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Lagrangean Structure and Propagation of Singularities in Multidimensional Compressible Flow

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Outline:

1. **Hoff solutions**
2. **Results**
3. **On proofs**

1 Hoff solutions

Navier-Stokes equations

$$\begin{cases} \rho_t + \mathbf{div}(\rho \mathbf{u}) = 0 \\ \\ (\rho \mathbf{u}^j)_t + \mathbf{div}(\rho \mathbf{u}^j \mathbf{u}) + \mathbf{P}(\rho)_{x_j} \\ \qquad\qquad = \mu \mathbf{\Delta u}^j + \lambda \mathbf{div} \mathbf{u}_{x_j} + \rho \mathbf{f}^j \\ \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0), \\ \\ \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{n} = \mathbf{2}, \mathbf{3}. \end{cases}$$

Assumptions

$$\begin{cases} \mathbf{P} \in \mathbf{C}^2([\mathbf{0}, \bar{\rho}]), \quad \mathbf{P}'(\tilde{\rho}) > \mathbf{0}, \quad \textbf{for some } \tilde{\rho} \in (\mathbf{0}, \bar{\rho}) \\ (\rho - \tilde{\rho})[\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho})] > \mathbf{0}, \quad \rho \neq \tilde{\rho} \ ; \end{cases}$$

$$\int_0^\infty \left(||\mathbf{f}(\cdot,\mathbf{t})||_2 + ||\mathbf{f}(\cdot,\mathbf{t})||_2^2 + \sigma(\mathbf{t})^\gamma ||\mathbf{f_t}(\cdot,\mathbf{t})||_2^2 \right) \mathbf{dt}$$

$$+ \quad \sup\nolimits_{\mathbf{t}\geq \mathbf{0}} ||\mathbf{f}(\cdot,\mathbf{t})||_\mathbf{p} \quad \leq \quad \mathbf{C_f} \ < \ \infty$$

$$\boxed{\textbf{some } \mathbf{p} > \mathbf{n}}, \quad \boxed{\sigma(\mathbf{t}) := \min\{\mathbf{1},\mathbf{t}\}}, \quad \gamma = \begin{cases} 3, & \mathbf{n}=\mathbf{2} \\ 5, & \mathbf{n}=\mathbf{3} \end{cases}$$

$$\begin{cases} \lambda,\, \mu > \mathbf{0}\,, & \mathbf{n}=\mathbf{2} \\ \mathbf{0} < \lambda < \frac{\mathbf{5}}{\mathbf{4}}\mu\,, & \mathbf{n}=\mathbf{3} \end{cases}$$

$$\int_{\mathbb{R}^\mathbf{n}} \left[\ \rho_\mathbf{0}|\mathbf{u_0}|^\mathbf{2} + |\rho_\mathbf{0} - \tilde{\rho}|^\mathbf{2}\ \right] \mathbf{dx} \quad \leq \ \mathbf{C_0} \quad < \ \infty$$

$$\rho_\mathbf{0} \ \geq \ \mathbf{0} \quad \textbf{a.e.}\,, \qquad ||\rho_\mathbf{0}||_\infty \ < \ \bar{\rho}$$

EXISTENCE OF SOLUTION
(D. HOFF 1995, 1997, 2005):

Given $\rho_1 \in (\widetilde{\rho}, \overline{\rho})$, **there are positive numbers** ε **and** $\mathbf{C}$ **depending on** $\widetilde{\rho}, \rho_1, \overline{\rho}, \mathbf{P}, \lambda, \mu,$ **and** $\mathbf{p}$, **and there is a universal positive constant** θ **such that, given initial data** $(\rho_\mathbf{0}, \mathbf{u_0})$ **and external force** $\mathbf{f}$ **satisfying**

$$\mathbf{0} \le \operatorname{ess\,inf} \rho_\mathbf{0} \le \operatorname{ess\,sup} \rho_\mathbf{0} \le \rho_\mathbf{1},$$

and

$$\mathbf{C_0} + \mathbf{C_f} \le \varepsilon\,,$$

the above initial–value problem has a global weak solution $(\rho, \mathbf{u})$ **with the following properties:**

