Some evolution equations with non-local terms*

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1 Introduction: the setup.

Our goal is to describe the properties of the real valued solutions of the [1]Cauchy Problem (or initial value problem) associated to certain evolution equations. More precisely, consider

$$\begin{aligned} \partial_t u &= F(t,u) \in Y, \\ u &= u(t) \in X,\ t \in [0,T_0],\ T_0 > 0, \\ u(0) &= \phi \in X, \end{aligned} \tag{1}$$

*This is an ongoing work

[1]In general real valued solutions of the equtions that appear below. For instance, in the case if the Schrödinger equations, complex solutions cannot be avoided.

where X and Y are Banach spaces (of functions in general), where $Y \hookrightarrow X$ [2] (such as $H^s(\mathbb{R}^n)$, the usual Sobolev Spaces of L^2type, which arise in many practical applications),

$$F : [0, T_0] \times Y \longrightarrow X \tag{2}$$

where $T_0 \geq 0$, F is Lipschtz in some sense (which will be defined when needed), with the additional restriction that it is non-local. We say that F is local if supp$(F(v)) \subseteq$ supp(v). Otherwise F is non-local. Note that convolution operators, that is, operators of the form

$$(T_f(g))(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy, \tag{3}$$

are non-local in general[3].

One important subclass of such equations is that of the Benjamin-Ono type, that is, those of the form

$$\partial_t u(t) + \sigma L u(t) + G(u(t)) = 0, \tag{4}$$

or

$$\partial_t (u(t) + \sigma L u(t)) \quad + G(u(t)) = 0,^{4} \tag{5}$$

where L is a, possibly unbounded linear operator, G is, in general,

a nonlinear function of its arguments and σ denotes the Hilbert

[2]The symbol $\hookrightarrow$ indicates that the injection is continuous and dense.

[3]Sometimes, they have to be defined using principal values, as in the case of the Hilbert transform, but even so they are still non-local.

[4]Please observe where the pharenthesis following ∂_t are placed. That makes all the difference.

transform

$$(\sigma f)(x) = pv\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{y-x}dy. \tag{6}$$

This operator is responsible for non-locality in (4) and (5). In the case of (4), with $L = \partial_x^2$ and $G(u) = u\partial_x u$, we obtain the original Benjamin-Ono equation (BO), while, with the same nonlinearity, and $L = \partial_x$, we have a BO type version of the Benjamin-Bona-Mahony equation (BBM).

Before proceeding, it is important to understand what we mean by a solution of (1)[5]. We will say that (1) is locally well posed if there exists a $T \in [0, T_0]$ and a unique $u \in C([0,T], Y)$ such that $u(0) = \phi$,and the derivative with respect to time (the variable t [6]), is taken with respect to the topology of X, that is,

$$\lim_{h\longrightarrow 0}\left\|\frac{u(t+h)-u(t)}{h} - F(t, u(t))\right\|_X = 0. \tag{7}$$

Moreover, the solution must depend continuously on the initial data (and on any other relevant parameters occurring in the equation), in appropriate topologies. In what follows we will consider only the initial data. Then what we what mean by continuous dependence is: assume that $u_0^{(j)} \in Y$, $j = 1, 2, 3, ..., \infty$, let $u^{(j)}$ be the corresponding solutions. Suppose that

$$\lim_{j\to\infty}\left\|u_0^{(j)} - u_0^{(\infty)}\right\|_s = 0. \tag{8}$$

[5] The following definition has nothing to do with non-locality.

[6] Which sometimes is not really time, but represents some other variable that, unfortunaly, apears in the equations as a first derivative.

Then, for all $T' \in (0, T)$ we have,

$$\lim_{j \to \infty} \sup_{t \in [0,T']} \left\| u^{(j)}(t) - u^{(\infty)}(t) \right\|_s = 0. \tag{9}$$

In case the preceding properties are valid for all $T > 0$, we say that the problem is globally well-posed. Otherwise it is ill-posed. It deserves notice, that this definition is a refinement of the definition of well-posednesss due to Hadamard, because it contains the notion of persistence([7]): the solution lives in the same space as does the initial condition. Below we will be concerned with this question.

