Weierstrass $\wp$ traveling solutions for BBM type equations

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Outline

Summary

- to obtain traveling wave solutions to many nonlinear dispersive eq. with dissipation
- we apply the derivation to BBM
- via reductions to 1$^{\text{st}}$ kind Abel, with polynomial nonlinearities and dissipation
- we explain why such integration via $\wp$ functions can be performed via genus of curves
- we show equivalence between nonlinear dissipative PDEs and classical ODE theory
- we present graphs of closed form solutions of $\wp$ functions from which in limiting cases classical solutions can be obtained

Ansatz

- certain classes of PDEs can be reduced via via traveling wave reduction $\zeta = x - vt$ into the ODE

$$u_{\zeta\zeta} + f_2(u)u_\zeta + f_3(u) + f_1(u)u_\zeta^2 + f_0(u)u_\zeta^3 = 0 \tag{1}$$

Examples KdV- type

KdV-Burgers: $\delta u_{\zeta\zeta\zeta} = \nu u_{\zeta\zeta} - \alpha u u_\zeta - c u_\zeta$

Gardner: $\delta u_{\zeta\zeta\zeta} = \nu u_{\zeta\zeta} - \alpha u u_\zeta - c u_\zeta - \beta u^2 u_\zeta$

Fisher: $u_\zeta = u_{\zeta\zeta} + \alpha u(1-u)$

Ginzburg-Landau: $-c u_\zeta = \epsilon u + \nu_1 u_{\zeta\zeta} + \nu_3 u|u|^2 + \nu_5 u|u|^4$

- after reduction $f_0(u) = f_1(u) = 0$
- $f_2(u), f_3(u)$ are polynomials

Abel

- letting $u_\zeta = \eta(u)$, we obtain a 2nd kind Abel's equation

$$\eta \frac{d\eta}{du} + f_2(u)\eta + f_3(u) + f_1(u)\eta^2 + f_0(u)\eta^3 = 0 \qquad (2)$$

- 2nd kind can be transformed into a 1st kind via $\eta = \frac{1}{y}$

$$\frac{dy}{du} = f_0(u) + f_1(u)y + f_2(u)y^2 + f_3(u)y^3 \qquad (3)$$

- it is still not known how to integrate it for general $f_i(u)$, for special cases, see Kamke [3] (normal, canonical form)

BBM equation

- The Benjamin-Bona-Mahony (1972) equation [1] or regularized long-wave equation (small wave amplitude, large wavelength, inviscid, incompressible flow)

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{4}$$

  is a popular alternative to the Korteweg-de Vries (KdV) (1895) [4] for modeling long waves in a wide class of nonlinear dispersive systems.
- A generalization to BBM to include a viscous term is given by

$$u_t + u_x + uu_x - u_{xxt} = \nu u_{xx} \tag{5}$$

  where $\nu > 0$ is transformed kinematic viscosity.

BBM equation

- Benjamin *et. al* showed (5) has similar properties as KdV but solutions have better smoothness properties
- moreover, the linearized version has dispersion relation for which c_p, c_g are bounded for $\forall k$, so fine scales features tend not to propagate
- existence and stability of the solitary waves solutions of (5) has been investigated by Benjamin *et. al* , Bona *et. al* , Pritchard, Scott, Tzvetkov, etc.
- here, we will find the general solutions of (5) using Weierstrass $\wp$ functions without simplifying assumptions

Abel for BBM

- using $\zeta = x - vt$ and integrating once (keeping the integration constant)

$$(1-c)u + \frac{u^2}{2} + cu_{\zeta\zeta} = \nu u_\zeta + A \tag{6}$$

- which leads to polynomials

$$\begin{aligned} f_2(u) &= -\frac{\nu}{c} \\ f_3(u) &= \frac{1}{2c}u^2 + \frac{1-c}{c}u - \frac{A}{c} \end{aligned} \tag{7}$$

- in Abel's equation

$$\frac{dy}{du} = f_2(u)y^2 + f_3(u)y^3 \tag{8}$$

Weierstrass no dissipation

- if $f_2(u) = 0$ then (8) is separable with solution given by the elliptic curve

$$\eta^2 = q_3(u) \equiv -\frac{u^3}{3c} - \frac{1-c}{c}u^2 + \frac{2A}{c}u - 2B \qquad (9)$$