- ***(energy estimate)***

$$\sup_{\mathbf{t}>\mathbf{0}} \int_{\mathbb{R}^\mathbf{n}} \Big[\ \rho(\mathbf{x},\mathbf{t})|\mathbf{u}(\mathbf{x},\mathbf{t})|^\mathbf{2} + |\rho(\mathbf{x},\mathbf{t}) - \tilde{\rho}|^\mathbf{2}$$
$$+\ \sigma(\mathbf{t})|\nabla \mathbf{u}(\mathbf{x},\mathbf{t})|^\mathbf{2}\Big]\ \mathbf{dx}$$

$$+ \int_\mathbf{0}^\infty \int_{\mathbb{R}^\mathbf{n}} \Big[\ |\nabla \mathbf{u}|^\mathbf{2} + \sigma(\mathbf{t})^\mathbf{n} |\nabla \dot{\mathbf{u}}|^\mathbf{2}\ \Big]\,\mathbf{dxdt}$$

$$\le\ \mathbf{C}\,(\mathbf{C_0} + \mathbf{C_f})^\theta < \infty$$

$\dot{\mathbf{u}}$ ***is the 'convective (material) derivative':***

$$\dot{\mathbf{u}}^{\mathbf{j}} := \mathbf{u}^{\mathbf{j}}_{\mathbf{t}} + \mathbf{u} \cdot \nabla \mathbf{u}^{\mathbf{j}}.$$

- $$\int_0^\infty \int_{\mathbb{R}^n} \sigma(\mathbf{t})\rho|\dot{\mathbf{u}}|^2 \, \mathbf{dxdt} \le \mathbf{C}\,(\mathbf{C_0} + \mathbf{C_f})^\theta$$

  if $\inf \rho_0 > \mathbf{0}$ $\quad (\inf \equiv \operatorname{ess\,inf})$

- $$\boxed{\mathbf{C}^{-1} \inf \rho_0 \le \rho \le \bar{\rho}} \qquad \textit{a.e.} \qquad (\mathbf{C} > \mathbf{0})$$

- ***Hölder continuity: For any*** $\tau > \mathbf{0}$, ***we have that*** $\mathbf{u}$,

  $$\boxed{\mathbf{F} := (\mu + \lambda)\mathbf{div}\,\mathbf{u} - (\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho}))}$$

  and

  $\omega^{\mathbf{j},\mathbf{k}} = \mathbf{u}^{\mathbf{j}}_{\mathbf{x_k}} - \mathbf{u}^{\mathbf{k}}_{\mathbf{x_j}}$ ***(vorticity matrix)***

  are Hölder continuous in $\mathbb{R}^{\mathbf{n}} \times [\tau, \infty)$.

- ***The solution*** $(\rho, \mathbf{u})$ ***is obtained as the limit as*** $\delta \to \mathbf{0}$ ***of smooth approximate solutions*** $(\rho^\delta, \mathbf{u}^\delta)$ ***satisfying the above estimates with constants which are independent of*** δ, $\quad \rho_0^\delta = \mathbf{j}_\delta * \rho_0 + \delta$, $\quad \mathbf{u}_0^\delta = \mathbf{j}_\delta * \mathbf{u}_0$.

Weak solution:

$$\int_{\mathbb{R}^n} \rho(\mathbf{x},\cdot)\varphi(\mathbf{x},\cdot)\mathbf{dx}\Big|_{\mathbf{t_1}}^{\mathbf{t_2}} = \int_{\mathbf{t_1}}^{\mathbf{t_2}}\int_{\mathbb{R}^n}(\rho\varphi_{\mathbf{t}} + \rho\mathbf{u}\cdot\nabla\varphi)\mathbf{dxdt}$$

and

$$\int_{\mathbb{R}^n}(\rho\mathbf{u^j})(\mathbf{x},\cdot)\varphi(\mathbf{x},\ \cdot)\mathbf{dx}\Big|_{\mathbf{t_1}}^{\mathbf{t_2}}$$

$$= \int_{\mathbf{t_1}}^{\mathbf{t_2}}\int_{\mathbb{R}^n}[\rho\mathbf{u^j}\varphi_{\mathbf{t}} + \rho\mathbf{u^j}\mathbf{u}\cdot\nabla\varphi + \mathbf{P}(\rho)\varphi_{\mathbf{x_j}}]\mathbf{dxdt}$$

$$- \int_{\mathbf{t_1}}^{\mathbf{t_2}}\int_{\mathbb{R}^n}[\mu\nabla\mathbf{u^j}\cdot\nabla\varphi + \lambda(\mathrm{div}\,\mathbf{u})\varphi_{\mathbf{x_j}}]\mathbf{dxdt}$$

$$+ \int_{\mathbf{t_1}}^{\mathbf{t_2}}\int_{\mathbb{R}^n}\rho\mathbf{f^j}\varphi\,\mathbf{dxdt}$$

for all $\mathbf{t_2} \geq \mathbf{t_1} \geq \mathbf{0}$ **and all** $\varphi \in \mathbf{C^1}(\mathbb{R}^n \times [\mathbf{t_1}, \mathbf{t_2}])$ **with compact support.**

<u>Motivation for the definition of F</u>/Piecewise smooth weak solution:

Let $\mathcal{S}$ be a hypersurface in $\mathbf{x}-\mathbf{t}$ space and suppose that $(\rho, \mathbf{u})$ is a weak solution that is C^1 in the complement of $\mathcal{S}$, it has a uniquely defined flux $\mathbf{X}(\mathbf{t}, \mathbf{x})$ $(\partial\mathbf{X}/\partial\mathbf{t} = \mathbf{u}(\mathbf{X}, \mathbf{t}),\ \mathbf{X}(\mathbf{0}, \mathbf{x}) = \mathbf{x})$, and such that it has one-sided limits with respect to $\mathcal{S}$. (Recall that $\mathbf{u}$ is continuous (in fact Hölder continuous) for $\mathbf{t} > 0$.) Then

$$\mathcal{S} \cap \{\mathbf{t} = \mathbf{t_0}\} \ = \mathbf{X}(\mathbf{t_0}, \cdot)(\mathcal{S} \cap \{\mathbf{t} = \mathbf{0}\})\,,$$

and the following jump conditions hold along $\mathcal{S}$:

$$[\mathbf{u}^{\mathbf{j}}_{\mathbf{x_k}}] \ = \ [\mathbf{u}^{\mathbf{k}}_{\mathbf{x_j}}] \quad \text{and} \quad [\mathbf{P}(\rho)] \ = \ (\lambda + \mu)[\mathbf{divu}]$$

<u>Rankine-Hugoniot conditions</u>

i.e.

$$[\omega] \ = \ [\mathbf{F}] \ = \ \mathbf{0}.$$

Questions

For Hoff solutions we ask the following questions:

- **Does** $\mathbf{u}$ **have a flux and it is unique ??** **i.e.**
Is there a unique map $(\mathbf{t}, \mathbf{x}) \in [\mathbf{0}, \infty) \times \mathbb{R}^{\mathbf{n}} \mapsto \mathbf{X}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^{\mathbf{n}}$ **such that**
$$\begin{cases} \dfrac{\partial}{\partial \mathbf{t}} \mathbf{X}(\mathbf{t}, \mathbf{x}) = \mathbf{u}(\mathbf{X}(\mathbf{t}, \mathbf{x}), \mathbf{t}), & \mathbf{t} > \mathbf{0}, \ \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \\ \mathbf{X}(\mathbf{0}, \mathbf{x}) = \mathbf{x}, & \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \quad ? \end{cases}$$

- **Given** $\mathcal{M}$ **a continuous hypersurface in** $\mathbb{R}^{\mathbf{n}}$, **is** $\mathcal{M}^{\mathbf{t}} := \mathbf{X}(\mathbf{t}, \cdot)(\mathcal{M})$ **a continuous hypersurface?**

- **If** ρ_0 **has a one-sided limit with respect to** $\mathcal{M}$ **at a point** $\mathbf{x}_0$, **does** $\rho(\cdot, \mathbf{t})$ **have a one-sided limit at** $\mathbf{X}(t, \mathbf{x}_0)$ **with respect to** $\mathcal{M}^{\mathbf{t}}$ **?**

2 Results

Theorem 1 (Hoff-Santos) – Lagrangean structure:
Assume also that

$$\sup_{0\le t\le \tau_0}\int_{\mathbb{R}^n}|\nabla f(x,t)|^2 dx+\int_0^{\tau_0}\int_{\mathbb{R}^n}|f_t(x,t)|^2 dxdt<\infty$$

and

$$\boxed{u_0\in H^s(\mathbb{R}^n) \quad \textit{where} \quad s>0 \ \textit{for}\ n=2 \ \textit{and}\ s>1/2 \ \textit{for}\ n=3}.$$

Let V ***be an open set in*** $\mathbb{R}^n$ ***and assume that***

$$\boxed{\inf \rho_0|_V \ge \underline{\rho}>0}.$$

Then

a) ***For each*** $y\in V$ ***there is a unique curve*** $X(\cdot,y)$,

$$X(t,y)=y+\int_0^t u(X(\tau,y),\tau)d\tau.$$

b) ***For each*** $t>0$, $V^t\equiv X(t,\cdot)V$ ***is open and the map*** $y\mapsto X(t,y)$ ***is a homeomorphism of*** V ***onto*** V^t, ***and it is locally Hölder continuous.***

c) ***Let*** $\mathcal{M} \subset\subset \mathbf{V}$ ***be a*** $\mathbf{C}^{\alpha}$ ***hypersurface,*** $\alpha \in [\mathbf{0}, \mathbf{1})$***. Then for any*** $\mathrm{t} > \mathbf{0}$***,*** $\mathcal{M}^{\mathrm{t}} \equiv \mathbf{X}(\mathbf{t}, \cdot)\mathcal{M}$ ***is a*** $\mathbf{C}^{\beta}$ ***hypersurface,*** $\beta = \alpha \mathbf{e}^{-\mathbf{C}\mathbf{t}^{\gamma}}$***.***

d) ***There is a positive number*** $\underline{\widetilde{\rho}}$ ***such that, for all*** $\mathrm{t} > \mathbf{0}$***,***

$$\inf \rho(\cdot, \mathbf{t})|_{\mathbf{V}^{\mathbf{t}}} \;\geq\; \underline{\widetilde{\rho}} > \mathbf{0}.$$

