Questão 1.

$$\begin{aligned}\mathcal{L}\{y''+2y'+5y\} &= \mathcal{L}\{2\delta(t-\pi)\}\\ \mathcal{L}\{y''\}+\mathcal{L}\{2y'\}+\mathcal{L}\{5y\} &= 2\mathcal{L}\{\delta(t-\pi)\}\end{aligned}$$

0,2 pontos até aqui.

$$s^2\mathcal{L}\{y\}-sy(0)-y'(0)+2[s\mathcal{L}\{y\}-y(0)]+5\mathcal{L}\{y\}=2\mathrm{e}^{\pi s}$$
$$(s^2+2s+5)\mathcal{L}\{y\}-s-2=2\mathrm{e}^{\pi s}$$
$$\mathcal{L}\{y\}=\frac{s+2}{s^2+2s+5}+\frac{2\mathrm{e}^{\pi s}}{s^2+2s+5}$$

$+\,\mathbf{0,6}$

$$\mathcal{L}\{y\} = \frac{s+2}{(s+1)^2+4}+\frac{2\mathrm{e}^{\pi s}}{(s+1)^2+4}$$

$+\,\mathbf{0,2}$

$$=\frac{s+1}{(s+1)^2+4}+\frac{1}{(s+1)^2+4}+\frac{2\mathrm{e}^{\pi s}}{(s+1)^2+4}$$

$+\,\mathbf{0,2}$

$$\begin{aligned}y &= \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+4}+\frac{1}{(s+1)^2+4}+\frac{2\mathrm{e}^{\pi s}}{(s+1)^2+4}\}\\ &= \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+4}\}+\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+4}\}+\mathcal{L}^{-1}\{\frac{2\mathrm{e}^{\pi s}}{(s+1)^2+4}\}\end{aligned}$$

$+\,\mathbf{0,2}$

$$y=\mathrm{e}^{-t}\cos 2t+\frac{1}{2}\mathrm{e}^{-t}\mathrm{sen}2t+u_\pi(t)\mathrm{e}^{-(t-\pi)}\mathrm{sen}2(t-\pi).$$

$+\,\mathbf{0,6}$

Questão 2. (a) Tomando $F(s)=\frac{1}{s^2-1}$, temos $G(s)=F(s)/s$, **0,2**
logo, pela fórmula dada, vem que $\mathcal{L}^{-1}\{G\}=\int_0^t f(\tau)d\tau$, onde

$$f=\mathcal{L}^{-1}\{F\}=\mathcal{L}^{-1}\{\tfrac{1}{s^2-1}\}$$

$+\,\mathbf{0,2}$

$$=\mathrm{senh}t.$$

$+\,\mathbf{0,2}$

Assim,

$$\begin{aligned}\mathcal{L}^{-1}\{G\} &= \int_0^t \mathrm{senh}\tau d\tau\\ &= \mathrm{cosh}\tau\Big|_0^t\end{aligned}$$

$+\,\mathbf{0,2}$

$$=\mathrm{cosh}t-1.$$

+ 0, 2

(b) Sejam $G(s) = \frac{1}{s+1}$ e $H(s) = \frac{1}{s+2}$. Então $F(s) = G(s)H(s)$ e

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{-2t}, \qquad g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-t}.$$

+ 0, 4

Portanto,

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s)H(s)\} &= \int_0^t e^{-(t-\tau)}e^{-2\tau}d\tau \\ &= e^{-t}\int_0^t e^{-\tau}d\tau = -e^{-t}e^{-\tau}\Big|_0^t = e^{-t} - e^{-2t}.\end{aligned}$$

+ 0, 6

Questão 3.