- therefore the well known solutions for nondisipative BBM are found easily from the elliptic equation

$$u_\zeta^2 = q_3(u) \qquad (10)$$

- which can be transformed in standard form

$$\hat{u}_\zeta^2 = 4\hat{u}^3 - g_2\hat{u} - g_3 \qquad (11)$$

- via transformation $u = -\sqrt[3]{12c}\hat{u} - (1-c)$

Classical solutions

- which has solution $\hat{u}(\zeta) = \wp(\zeta, g_2, g_3)$ with invariants

$$g_2 = \sqrt[3]{\frac{12}{c^2}}(c-1)^2 > 0 \tag{12}$$

$$g_3 = \frac{2(1-c)^3}{3c} \tag{13}$$

two limiting cases with assumption $A = 0, B = 0$:

$c \neq 1$

$c > 1 \rightarrow g_3 < 0$ fast waves solitary
$0 < c < 1 \rightarrow g_3 > 0$ slow waves periodic

$c = 1$

Jacobian elliptic functions with modulus $k = \sin\frac{5\pi}{12}$

Classical solutions

$c \neq 1$

$$u(x,t) = 3(c-1)\mathrm{Sech}^2\left[\frac{\sqrt{(c-1)/c}}{2}(x-ct)\right] \text{ if } c > 1 \qquad (14)$$

$$u(x,t) = -\frac{3c}{1+c}\mathrm{Sec}^2\left[\frac{\sqrt{c}}{2}\Big(x - t/(1+c)\Big)\right] \text{ if } 0 < c < 1 \qquad (15)$$

$c = 1$

$$u(x,t) = C\left[1 - \sqrt{3}\,\frac{1 \mp \mathrm{cn}\Big(\sqrt{\mathrm{C}}/\sqrt[4]{3}(\mathrm{x}-\mathrm{t}),\mathrm{k}\Big)}{1 \pm \mathrm{cn}\Big(\sqrt{\mathrm{C}}/\sqrt[4]{3}(\mathrm{x}-\mathrm{t}),\mathrm{k}\Big)}\right] \qquad (16)$$

Graphs of classical solutions

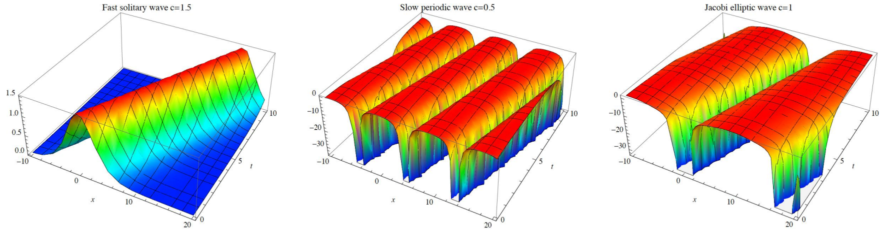

Figure : Traveling waves $\nu = 0$, left $c = 1.5$; middle $c = 0.5$; right $c = 1$

Weierstrass for dissipative BBM

- consider (8) in the form of non-authonomous eq $F(y, y_u, u) = 0$
- Poincare [6] proved in 1885 that any non-autonomous eq having genus $p = 1$ is integrable via Weierstrass $\wp$ functions, after linear fractional transformation
- $\forall$ elliptic functions $f(z) = A(\wp) + B(\wp)\wp'$, see Whittaker [7]

Lemke's transformation

- progress of integration of (8) is based on Lemke transformation $v = \int f_2(u)du = \frac{\nu}{c}u + D$ [5]

$$\frac{dy}{dv} = y^2 + g(v)y^3, \qquad (17)$$

- where $g(v) = a_2 v^2 + a_1 v + a_0 = \frac{f_3(v)}{f_2(v)}$
- letting $y = -\frac{1}{z}\frac{dz}{dv}$, we obtain the 2^{nd} order non-autonomous system

$$z^2 \frac{d^2 v}{dz^2} + g(v) = 0 \qquad (18)$$

- since $g(v)$ has no singularities $\rightarrow$ v has only poles of $O(2)$ $\rightarrow$ solution is an elliptic function
- Ince [2] proposed solutions to (18) of the type

$$v = Ez^p \omega(z^q) + F \tag{19}$$

which by substitution give $p = \frac{2}{5}$, $q = \frac{1}{5}$

- E, F are arbitrary constants to be determined next
- we will show next that the function ω satisfies an elliptic equation, while due to Lemke's transformation z satisfies a linear equation (23)

- all the above transformations can be combined into

$$u(\zeta) = \sigma - e^{-n\zeta}\omega(z(\zeta)) \tag{20}$$

- which by substitution into (5) leads to

$$(z')^2\ddot{\omega} + \left(z'' - (2n + \frac{\nu}{c})z'\right)\dot{\omega} + \left(n^2 + \frac{1 - c + \sigma + n\nu}{c}\right)\omega = \frac{1}{2c}e^{-n\zeta}\omega^2 \tag{21}$$

- The free term was eliminated by setting $A = \frac{\sigma^2}{2} + \sigma(1 - c)$.