Theorem 2 (Hoff-Santos) – One-sided limits:

If ρ_0 ***has a one-sided limit at*** x_0 ***from a side of*** $\mathcal{M}$***, then for each*** $\mathrm{t} > \mathbf{0}$***,*** $\rho(\cdot, \mathbf{t})$ ***and*** $\mathrm{div}\, \mathbf{u}(\cdot, \mathbf{t})$ ***have one-sided limits at*** $\mathbf{X}(\mathbf{t}, \mathrm{x}_0)$ ***from the same side of*** $\mathbf{X}(\mathbf{t}, \cdot)\mathcal{M}$***.***

If both one-sided limits $\rho_0(\mathrm{x}_0\pm)$ ***of*** ρ_0 ***at*** x_0 ***with respect to*** $\mathcal{M}$ ***exist, then for each*** $\mathrm{t} > \mathbf{0}$ ***the jumps in*** $\mathrm{P}(\rho(\cdot, \mathbf{t}))$ ***and*** $\mathrm{div}\, \mathbf{u}(\cdot, \mathbf{t})$ ***at*** $\mathbf{X}(\mathbf{t}, \mathrm{x}_0)$ ***satisfy the Rankine–Hugoniot condition***

$$[\mathbf{P}(\rho(\mathbf{X}(\mathbf{t}, \mathbf{x_0}), \mathbf{t}))] = [(\mu + \lambda)\mathrm{div}\, \mathbf{u}(\mathbf{X}(\mathbf{t}, \mathbf{x_0}), \mathbf{t})].$$

(Indeed,

$$\mathbf{F} := (\mu + \lambda)\, \boldsymbol{div}\, \mathbf{u} - (\, \mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho})\,)$$

is (Hölder) continuous.)

Theorem 3 (Hoff-Santos) – Time evolution of discontinuities:

The map $\mathbf{t} \mapsto \rho(\mathbf{X}(\mathbf{t}, \mathbf{x_0})+, \mathbf{t})$ ***is in*** $\mathbf{C}([\mathbf{0}, \infty)) \cap \mathbf{C^1}((\mathbf{0}, \infty))$ ***and the map*** $\mathbf{t} \mapsto \operatorname{div} \mathbf{u}(\mathbf{X}(\mathbf{t}, \mathbf{x_0})+, \mathbf{t})$ ***is locally Hölder continuous on*** $(\mathbf{0}, \infty)$***.***

If both one-sided limits $\rho_0(\mathbf{x_0}\pm)$ ***of*** ρ_0 ***at*** $\mathbf{x_0}$ ***with respect to*** $\mathcal{M}$ ***exist, then the jump in the logarithm of*** ρ ***satisfies the representation***

$$[\log \rho(\mathbf{X}(\mathbf{t}, \mathbf{x_0}), \mathbf{t})] = \exp\left(-(\mu + \lambda)^{-\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{a}(\tau) \mathbf{d}\tau\right) [\log \rho_{\mathbf{0}}(\mathbf{x_0})]$$

where

$$\mathbf{a}(\tau) = \frac{[\mathbf{P}(\rho(\mathbf{X}(\tau, \mathbf{x_0}), \tau))]}{[\log \rho(\mathbf{X}(\tau, \mathbf{x_0}), \tau)]}\,.$$

3 "Proofs"

Theorem 1 – Lagrangean structure:

A vector field $\mathbf{u}$ ***in*** $\mathbb{R}^n$ ***is said to be log-Lipschitzian*** **(LL)** ***if***

$$\langle \mathbf{u} \rangle_{LL} \equiv \sup_{0<|\mathbf{x}-\mathbf{y}|\leq 1} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{y}| \log |\mathbf{x}-\mathbf{y}|} < \infty.$$

Example. ***Let*** $\mathbf{w} \in \mathbf{L}^{\mathbf{p}}(\mathbb{R}^{\mathbf{n}}) \cap \mathbf{L}^{\infty}(\mathbb{R}^{\mathbf{n}})$, ***where*** $\mathbf{p} \in [\mathbf{1}, \infty)$, ***and*** $\mathbf{\Gamma}$ ***the fundamental solution of the Laplacian equation in*** $\mathbb{R}^{\mathbf{n}}$***. Then*** $\nabla \mathbf{\Gamma} * \mathbf{w} \in \mathbf{LL}(\mathbb{R}^{\mathbf{n}})$ ***and***

$$\langle \nabla \mathbf{\Gamma} * \mathbf{w} \rangle_{\mathbf{LL}} \leq \mathbf{C}(||\mathbf{w}||_{\mathbf{p}} + ||\mathbf{w}||_{\infty})$$

where $\mathbf{C} = \mathbf{C}(\mathbf{n}, \mathbf{p})$***. If*** $p < n$, ***we also have***

$$||\nabla \mathbf{\Gamma} * \mathbf{w}||_{\infty} \leq \mathbf{C}(||\mathbf{w}||_{\mathbf{p}} + ||\mathbf{w}||_{\infty}) \,.$$