2 Some examples.

• The Simplest Equation of BO Type.

Consider,

$$\begin{cases} \partial_t u(t) + \sigma u(t) = 0, \\ u(0) = \phi \in L^2(\mathbb{R}). \end{cases} \tag{10}$$

[7]Which, as far as I know, was introduced some time ago by Tosio Kato. The point is that this is a non trivial condition, as we shall see in what follows.

Recall, if $f \in L^2(\mathbb{R})$**,**

$$(\sigma f)^{\wedge}(\xi) = isgn(\xi)\,\widehat{f}(\xi)\,,\ \xi - a.e., \tag{11}$$

where,

$$sgn(\xi) = \begin{cases} -1\, se\, \xi < 0, \\ 0\, se\, \xi = 0, \\ 1\, se\ \ \xi > 0 \end{cases} \tag{12}$$

and $\widehat{g}$ **denotes the Fourier transform of the tempered distribution** g**. In case** g **belongs to** $L^1(\mathbb{R})$**, our definition is**

$$\widehat{g}(\xi) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{\mathbb{R}} g(x) \exp(-i\xi x)\, dx. \tag{13}$$

In particular, σ **is a unitary operator in** $L^2(\mathbb{R})$ **(and in any** $H^s(\mathbb{R})\,, s \in R$**). Moreover,**

$$\sigma^2 = -1. \tag{14}$$

It is easy to verify that the unique solution of (10), with $X = Y = L^2(\mathbb{R})$ **is given by the unitary group**

$$u(t) = \exp(-\sigma t)\,\phi = (\cos t - \sigma \sin t)\,\phi,\ \ \phi \in L^2(\mathbb{R})\,. \tag{15}$$

Now consider the problem

$$\begin{cases} \partial_t u(t) + \sigma u(t) = 0, \\ u(0) = \phi \in L^2_1(\mathbb{R})\,, \end{cases} \tag{16}$$

where

$$L_s^2(\mathbb{R}^n) = (H^s(\mathbb{R}^n))^{\wedge}. \tag{17}$$

Thus ϕ **and** $x\phi \in L^2(\mathbb{R})$. **In particular,** $u(t) \in L^2(\mathbb{R})$ **for all** $t \in \mathbb{R}$. **Next, we consider persistence. We must verify if** $xu(t) \in L^2(\mathbb{R})$. **Take** $t_0 \neq 0$,**such that** $\sin t_0 \neq 0$ **(which is not difficult to do). Then look at**

$$xu(t_0) = (\cos t_0)\, x\phi - (\sin t_0)\, x\sigma\phi. \tag{18}$$

Thus

$$xu(t_0) \in L^2(\mathbb{R}) \Longleftrightarrow x\sigma\phi \in L^2(\mathbb{R}). \tag{19}$$

However

$$\begin{aligned} x(\sigma\phi)(x) &= pv\frac{1}{\pi}\int_{\mathbb{R}} \frac{x-y}{y-x}\phi(y)\,dy + pv\frac{1}{\pi}\int_{\mathbb{R}} \frac{y\phi(y)}{y-x}dy \\ &= -\frac{1}{\pi}\int_{\mathbb{R}} \phi(y)\,dy + \sigma(y\phi). \end{aligned} \tag{20}$$

(Note that $\phi \in L^1(\mathbb{R})$.**) Combining (18), (19) and (20) we conclude**

$$xu(t_0) \in L^2(\mathbb{R}) \Longleftrightarrow \widehat{\phi}(0 = 0). \tag{21}$$

Note this this apparently harmless equation, shows that we may expect trouble with persistence whenever the Hilbert transform and weighted spaces are involved in the problem. Finally, it is easy to see that problem (16) is globally well posed if and only $L_1^2(\mathbb{R})$ **is replaced by**

$$\mathfrak{F}_1(\mathbb{R}) = \left\{ \phi \in L_1^2(\mathbb{R}) \middle| \widehat{\phi}(0 = 0) \right\}. \tag{22}$$

This does not bode well in the case of the Benjamin-Ono equation in weighed Sobolev spaces.