By letting

$$z'' - \left(2n + \frac{\nu}{c}\right) z' = 0 \tag{22}$$

we obtain

$$z'(\zeta) = c_1 e^{(2n+\frac{\nu}{c})\zeta} \tag{23}$$

We also choose $\sigma = -(n^2 c + n\nu + 1 - c)$ which cancels the linear term in (21).

We are left to solve

$$(z')^2\ddot{\omega} - \frac{1}{2c}e^{-n\zeta}\omega^2 = 0 \qquad (24)$$

subject to (23). If $n = -\frac{2\nu}{5c}$, then

$$\sigma = \frac{14\nu^2}{25c} + c - 1 \qquad (25)$$

By substituting (23) into (24), we obtain

$$\ddot{\omega} = \frac{1}{2cc_1^2}\omega^2. \qquad (26)$$

Letting $c_1 = \frac{1}{2\sqrt{3c}}$, we arrive at the elliptic equation

$$\ddot{\omega} = 6\omega^2 \tag{27}$$

which by multiplication by $\dot{\omega}$ and integration becomes

$$(\dot{\omega})^2 = 4\omega^3 - g_3. \tag{28}$$

Its solution is

$$\omega(z) = \wp(z + c_3, 0, g_3) \tag{29}$$

with invariants $g_2 = 0$, and g_3.

Then, the general solution to (5) is

$$u(\zeta) = \frac{14\nu^2}{25c} \pm \sqrt{\left(\frac{14\nu^2}{25c}\right)^2 - 2A - e^{\frac{2\nu\zeta}{5c}}\wp\left(c_4 + \frac{5\sqrt{3c}}{6\nu}e^{\frac{\nu\zeta}{5c}}, 0, g_3\right)} \quad (30)$$

If $A = 0 \rightarrow c - 1 = \pm\frac{14\nu^2}{25c}$, and one selects the lower branch of the radical, we obtain

$$\begin{aligned} u(\zeta) &= -e^{\frac{2\nu\zeta}{5c}}\wp\left(c_4 + \frac{5}{\nu}\sqrt{\frac{c}{3a}}e^{\frac{\nu\zeta}{5c}}, 0, g_3\right) \text{ if } g_3 \neq 0 \\ u(\zeta) &= -\frac{e^{\frac{2\nu\zeta}{5c}}}{\left(c_6 \pm \frac{5\sqrt{3c}}{6\nu}e^{\frac{\nu\zeta}{5c}}\right)^2} \text{ if } g_3 = 0. \end{aligned} \quad (31)$$

Graphs of $\wp$ solutions

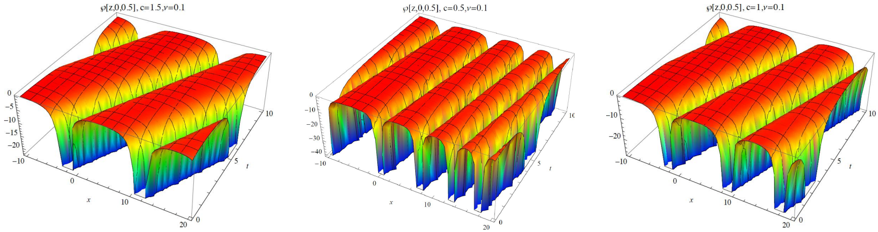

Figure : Weierstrass solutions $\nu = 0.1$ left $c = 1.5$; middle $c = 0.5$; right $c = 1$

Graphs of kink solutions

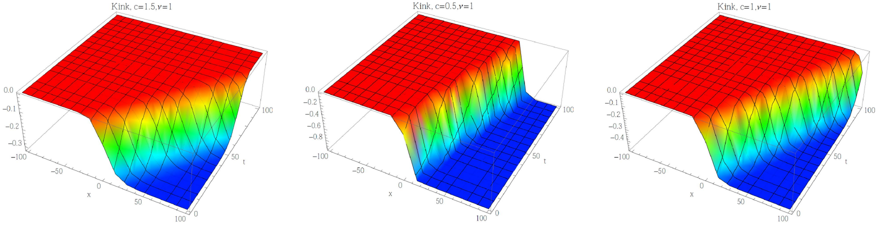

Figure : Kink solutions $\nu = 1$ left $c = 1.5$; middle $c = 0.5$; right $c = 1$

Nonlinear Waves Lab

- $16' \times 4' \times 4'$
- $3'$ water max
- $192ft^3 \approx 5500l$
- 1.5 yrs to build
- attracted $> 120k$ USD
- shallow and deep water for UWV research, supercavitation

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