Lagrangean structure of log-Lipschtzian vector fields
("Generalized Picard's theorem"):

If for each $t \geq 0$, $\mathbf{u}(\mathrm{x}, t)$ ***is a vector field in*** $\mathbb{R}^n$ ***such that***

$$\langle \mathbf{u}(\cdot, t) \rangle_{LL} \ \in \ L^1_{loc}([0, \infty))$$

then for every $\mathrm{x} \in \mathbb{R}^n$ ***there exists a unique map*** $\mathbf{X}(\cdot, \mathbf{x}) \in \mathbf{C}([\mathbf{0}, \mathbf{t_x}); \mathbb{R}^{\mathbf{n}})$, $\mathbf{t_x} > \mathbf{0}$***, such that***

$$\mathbf{X}(\mathbf{t}, \mathbf{x}) = \mathbf{x} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{u}(\mathbf{X}(\tau, \mathbf{x}), \tau)\, \mathbf{d}\tau, \qquad (\mathbf{0} \leq \mathbf{t} < \mathbf{t_x}).$$

Gronwall type inequality (Osgood's lemma):

Let $\eta \geq \mathbf{0}$ ***be a mensurable function and locally bounded in*** $[0, \infty)$, $\mathbf{a} \geq \mathbf{0}$, ***and*** $\mathbf{0} \leq \mathbf{g} \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}([\mathbf{0}, \infty))$, ***such that***

$$\eta(\mathbf{t}) \leq \mathbf{a} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{g}(\tau)\, \mathbf{m}(\eta(\tau))\, \mathbf{d}\tau, \qquad \mathbf{t} \in [\mathbf{0}, \infty),$$

where

$$m(r) = \begin{cases} r(1 - \log r), & 0 < r \leq 1 \\ r, & 1 \leq r < \infty\,. \end{cases}$$

Assume that $\eta \leq 1$. ***Then***

$$|\eta(\mathbf{t})| \leq \exp\left(\mathbf{1} - \mathbf{e}^{-\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{g}\, \mathbf{d}\tau}\right) \mathbf{a}^{\exp\left(-\int_{\mathbf{0}}^{\mathbf{t}} \mathbf{g}\, \mathbf{d}\tau\right)}$$

in the case that $\mathbf{a} \neq \mathbf{0}$, ***and*** $\eta(\mathbf{t}) \equiv \mathbf{0}$ ***if*** $\mathbf{a} = \mathbf{0}$.

Decompose the velocity u by writing

$$\begin{aligned}\mathbf{\Delta u^j = u^j_{x_k,x_k}} &= \mathbf{u^k_{x_k,x_j}} + (\mathbf{u^j_{x_k}} - \mathbf{u^k_{x_j}})_{\mathbf{x_k}} \\ &= \operatorname{div} \mathbf{u_{x_j}} + \omega^{\mathbf{j,k}}_{\mathbf{x_k}} \\ &= (\mu+\lambda)^{-1}\mathbf{F_{x_j}} + \omega^{\mathbf{j,k}}_{\mathbf{x_k}} + (\mu+\lambda)^{-1}\mathbf{P}(\rho)_{\mathbf{x_j}} \\ &\equiv \mathbf{\Delta u^j_{F,\omega}} + \mathbf{\Delta u^j_P}\ ,\end{aligned}$$

so that

$$\boxed{\mathbf{u} = \mathbf{u_{F,\omega}} + \mathbf{u_P}}$$

where $\mathbf{u_{F,\omega}}$, $\mathbf{u_P}$ **are defined by**

$$\mathbf{\Delta u^j_{F,\omega}} = (\mu+\lambda)^{-1}\,\mathbf{F_{x_j}} + (\omega^{\mathbf{j,k}})_{\mathbf{x_k}},$$

$$\begin{aligned}\mathbf{\Delta u^j_P} &= (\mu+\lambda)^{-1}\,(\,\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho})\,)_{\mathbf{x_j}} \\ \mathbf{u_p} &= (\mu+\lambda)^{-1}\,\nabla\mathbf{\Gamma} * (\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho}))\,.\end{aligned}$$

Then

$$\mathbf{u_P}(\cdot,\mathbf{t}) \in \mathbf{LL} \qquad (\ \mathbf{P}(\rho(\cdot,\mathbf{t})) - \mathbf{P}(\tilde{\rho}) \ \in\ \mathbf{L^2} \cap \mathbf{L^\infty}\ )$$