• The Benjamin-Bona-Mahony Equation (BBM).

Consider the equation,

$$\partial_t u + u\partial_x u + \partial_{txx} u = 0 \tag{23}$$

This is an alternate model for the KdV equation. Note that it can be rewritten as

$$\left(1-\partial_x^2\right)\partial_t u = -\frac{1}{2}\partial_x\left(u^2\right). \tag{24}$$

However, since the operator $\left(1-\partial_x^2\right)$is invertible in general (for instance, in the Sobolev Spaces $H^s(\mathbb{R})$, $s \in R$), we obtain

$$\partial_t u = -\frac{1}{2}\left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2\right). \tag{25}$$

Due to the presence of the (infinitely) smoothing operator

$$\left(\left(1-\partial_x^2\right)^{-1} f\right)(x) = \frac{1}{2}\int_{-\infty}^{\infty}\left(\exp|x-y|\right) f(y)\,dy, \tag{26}$$

it is easy to prove global well-posedness in $H^s(\mathbb{R})$,$s \in Z^+ = \{1, 2, ...\}$,and then apply nonlinear interpolation to obtain the result for all $s \geq 1$. For details see [4], [5] and [6]. Moreover it is also easy to show that the Cauchy problem is locally well-posed in $L^2(\mathbb{R})$. Global well-posedness, in this space, is more difficult. It was proved by Bona

and Tzvetkov [7]. It deserves remark that their work also suggests that the problem may ill-posed (even locally) for $s < 0$. This was confirmed by M. Panthee ([18]).

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• The Hirota-Satsuma.

This equation is also a model for the phenomena described by K-dV and has some similarity with BBM, because it also contains the resolvent of a Schrödinger operator. This operator, however, is more difficult to handle, because it is not a free operator, as in BBM, but a full Schrödinger operator whose potential happens to be the solution we are seeking.

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3 Total ReKall or the Benjamin-Ono (BO) Equation.

The title of this example may surprise you, but to tell the truth this is one of the most fascinating equations that I have ever encountered in my life and while writing I am recalling our trajectory together. Of course, many other authors have worked with it. Back in 1985, I returned to Berkeley, to work with Tosio Kato who had been my thesis advisor. As usual, he told me to look for a problem.

He always wanted his students or pos-docs to find their own problems (and so do I, having been his student). Thus, I found BO in a book by Ablowitz et al [1], thought it seemed to be interesting and showed it to Kato. He said go ahead, work on it. The equation is

$$\partial_t u + u\partial_x u + \sigma\partial_x^2 u = 0, \tag{27}$$

which is very similar to KdV

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \tag{28}$$

but is full of surprises. Recall that σ denotes the Hilbert transform as defined in (11). The reason for the title above is that I have worked on this equation, on and off, for many years, and as I write this, I recall the trajectory of this "magical" equation in my life. Back in 2001, I was able to prove a remarkable result[8]. A bit later, my friends Fonseca, Linares and Ponce [9] proved that, what I had shown, could be extended in terms of function spaces, but not in terms of the fundamental assumptions on the equation. Three assumption were needed and only three. No more[9]. For this proof, I thank them, because, they showed that, in terms of assumptions on the equation, my results are sharp. In what follows I will try to explain what this is all about.

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[8]"If nobody praises you, do, praise yourself". This is a translation of a passage In *The Praise to Folly,* by Erasmus of Rotterdan (1509). The interested reader may consult the Internet. Besides, Kato would have critized me for what I wrote. He told me once that only mathematicians, such as Gauss, could praise themselves.

[9]Herein lies a mistery, because in the case of K-dV only two such asumptions are needed.