O sistema dado é um sistema linear não-homogêneo:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t),$$

$$A = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 2\sqrt{t}\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{pmatrix}, \quad \mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

0,1

(a)

Autovalores:

$$\begin{vmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{vmatrix} = 0$$
$$(-\tfrac{3}{2} - r)(-\tfrac{1}{2} - r) + \tfrac{1}{4} = 0$$
$$r^2 + 2r + 1 = 0, \quad (r+1)^2 = 0$$
$$r = -1 : \quad \text{autovalor (repetido) de multiplicidade dois.}$$

+ 0, 3

Autovetores:

$$\begin{pmatrix} -\frac{3}{2} - (-1) & 1 \\ -\frac{1}{4} & -\frac{1}{2} - (-1) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$-\tfrac{1}{2}a + b = 0, \quad b = \tfrac{1}{2}a$$

Logo, os autovetores (associados ao único autovalor $r = -1$) são da forma

$$\begin{pmatrix} a \\ \frac{1}{2}a \end{pmatrix} = a\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, \quad a \in \mathbb{R}.$$

+ 0, 3

Tomando $a = 2$, obtemos a solução

$$\mathbf{x}^1 = \mathrm{e}^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{pmatrix}$$

do sistema homogêneo associado, $\mathbf{x}' = A\mathbf{x}$. $+\mathbf{0,4}$

<u>Solução linearmente de $\mathbf{x}^1$</u>:

$$\mathbf{x}^2 = t\mathrm{e}^{-t}V_1 + \mathrm{e}^{-t}V_2, \quad (A - (-1))V_2 = V_1, \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$+\mathbf{0,2}$

Calculando V_2:

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{cases} -\frac{1}{2}a + \ b & = 2 \\ -\frac{1}{4}a + \frac{1}{2}b & = 1 \end{cases}$$

$$-\tfrac{1}{2}a + b = 2, \quad b = 2 + \tfrac{1}{2}a.$$

Tomando $a = 0$, obtemos $V_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $+\mathbf{0,3}$

e

$$\mathbf{x}^2 = t\mathrm{e}^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathrm{e}^{-t} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2t\mathrm{e}^{-t} \\ (2+t)\mathrm{e}^{-t} \end{pmatrix}.$$

$+\mathbf{0,1}$

$$\Psi(t) = \begin{pmatrix} \mathbf{x}^1 & \mathbf{x}^2 \end{pmatrix} = \begin{pmatrix} 2\mathrm{e}^{-t} & 2t\mathrm{e}^{-t} \\ \mathrm{e}^{-t} & (2+t)\mathrm{e}^{-t} \end{pmatrix}$$

é uma matriz fundamental $+\mathbf{0,2}$

e a solução geral do sistema homogêneo é então

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^1 + c_2\mathbf{x}^2 \quad (c_1, c_2 \in \mathbb{R}) \\ &= \begin{pmatrix} \mathbf{x}^1 & \mathbf{x}^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \Psi(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} 2\mathrm{e}^{-t} & 2t\mathrm{e}^{-t} \\ \mathrm{e}^{-t} & (2+t)\mathrm{e}^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

$+\mathbf{0,2}$

(b)

$$\mathrm{e}^{-t} \begin{pmatrix} 2 & 2t \\ 1 & 2+t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2\sqrt{t}\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{pmatrix}$$

$$\begin{cases} u_1' + tu_2' & = \sqrt{t} \\ u_1' + (2+t)u_2' & = 1 \end{cases}$$

$$2u_2' = 1 - \sqrt{t}, \quad u_2' = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{t}$$

$$u_1' = \sqrt{t} - tu_2' = \sqrt{t} - \tfrac{1}{2}t + \tfrac{1}{2}t^{3/2}.$$

0,5

Uma solução particular do sistema dado (não-homogêneo):

$$\mathbf{x}_P = \Psi(t)U(t) \qquad (\Psi(t)U'(t) = \mathbf{g}(t))$$

+ 0,3

$$\begin{aligned} u_1 &= \int \left(\sqrt{t} - \tfrac{1}{2}t + \tfrac{1}{2}t^{3/2}\right) dt = \tfrac{2}{3}t^{3/2} - \tfrac{1}{4}t^2 + \tfrac{1}{5}t^{5/2} \ (+c\,) \\ u_2 &= \int \left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{t}\right) dt = \tfrac{1}{2}t - \tfrac{1}{3}t^{3/2} \ (+c\,) \end{aligned}$$