$\mathbf{u_{F,\omega}}(\cdot,\mathbf{t}) \in \mathbf{Lip}$ **(**$\mathbf{F}(\cdot,\mathbf{t})$ **e** $\omega(\cdot,\mathbf{t})$ **são Hölder contínuas).**

Besides,

$$\langle \mathbf{u_P}(\cdot,\mathbf{t})\rangle_{\mathbf{LL}} \ \le\ \mathbf{C}\,||\mathbf{P}(\rho(\cdot,\mathbf{t})) - \mathbf{P}(\tilde{\rho})||_{\mathbf{L^2}\cap\mathbf{L^\infty}} \ \in\ \mathbf{L^1_{loc}}([\mathbf{0},\infty)\,).$$

<u>**Question:**</u> $\langle \mathbf{u}_{\mathbf{F},\omega}(\cdot,\mathbf{t})\rangle_{\mathbf{Lip}} \in \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}([\mathbf{0},\infty))$ **?**

Notice that

$$\left\langle \mathbf{u}_{\mathbf{F},\,\omega} \right\rangle_{\mathbf{Lip}} \leq \mathbf{C}\,||\nabla\mathbf{F} + \nabla\omega||_{\mathbf{p}}\,.$$

On the other hand, the momentum equation can be written as

$$\boxed{\rho\dot{\mathbf{u}}^{\mathbf{j}} = \mathbf{F}_{\mathbf{x_j}} + \mu\omega^{\mathbf{j},\mathbf{k}}_{\mathbf{x_k}} + \rho\mathbf{f}^{\mathbf{j}}}\,.$$

so, taking div e rot we obtain the equations

$$\boxed{\boldsymbol{\Delta}\mathbf{F} = \mathbf{div}(\rho\dot{\mathbf{u}} - \rho\mathbf{f})} \qquad \boxed{\mu\boldsymbol{\Delta}\omega^{\mathbf{j},\mathbf{k}} = \mathbf{rot}(\rho\dot{\mathbf{u}} - \rho\mathbf{f})^{\ \mathbf{j},\mathbf{k}}}\,.$$

Therefore

$$||\nabla\mathbf{F}||_{\mathbf{p}}\,,\quad ||\mathbf{D}\omega||_{\mathbf{p}} \ \leq\ \mathbf{C}\,(||\rho\dot{\mathbf{u}}||_{\mathbf{p}} + ||\rho\mathbf{f}||_{\mathbf{p}})\,.$$

$$||\,(\rho\dot{\mathbf{u}})(\cdot,\mathbf{t})\,||_{\mathbf{p}} \ \in\ \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\,[\mathbf{0},\infty)\,)\ \ ??$$

Suppose $\inf \rho_0 > \mathbf{0}.$

$$||\dot{\mathbf{u}}||_{\mathbf{p}} \leq \mathbf{C}\,||\dot{\mathbf{u}}||_{\mathbf{2}}^{\mathbf{1}-\kappa}||\nabla\dot{\mathbf{u}}||_{\mathbf{2}}^{\kappa}$$

$$\boxed{\kappa = \mathbf{n}(\tfrac{\mathbf{1}}{\mathbf{2}} - \tfrac{\mathbf{1}}{\mathbf{p}})}$$

$$\int_{\mathbf{0}}^{\mathbf{1}} ||\dot{\mathbf{u}}||_{\mathbf{2}}^{\mathbf{1}-\kappa}||\nabla\dot{\mathbf{u}}||_{\mathbf{2}}^{\kappa}\mathbf{dt}$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} \left(t^{1-s}\int|\dot{\mathbf{u}}|^2 d\mathbf{x}\right)^{(1-\kappa)/2} \left(t^{2-s}\int|\nabla\dot{\mathbf{u}}|^2 d\mathbf{x}\right)^{\kappa/2} t^{(s-1-\kappa)/2dt}$$

$$\leq \left(\int_{\mathbf{0}}^{\mathbf{1}} \left(\mathbf{t}^{\mathbf{1-s}}\int|\dot{\mathbf{u}}|^{\mathbf{2}}\mathbf{dx}\right)^{\mathbf{1}-\kappa} \left(\mathbf{t}^{\mathbf{2-s}}\int|\nabla\dot{\mathbf{u}}|^{\mathbf{2}}\mathbf{dx}\right)^{\kappa} \mathbf{dt}\right)^{\mathbf{1/2}} \left(\int_0^1 t^{s-1-\kappa}dt\right)^{1/2}$$

$$\leq \mathbf{c}\,(\mathbf{C_0} + \mathbf{C_f})^{\theta} \quad \left(\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{t}^{\mathbf{s-1}-\kappa}\mathbf{dt}\right)^{\mathbf{1/2}} :$$