4 The Brinkman Flow.

Finally we turn to the properties of the real valued solutions of the Cauchy Problem associated to the Brinkman Flow ([8]), namely

$$\begin{cases} \phi\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v)=F\left(t,\rho\right), \\ (-\mu_{eff}\Delta+\frac{\mu}{k})v=-\nabla P(\rho), \\ (\rho(0),v(0))=(\rho_0,v_0), \end{cases} \tag{29}$$

which models fluid flow in certain types of porous media[10]. Here

$$\rho=\rho\left(t,x\right),\ v=v\left(t,x\right),\ t\geq 0,\ x\in\Omega\subseteq\mathbb{R}^n, \tag{30}$$

where Ω is a domain with a sufficiently smooth boundary, μ , k, and μ_{eff} denote the fluid viscosity, the porous media permeability and the pure fluid viscosity, respectively, while ρ is the fluid's density, v its velocity, P is the pressure, F is an external mass flow rate, and ϕ is the porosity of the medium. It should be observed that the second equation in (29), is a linear combination of Darcy's law and Stokes' law for fluid flows.

In what follows, for the sake of simplicity, we will be concerned (mostly) with the case $\Omega=\mathbb{R}$. Moreover, to simplify the notation, we will choose all the coefficients in (29) to be equal to 1. At the moment we are interested only in the mathematical structure of the system. At a later stage, if one desires, the constants can be

[10]One may, and must, sometimes, add boundary condititions (BC's). This makes the problem more difficult, because the BC's must be incorporateted into the differential operators involved, order to define them as bona fide operators in Banach spaces.

put back in, and various limiting cases can be studied. Thus our problem becomes:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \partial_x(\rho v) = F\left(t, \rho\right), \\ (-\partial_x^2 + 1)v = -\partial_x P(\rho), \\ (\rho(0), v(0)) = (\rho_0, v_0) \end{cases} \tag{31}$$

where $x \in \mathbb{R}$ **and** $t > 0$.

To handle (31), we compute $v\left(t, x\right)$ **using the second equation, (usually referred to as *Brinkman's condition*) to get**

$$v = -\left(1 - \partial_x^2\right)^{-1} \partial_x P\left(\rho\right) \tag{32}$$

and substitute it into the first one (which describes the variation of mass) to obtain the Cauchy Problem

$$\begin{cases} \partial_t \rho = \partial_x \left(\rho \left(1 - \partial_x^2\right)^{-1} \partial_x P\left(\rho\right)\right) \\ \rho\left(0\right) = \rho_0. \end{cases} \tag{33}$$

Then we solve (33), and compute v **using (32). Of course the following compatibility condition must be satisfied:**

$$v_0 = -\left(1 - \partial_x^2\right)^{-1} \partial_x P\left(\rho_0\right) \tag{34}$$

There is a number of ways of proving that (33) is locally well posed. We will mention two of them,

- **Kato's theory of quasi-linear equations and**

- **Parabolic Regularization.**

We start with Kato's method, which is the most convenient in terms of local well-posedness because it provides continuous dependence automatically[11]**. No such result exists in the context of parabolic regularization. We will indicate how to use Kato's quasi-linear theory to prove that (33) is locally well-posed in** $H^s(\mathbb{R})$ **for all** $s > 3/2$. **Before proceeding, it is convenient to make a few remarks about Kato's theory. Its aim is to establish sufficient conditions to ensure the local well-posedness for problems of the form**

$$\begin{cases} \partial_t u + A(t,u)\,u = f(t,u) \in X \\ u(0) = \phi \in Y \end{cases} \tag{35}$$

here X **and** Y **are Banach spaces, with** Y **continuously and densely embedded in** X **and** $A(t,u)$ **is bounded from** Y **into** X **and is the (negative) generator of a** C^0 **semigroup for each** $(t,u) \in [0.T] \times W$, W **open in** Y. **In its most general formulation,** X **and** Y **may be non-reflexive**[12] **Since we will deal exclusively with reflexive spaces, we will employ a simpler version, which can be found in ([14]). (See also ([15]) and ([10]).) The essential assumption of the theory is the existence of an isomorphism** S **from** Y **onto** X **such that**

$$SA(t,u)\,S^{-1} = A(t,u) + B(t,u) \tag{36}$$

[11] In fact a combination of the two methods seems to be the best weapon to prove global well-posedness. Another option is to use the Bona-Smith approximations to prove continuous dependence (see [**?**] and [11]).

