times \mathbb{R},$$

satisfying

$$\left\|p_n^0\right\|_1^2 + \left\|q_n^0\right\|_1^2 + |E_n^0|^2 + |E_n^1|^2 = 1, \quad \forall n \geq 0,$$

and such that

$$\int_0^T |f_n(t) + E_n(t)|^2 \, dt \to 0 \text{ as } n \to \infty.$$

Extracting a subsequence, still denoted $(p_n^0, q_n^0, E_n^0, E_n^1)_{n\geq 0}$, we have that

$$(p_n^0, q_n^0, E_n^0, E_n^1) \rightharpoonup (p^0, q^0, E^0, E^1) \quad \text{in } \mathcal{H};$$

that is,

$$\begin{array}{lll} p_n^0 \rightharpoonup p^0 & \text{in} & \widehat{H}^1_{even}(0,\pi), \\ q_n^0 \rightharpoonup q^0 & \text{in} & \widehat{H}^1_{odd}(0,\pi), \\ E_n^0 \to E^0 & \text{in} & \mathbb{R}, \\ E_n^1 \to E^1 & \text{in} & \mathbb{R}. \end{array}$$

Since the embedding $H^1(0,\pi) \hookrightarrow L^2(0,\pi)$ is compact, we have for a subsequence still denoted $(p_n^0, q_n^0, E_n^0, E_n^1)_{n\geq 0}$

$$p_n^0 \to p^0 \quad \text{in} \quad L^2(0,\pi),$$
$$q_n^0 \to q^0 \quad \text{in} \quad L^2(0,\pi),$$
$$E_n(t) = E_n^1 t + E_n^0 \to E(t) = E^1 t + E^0 \quad \text{in} \quad L^2(0,T).$$

On the other hand,

$$\left\|p_n^0\right\|_1^2 + \left\|q_n^0\right\|_1^2 + |E_n^0|^2 + |E_n^1|^2 \leqslant C^T \left(\int_0^T |f_n(t) + E_n(t)|^2 \, dt + \left\|p_n^0\right\|_0^2 + \left\|q_n^0\right\|_0^2 \right),$$

for all $T > 0$, i. e.,

$(p_n^0, q_n^0, E_n^0, E_n^1)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H} := \widehat{H}^1_{even} \times \widehat{H}^1_{odd} \times \mathbb{R} \times \mathbb{R}$.

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$$\begin{aligned} &p_n^0 \to p^0 \quad \text{in} \quad L^2(0,\pi),\\ &q_n^0 \to q^0 \quad \text{in} \quad L^2(0,\pi),\\ &E_n(t) = E_n^1 t + E_n^0 \to E(t) = E^1 t + E^0 \quad \text{in} \quad L^2(0,T). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left\|p_n^0\right\|_1^2 + \left\|q_n^0\right\|_1^2 + |E_n^0|^2 + |E_n^1|^2 \\ &\qquad \leqslant C^T \left(\int_0^T |f_n(t) + E_n(t)|^2 \, dt + \left\|p_n^0\right\|_0^2 + \left\|q_n^0\right\|_0^2 \right), \end{aligned}$$

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We infer that $(p_n^0, q_n^0, E_n^0, E_n^1)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}$, which allows us to conclude that

$$\left\|p^0\right\|_1^2 + \left\|q^0\right\|_1^2 + |E^0|^2 + |E^1|^2 = 1 \tag{7}$$

and

$$\int_0^T \left| \int_0^\pi (q + cq_{xx})dx + E(t) \right|^2 dt = 0, \tag{8}$$

where (p, q, E) is a solution of the problem.

Claim. For $T > 0$, let N_T denote the space of the (initial) states $(p^0, q^0, E^0, E^1) \in \mathcal{H}$ such that the corresponding solution (p, q, E) satisfies $\displaystyle\int_0^\pi (q + cq_{xx})dx + E(t) = 0$ in $L^2(0, T)$. Then, $N_T = \{0\}$ for all $T > 0$.

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If $N_T \neq \{0\}$, the map

$$(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T \to \mathcal{A}(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T$$

(where $\mathbb{C}N_T$ denotes the complexification of N_T) has at least one eigenvalue; that is, *there exist* $\lambda \in \mathbb{C}$ *and an initial state* $(p^0, q^0, E^0, E^1) \in \mathbb{C}N_T \setminus \{(0,0,0,0)\}$, *such that*

$$\begin{cases} \lambda p^0 = -q^0_x - c q^0_{xxx}, \\ \lambda q^0 = -p^0_x - a p^0_{xxx}, \\ \lambda E^0 = E^1, \\ \lambda E^1 = 0, \\ p^0_x(0) = p^0_x(\pi) = 0, \\ q^0(0) = q^0(\pi) = q^0_{xx}(0) = q^0_{xx}(\pi) = 0, \end{cases}$$

and $\int_0^\pi (q + c q_{xx}) dx + E(t) = 0$ in $(0, T)$.

We can prove that $(p^0, q^0, E^0, E^1) = (0,0,0,0)$.

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$$\begin{cases} \lambda p^0 = -q^0_x - cq^0_{xxx}, \\ \lambda q^0 = -p^0_x - ap^0_{xxx}, \\ \lambda E^0 = E^1, \\ \lambda E^1 = 0, \\ p^0_x(0) = p^0_x(\pi) = 0, \\ q^0(0) = q^0(\pi) = q^0_{xx}(0) = q^0_{xx}(\pi) = 0, \end{cases}$$

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We can prove that $(p^0, q^0, E^0, E^1) = (0,0,0,0)$.

Final comments

- S. Micu, J. H. Ortega, L. Rosier, B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete and Continuous Dynamical Systems 24 (2009), 273–313.
- J.-M. Coron, Local Controllability of a 1-D tank containing a fluid modeled by the shallow water equations, ESAIM Control Optim. Calc. Var. 8 (2002), 513–554.

  The motion of the fluid was described by the so-called shallow water (or Saint-Venant) equations, obtained from the Boussinesq system by letting $a = b = c = d = 0$.
- It would be interesting to prove a similar result for the full Boussinesq system (still with $b = d = 0$). This cannot be done through a simple linearization argument, since $((\eta\omega)_x, \omega\omega_x)$ is not expected to belong to $\widehat{H}^r_{even} \times \widehat{H}^r_{odd}$ for some r when $(\eta, \omega) \in \widehat{H}^s_{even} \times \widehat{H}^s_{odd}$ for some s.

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- S. Micu, J. H. Ortega, L. Rosier, B.-Y. Zhang, Control and stabilization of a family of Boussinesq systems, Discrete and Continuous Dynamical Systems 24 (2009), 273–313.
- J.-M. Coron, Local Controllability of a 1-D tank containing a fluid modeled by the shallow water equations, ESAIM Control Optim. Calc. Var. 8 (2002), 513–554.
  The motion of the fluid was described by the so-called shallow water (or Saint-Venant) equations, obtained from the Boussinesq system by letting $a = b = c = d = 0$.
- It would be interesting to prove a similar result for the full Boussinesq system (still with $b = d = 0$). This cannot be done through a simple linearization argument, since $((\eta\omega)_x, \omega\omega_x)$ is not expected to belong to $\widehat{H}^r_{even} \times \widehat{H}^r_{odd}$ for some r when $(\eta, \omega) \in \widehat{H}^s_{even} \times \widehat{H}^s_{odd}$ for some s.