<u>**finite**</u>**,** **since** $s > \begin{cases} \mathbf{0}, & \mathbf{n=2} \\ \mathbf{1/2}, & \mathbf{n=3} \end{cases}$ **e** $\mathbf{u_0} \in \mathbf{H}^{\mathbf{s}}(\mathbb{R}^{\mathbf{n}})$**,** **due to** **[Hoff] and** $\inf_{\mathbf{p>n}} \kappa = \kappa|_{\mathbf{p=n}} = \mathbf{n}(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{n}}) = \begin{cases} \mathbf{0}, & \mathbf{n=2} \\ \mathbf{1/2}, & \mathbf{n=3}. \end{cases}$

Since it may occur $\rho_0|_{\mathbf{V}^c} = 0$ we do not know if

$$\langle \mathbf{u}(\cdot, \mathbf{t}) \rangle_{\mathrm{LL}} \in \mathbf{L}^1_{\mathbf{loc}}([\mathbf{0}, \infty))$$

but

$$\begin{aligned} &|\mathbf{X_1}(\mathbf{t}, \mathbf{y}) - \mathbf{X_1}(\mathbf{t}, \mathbf{y})| \\ &\leq \int_0^t \mathbf{g}(\tau)\mathbf{m}(|\mathbf{X_2}(\tau, \mathbf{y}) - \mathbf{X_1}(\tau, \mathbf{y})|)\mathbf{d}\tau \end{aligned}$$

$$\mathbf{g}(\mathbf{t}) := \frac{|\mathbf{u}(\mathbf{X_2}(\mathbf{t}, \mathbf{y}), \mathbf{t}) - \mathbf{u}(\mathbf{X_1}(\mathbf{t}, \mathbf{y}), \mathbf{t})|}{\mathbf{m}(|\mathbf{X_2}(\mathbf{t}, \mathbf{y_2}) - \mathbf{X_1}(\mathbf{t}, \mathbf{y_1})|)} \in \mathbf{L}^1_{\mathrm{loc}}([\mathbf{0}, \infty))$$

In fact

$$\mathbf{g}(\mathbf{t}) \leq \mathbf{g_r}(\mathbf{t}) := \begin{cases} \langle \mathbf{u}(\cdot\,, \mathbf{t}) \rangle_{\mathbf{LL},\ \mathbf{B_r}(\mathbf{X}(\mathbf{t},\mathbf{y}))}, & \mathbf{0} \leq \mathbf{t} \leq \mathbf{t_r} << \mathbf{1} \\ \langle \mathbf{u}(\cdot\,, \mathbf{t}) \rangle_{\mathbf{LL},\ \mathbb{R}^n}, & \mathbf{t} > \mathbf{t_r} \end{cases}$$

and $\mathbf{g_r} \in \mathbf{L}^1_{\mathrm{loc}}([\mathbf{0}, \infty))$ where $\mathbf{r} > \mathbf{0}\,;\, \overline{\mathbf{B_r}(\mathbf{y})} \subset \mathbf{V}$.
Indeed,

$$\int_0^t ||\mathbf{u}(\cdot, \tau)||_\infty \, \mathbf{d}\tau \leq \mathbf{C}\, \mathbf{t}^\gamma\,.$$

Theorem 2 – One-sided limits

Recall that $\rho \equiv \rho(\cdot, t)$ **has a one-sided limit at** $\mathbf{x} \in \mathcal{M}^{\mathrm{t}}$ **from the "plus" side** $\mathcal{M}^{\mathrm{t}}_{+}$ **of** $\mathcal{M}_{\mathrm{t}}$ **if**

$$\begin{aligned}\mathrm{osc}(\rho; \mathbf{x}, \mathcal{M}^{\mathbf{t}}_{+}) :=\ & \lim_{\mathbf{r}\to\mathbf{0}} \Big[\operatorname{ess\,sup} \rho|_{\mathcal{M}^{\mathrm{t}}_{+}\cap \mathbf{B_r}(\mathbf{x})} \\ & \qquad - \operatorname{ess\,inf} \rho|_{\mathcal{M}^{\mathrm{t}}_{+}\cap \mathbf{B_r}(\mathbf{x})}\Big] \\ =\ & \mathbf{0}\,.\end{aligned}$$

In this case

$$\begin{aligned}\rho(\mathbf{x}+, \mathbf{t}) :=\ & \lim_{\mathbf{r}\to\mathbf{0}} \operatorname{ess\,sup} \rho(\cdot, \mathbf{t})|_{\mathcal{M}^{\mathrm{t}}_{+}\cap \mathbf{B_r}(\mathbf{x})} \\ =\ & \lim_{\mathbf{r}\to\mathbf{0}} \operatorname{ess\,inf} \rho(\cdot, \mathbf{t})|_{\mathcal{M}^{\mathrm{t}}_{+}\cap \mathbf{B_r}(\mathbf{x})}.\end{aligned}$$