[12] This is rather important, since it allows one to show that continuous dependence can be reduced to a question of existence and uniqueness in non-reflexive Banach spaces. See [13] and the references therein.

where $B(t,u) \in \mathcal{B}(X)$. This is, in fact, a condition on the commutator $[S, A(t,u)]$ because (33) can be rewritten as

$$[S, A(t,u)]\, S^{-1} = B(t,u). \tag{37}$$

There are also lesser requirements, involving Lipschitz conditions on the operators in question. For example, $A(t,u)$ must satisfy

$$\|A(t,w) - A(t,\widetilde{w})\|_{\mathcal{B}(Y,X)} \leq \mu \|w - \widetilde{w}\|_X. \tag{38}$$

for all pairs (t,w), $(t,\widetilde{w})$ in $[0.T] \times W$. Both $B(t,u)$ and $f(t,u)$ must satisfy similar conditions. We are now in position to state the main result of this section.

Theorem 1

Let

$$A(\rho) f = -\partial_x \left(f \left(1 - \partial_x^2\right)^{-1} \partial_x P(\rho) \right) + F(t,\rho), \tag{39}$$

so that the PDE in (33) can be written as

$$\partial_t \rho + A(\rho)\rho = F(t,\rho). \tag{40}$$

Let $\rho_0 \in H^s(\mathbb{R})$, $s > 3/2$ and assume that P and F satisfy the following assumptions.
(a) P maps $H^s(\mathbb{R})$ into itself, $P(0) = 0$ and is Lipchitz in the following sense:

$$\|P(\rho) - P(\widetilde{\rho})\|_s \leq L_s\left(\|\rho\|_s, \|\widetilde{\rho}\|_s\right) \|\rho - \widetilde{\rho}\|_s \tag{41}$$

where $L_s : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and monotone non-decreasing with respect to each of its arguments.
(b) $F : [0,T_0]\times H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R})\,,\ F(t,0) = 0$ and satisfies the following Lipschtz condition:

$$\|F(t,\rho) - F(t,\widetilde{\rho})\|_s \leq M_s\left(\|\rho\|_s\,, \|\widetilde{\rho}\|_s\right)\|\rho - \widetilde{\rho}\|_s\,. \tag{42}$$

Then (33) is locally well-posed in the sense described in Section 1.

The proof of this result has appeared in several works and we refrain from presenting it here. (See [10], [2]and [3].) It should be noted, however, that the previous result holds in $H^s(\mathbb{R}^n)$, with $n > \frac{n}{2} + 1$ and the obvious changes needed to accommodate the general case. It is also true when $H_0^s(\Omega)\,,\ \Omega \subset \mathbb{R}^n$ is a domain with a smooth boundary, and also in $H^s\left(S^1\right)$. This is due to the fact that all the crucial estimates used in the proof of the theorem are also true in the situations we have just mentioned. (See Lemma A4 of [17] and Appendix B of [11].)

From now on, to simplify our lives even further, we will assume that $F = 0$. One might as well ask, why use such a "complicated method" [13], instead of simply (33) into an equivalent integral equation, applying Banach's Fixed Point Theorem and Gromwal's inequality to solve the problem locally. The answer is : you can't. Well, why not? Assume that $\rho \in H^2(\mathbb{R})$ and, P maps $H^2(\mathbb{R})$ into

[13] Which, in view of the difficulty of the problem, is actually very simple.

itself even and look at

$$\partial_t \rho = \partial_x \underbrace{\left(\rho \left(1 - \partial_x^2\right)^{-1} \underbrace{\partial_x P(\rho)}_{H^1} \right)}_{H^2} \tag{43}$$

so if we start out with $\rho \in H^2(\mathbb{R})$**,** $\partial_t \rho$ **must belong to** $H^1(\mathbb{R})$[14]**. Thus the integral equation obtained naively by integrating (33) cannot be used to solve the problem. Thus we regularize the equation. Consider**

$$\begin{cases} \partial_t \rho^{(\mu)} = \partial_x \left(\rho^{(\mu)} \left(1 - \partial_x^2\right)^{-1} \partial_x P(\rho^\mu) \right) + \mu \partial_x^2 \rho^{(\mu)} \\ \rho^\mu(0) = \rho_0. \end{cases} \tag{44}$$

which is equivalent to

$$\rho^{(\mu)}(t) = U_\mu(t)\rho_0 + \int_0^t U_\mu(t-t') \left[A\left(\rho^{(\mu)}(t')\right) \rho^{(\mu)}(t') \right] dt'. \tag{45}$$