+ 0,2

$$\begin{aligned} \mathbf{x}_P &= \Psi(t)U(t) \\ &= \mathrm{e}^{-t}\begin{pmatrix} 2 & 2t \\ 1 & 2+t \end{pmatrix}\begin{pmatrix} \tfrac{2}{3}t^{3/2} - \tfrac{1}{4}t^2 + \tfrac{1}{5}t^{5/2} \\ \tfrac{1}{2}t - \tfrac{1}{3}t^{3/2} \end{pmatrix} \\ &= \mathrm{e}^{-t}\begin{pmatrix} 2\left[\tfrac{2}{3}t^{3/2} - \tfrac{1}{4}t^2 + \tfrac{1}{5}t^{5/2}\right] + 2t\left[\tfrac{1}{2}t - \tfrac{1}{3}t^{3/2}\right] \\ \left[\tfrac{2}{3}t^{3/2} - \tfrac{1}{4}t^2 + \tfrac{1}{5}t^{5/2}\right] + (2+t)\left[\tfrac{1}{2}t - \tfrac{1}{3}t^{3/2}\right] \end{pmatrix} \\ &= \mathrm{e}^{-t}\begin{pmatrix} \tfrac{4}{3}t^{3/2} + 2(\tfrac{1}{5} - \tfrac{1}{3}t)t^{5/2} + \tfrac{1}{2}t^2 \\ (\tfrac{1}{5} - \tfrac{1}{3}t)t^{5/2} + \tfrac{1}{4}t^2 + t \end{pmatrix} \end{aligned}$$

+ 0,5

Questão 4. (a) Notemos que

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n}.$$

Como $\frac{1}{n}\ln n = f(n)$ onde $f(x) = \frac{\ln x}{x}$ e

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0 \quad (\text{L'Hospital}),$$

+ 0,3

segue que

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^0 = 1,$$

já que a função exponencial é contínua. **+ 0,2**

(b) Observemos que

$$a_n = \frac{\sqrt{3}\,5^{n+1} + \pi 7^{n-2}}{8^{n-1}} = 25\sqrt{3}\left(\frac{5}{8}\right)^{n-1} + \frac{\pi}{7}\left(\frac{7}{8}\right)^{n-1}.$$

+ 0,3

Logo,

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n &= 25\sqrt{3}\sum_{n=2}^{\infty}\left(\frac{5}{8}\right)^{n-1} + \frac{\pi}{7}\sum_{n=2}^{\infty}\left(\frac{7}{8}\right)^{n-1} \\
&= 25\sqrt{3}\sum_{k=1}^{\infty}\left(\frac{5}{8}\right)^{k} + \frac{\pi}{7}\sum_{k=1}^{\infty}\left(\frac{7}{8}\right)^{k} \\
&= 25\sqrt{3}\frac{5}{8}\sum_{k=1}^{\infty}\left(\frac{5}{8}\right)^{k-1} + \frac{\pi}{7}\frac{7}{8}\sum_{k=1}^{\infty}\left(\frac{7}{8}\right)^{k-1} \\
&= \frac{125\sqrt{3}}{8}\frac{1}{1-\frac{5}{8}} + \frac{\pi}{8}\frac{1}{1-\frac{7}{8}} \\
&= \frac{125}{\sqrt{3}} + \pi
\end{aligned}$$

+ 0,7

(c) Fazendo $a_k = (-1)^k \frac{k!}{k^k}$, temos

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)!}{(k+1)^{k+1}}\frac{k^k}{k!} = \frac{k^k}{(k+1)^k}$$

0,3

$$= \left(\frac{k}{k+1}\right)^k = \frac{1}{\left(1+\frac{1}{k}\right)^k}.$$

Como

$$\lim_{k\to\infty}\frac{1}{\left(1+\frac{1}{k}\right)^k} = \frac{1}{e} < 1,$$

+ 0,4

pelo Teste da Razão obtemos que a série é absolutamente convergente.

+ 0,3