Then we can show that (see [2] and [3])

Theorem 2 ***Assume that*** $\mu > 0$ ***and that*** P ***satisfy (42) for all*** $s > 1/2$***. Then (33) is locally well-posed in*** $H^s(\mathbb{R})$***. Moreover, if*** $(0, T_\mu]$ ***is an interval of existence, then*** $\rho^{(\mu)} \in C((0, T_\mu]; H^\infty(\mathbb{R}))$***, where*** $H^\infty(\mathbb{R}) = \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R})$ ***provided with its natural Frechet space topology.***

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[14] This shows clearly why we really need at least two Banach spaces to begin with.

It should be noted that the proof (even in $\mathbb{R}^n$) depends heavily on the inequality

$$\|U_\mu(t)\phi\|_{r+\lambda} \leq K_\lambda \left[1+\left(\frac{1}{2\mu t}\right)^\lambda\right]^{1/2} \|\phi\|_r \tag{46}$$

where $K_\lambda > 0$ depends only on λ and holds for all $\phi \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$,

$\lambda \geq 0,$ and $\mu,\, t > 0.$(See [11], [12], [2] and [3] for example.) An easy bootstrapping argument combining (45) and (46), (with λ fixed in the interval $(1,2)$, so that the RHS of (??) is locally integrable near $t = 0)$, implies the last statement of 2.

Next, the, usual limiting process involved in the method of parabolic regularization (see [11] and [12]) we are able to show existence and uniqueness of solutions in $AC\left([0,T]; H^{s-1}(\mathbb{R})\right) \cap L^\infty([0,T]; H^s)$. Due to technical reasons (lack of invariance under certain changes of variables, see [12] and [17]), so far we were unable to prove that the solution we obtained in this way actually belongs to $C([0,T]; H^s(\mathbb{R})) \cap C^1\left([0,T]; H^{s-1}(\mathbb{R})\right)$, $s > 1/2$ as we would have liked. However, combining what we already have, with the results in Theorem 1, proved using Kato´s theory when $s > 3/2$, we see that the solutions must coincide, due to uniqueness, if $s > 3/2$. Now, parabolic regularization is useful in establishing the *a priori* estimates that imply global well-posedness, because we can differentiate all the Sobolev norms with respect to time, obtain the desired global estimates (if we are lucky) and then repeat the limiting process in each interval

$[0,T]$. **We will illustrate this by means of a funny inequality. To simplify the notation we write** $\rho^{(\mu)} = \rho$.

$$\begin{aligned}\partial_t \left\|\rho\left(t\right)\right\|_0^2 &= 2\left(\rho\left(t\right)\left|\partial_t \rho\left(t\right)\right.\right)_0 = \\ &= 2\mu\left(\rho\left(t\right)\left|\partial_x^2 \rho\left(t\right)\right.\right)_0 + 2\left(\rho\left(t\right)\left|\partial_x\left(\rho\left(t\right)\left(1-\partial_x^2\right)^{-1}\partial_x P\left(\rho\right)\right)\right.\right)_0 .\end{aligned} \tag{47}$$

Integration by parts shows that $\mu\left(\rho\left(t\right)\left|\partial_x^2 \rho\left(t\right)\right.\right) \leq \mu\left\|\partial_x \rho\left(t\right)\right\|_0^2 \leq 0$**, so that we can discard the first term in the third member of (47). But then,because** $\left\|\left(1-\partial_x^2\right)^{-1} P\left(\rho\left(t\right)\right)\right\|_0 \leq \left\|P\left(\rho\left(t\right)\right)\right\|_0$ **we obtain,**

$$\begin{aligned}&\partial_t \left\|\rho\left(t\right)\right\|_0^2 \leq 2\left(\rho\left(t\right)\left|\partial_x\left(\rho\left(t\right)\left(1-\partial_x^2\right)^{-1}\partial_x P\left(\rho\left(t\right)\right)\right)\right.\right)_0 \\ &= -\left(\rho\left(t\right)\partial_x\rho\left(t\right)\left|\left(1-\partial_x^2\right)^{-1}\partial_x^2 P\left(\rho\left(t\right)\right)\right.\right)_0 \\ &= +\left(\rho\left(t\right)^2\left|\left(1-\partial_x^2\right)^{-1}\left(\partial_x^2-1+1\right)P\left(\rho\left(t\right)\right)\right.\right)_0 \\ &= -\left(\rho\left(t\right)^2\left|P\left(\rho\left(t\right)\right)\right.\right) + \left(\rho\left(t\right)^2\left|\left(1-\partial_x^2\right)^{-1}P\left(\rho\left(t\right)\right)\right.\right) \\ &\leq -\left(\rho\left(t\right)^2\left|P\left(\rho\left(t\right)\right)\right.\right)_0 + \left\|\rho\left(t\right)^2\right\|_0\left\|P\left(\rho\left(t\right)\right)\right\|_0 \\ &\leq -\left(\rho\left(t\right)^2\left|P\left(\rho\left(t\right)\right)\right.\right)_0 + \frac{\left\|\rho\left(t\right)^2\right\|_0^2 + \left\|P\left(\rho\left(t\right)\right)\right\|_0^2}{2} = 2\left\|\rho\left(t\right)^2 - P\left(\rho\left(t\right)\right)\right\|_0^2\end{aligned} \tag{48a}$$

Thus,

$$\partial_t \left\|\rho\left(t\right)\right\|_0^2 \leq 2\left\|\rho\left(t\right)^2 - P\left(\rho\left(t\right)\right)\right\|_0^2 . \tag{49}$$

So, if $P\left(\rho\right) = \rho^2$**, it follows that** $\partial_t \left\|\rho\left(t\right)\right\|_0^2 \leq 0$ **which, in turn implies that** $\left\|\rho\left(t\right)\right\|_0^2 \leq \left\|\rho_0\right\|_0^2$**. This argument shows that** $P\left(\rho\right) = \rho^2$ **is a**

natural choice for the function $P(\rho)^{15}$.Thus we that we might be able to show global well posedness with $P(\rho)=\rho^2$ or with luck even in case $P(\rho)=\rho^{2k}$, $k=1,2,3...$Well, this is true, but we still lack some information. Recall that in the case of Hamiltonian systems, with enough conserved quantities we obtain *a priori estimates for the derivatives of the solutions* using these quantities. Here we to use another device, known as Comparison Principle, which we state in $\mathbb{R}^n$.

Theorem 3 ***(Comparison Principle). Let $(\rho, \vec{\mathrm{v}})$ and $(\eta, \vec{\mathrm{w}})$ be solutions of (31) with $P(\rho)=\rho^{2k}, P(\eta)=\eta^{2k}, k=1,2,3....$; and initial values $(\rho_0, \vec{v}_0)$ and $(\eta_0, \vec{w}_0)$ respectively. Then***

$$0 \leq \eta_0(x) \leq \rho_0(x) \text{ in } \mathbb{R}^n \Rightarrow 0 \leq \eta(x,t) \leq \rho(x,t) \text{ in } \mathbb{R}^n \times [0, T_0] \tag{50}$$

Theorem 4 ***(Global Solution). Let $s>\frac{n}{2}+1$, $P(\rho)=\rho^{2k}$, $F \equiv 0$ and $\rho_0 \in H^s(\mathbb{R}^n)$ with $0 \leq \rho_0(x) \leq 1$ in $\mathbb{R}^n$. Then (31) is globally well-posed in the sense described in Chapter 1 and satisfies $0 \leq \rho(x,t) \leq 1$, $\forall t \geq 0$.***

I will not submit you to gruesome torture that the proofs of the theorems may cause. In the final version they will appear

[15] Note that so far we requireded very little about $P(\rho)$!

in appendices. To the talk, I should remark that we solved the problem with bore-like data and we will submit it for publication as soon as we have checked it completely. Finally the next steps consists in studying what happens when we introduce boundaries, for example in

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \, | y > 0 \right\},$$

and

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \, | 0 < y < a, \;\; a > 0 \right\}.$$

References

[1] Ablowitz et al.,

[2] E. A. Alarcon and R. J. Iorio, Jr., *On the Cauchy Problem Associated to Brinkman Flow: the one dimensional theory,* Mat. Comtemp. 27 (2004), 1-17.

[3] E. A. Alarcon, R. J. Iorio, Jr. and M. Molina del Sol, *On the Cauchy Problem Associated to the Brinkman Flow in* $\mathbb{R}^n$, Appl. Anl. Discrete Math., 6 (2012), 214-237.

[4] T. B. Benjamin, J. L. Bona, and J. J. Mahony, *Model Equations for Long Waves in Nonlinear Dispersive Systems*", Philosophical Transactions of the Royal Society of London. Se-

ries A, Mathematical and Physical Sciences 272 (1220), (1972) 47–78.

[5] J. L Bona ,,,Scott, *The Korteweg - de Vries equation in fractional order Sobolev spaces* - Bona, Scott - 1976

[6] J. L. Bona and R. Smith, *The Initial Value problem for the Korteweg-de Vries Equation,* Phil. Trans. Roy. Soc. London, Ser. A 278 (1975), 555-604.

[7] J. L. Bona and N. Tzvetkov,*Sharp Well-posedness Results for the BBM Equation*, Discrete and Continuous Dynamical Systems, 23 (2009), 1241-1252.

[8] H. C. Brinkman, *A Calculation of the Viscous Force Exerted by a Flowing Fluid on a Dense Swarm of Particles*, Appl. Scientific Research, A1 (1947), 24-27.8.

[9] G. Fonseca, F. Linares and G. Ponce, *The IVP for the Benjamin-Ono equation in weighted Sobolev spaces II*, J. Funct. Analysis 262, pp. 2031-2049, 2012

[10] R. J. Iorio, Jr., *On Kato 's Theory of Quasilinear Equations*, Segunda Jornada de EDP e Análise Numérica, Publicação do IMUFRJ, Rio de Janeiro, Brasil, (1996).153-178.

[11] R. J. Iorio, Jr. and Valéria de Magalhães Iorio, *Fourier Analysis and Partial differential Equations*, Cambridge Studies in Advanced Mathematics, Cambridge University Press (2001).

[12] R. J. Iorio, Jr., ***KdV, BO and Friends in Weighted Sobolev Spaces, Functional Analytic Methods for Partial Differential Equations***, **H. Fujita, T. Ikebe, S. T. Kuroda (Eds.), Lecture Notes in Mathematics 1450, Springer-Verlag (1990), 104-121.**

[13] **T. Kato,** ***Abstract Evolution Equations, Linear and Quasilinear, Revisited,*** **Lecture Notes in Mathematics 1540, Springer-Verlag (1992), 103-127.**

[14] **T. Kato,** ***Quasilinear Equations of Evolution, with Applications to Partial Differential Equations***, **Lecture Notes in Mathematics 448, Springer (1975), 25-70.**

[15] **T. Kato,** ***Linear and Quasilinear Evolution Equations of Hyperbolic Type, Hyperbolicity (1976), 125-191*** **CIME, II Ciclo, Cortona.and**

[16] **T. Kato,** ***Perturbation Theory for Linear Operators,*** **3rd, Springer-Verlag (1995).**

[17] **T. Kato,** ***On the Cauchy Problem for the (Generalized) KdV equation***, **Studies in Applied Mathematics, in Mathematics Supplementary Studies, vol. 8, Academic Press (1983), 93-12.**

[18] **M. Panthee,** ***On the Ill-Posedness Result for the BBM Equation***, **Discrete and Continuous Dynamical Systems Volume 30, Number 1, May 2011.**

[19] **M. Reeed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, (1975).**