Write the conservation of mass $\rho_{\mathrm{t}} + \mathrm{div}(\rho\mathbf{u}) = \mathbf{0}$ **as**

$$\dot{\rho} = -\rho\, \mathbf{divu} = -\rho\,(\lambda+\mu)^{-1}\,[\mathbf{F} + (\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho}))]$$

$$(\lambda+\mu)\dot{\rho} = -\rho\ [\mathbf{F} + (\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho}))]$$

Since $\rho(\cdot, \mathbf{t})|_{\mathbf{V}^{\mathrm{t}}}$ **is strictly positive, we may divide by** $\rho(\mathbf{X}(\mathbf{t}, \mathbf{y}), \mathbf{t})$**, for** $\mathbf{y} \in \mathbf{V}$**, to obtain that**

$$(\lambda+\mu)\frac{\mathbf{d}}{\mathbf{dt}} \log \rho(\mathbf{X}(\mathbf{t}, \mathbf{y}), \mathbf{t}) + \mathbf{P}(\rho(\mathbf{X}(\mathbf{t}, \mathbf{y}), \mathbf{t})) - \mathbf{P}(\tilde{\rho})$$

$$= -\mathbf{F}(\mathbf{X}(\mathbf{t}, \mathbf{y}), \mathbf{t})\ .$$

Then for $\mathbf{y_1, y_2 \in V}$**,**

$$(\lambda+\mu)\frac{\mathbf{d}}{\mathbf{dt}}\log\rho(\mathbf{X}(\mathbf{t},\cdot),\mathbf{t})\Big|_{\mathbf{y_1}}^{\mathbf{y_2}} + \mathbf{a}(\mathbf{t})\log\rho(\mathbf{X}(\mathbf{t},\cdot),\mathbf{t})\Big|_{\mathbf{y_1}}^{\mathbf{y_2}}$$

$$= -\mathbf{F}(\mathbf{X}(\mathbf{t},\cdot),\mathbf{t})\Big|_{\mathbf{y_1}}^{\mathbf{y_2}} ,$$

$$\mathbf{a}(\mathbf{t}) = \frac{\mathbf{P}(\rho(\mathbf{X}(\mathbf{t},\cdot),\mathbf{t}))\Big|_{\mathbf{y_1}}^{\mathbf{y_2}}}{\log\rho(\mathbf{X}(\mathbf{t},\cdot),\mathbf{t})\Big|_{\mathbf{y_1}}^{\mathbf{y_2}}} .$$

Then (writing $\sup \equiv \operatorname{ess\,sup}$ **and** $\inf \equiv \operatorname{ess\,inf}$ **)**

$$\begin{aligned} &(\sup - \inf)\log\rho(\cdot,\mathbf{t})\Big|_{\mathbf{X}(\mathbf{t},\cdot)(\mathbf{B_r}(\mathbf{x_0})\cap\mathcal{M}_+)} \\ \le\ & \mathbf{e}^{-(\lambda+\mu)^{-1}\int_{\mathbf{0}}^{\mathbf{t}}\mathbf{a}(\tau)\mathbf{d}\tau}(\sup - \inf)\log\rho_{\mathbf{0}}\Big|_{\mathbf{B_r}(\mathbf{x_0})\cap\mathcal{M}_+} \\ & + \int_{\mathbf{0}}^{\mathbf{t}}\mathbf{e}^{-(\lambda+\mu)^{-1}\int_{\tau}^{\mathbf{t}}\mathbf{a}(\xi)\mathbf{d}\xi}(\sup - \inf)\mathbf{F}(\cdot,\tau)\Big|_{\mathbf{X}(\tau,\cdot)(\mathbf{B_r}(\mathbf{x_0})\cap\mathcal{M}_+)}\mathbf{d}\tau\ . \end{aligned}$$

The first term on the right goes to zero as $\mathbf{r \to 0}$ **by the assumption that** ρ_0 **has a one-sided limit at** $\mathbf{x_0}$ **and the second goes to zero by the Hölder continuity of** $\mathbf{F}(\cdot,\tau)$ **for** $\tau > \mathbf{0}$**.**
■

Theorem 3 – Time evolution of discontinuites

Let $\mathbf{r > 0}$ such that $\mathbf{r < \mathrm{dist}(x_0, \partial V)}$, $\{\mathbf{x_h}\}_{\mathbf{h>0}} \in \mathcal{M}_+$ and $\{\mathbf{r_h}\} > \mathbf{0}$ such that $\mathbf{x_h \to x_0}$, $\mathbf{r_h \to 0}$ as $\mathbf{h \to 0}$, and $\mathbf{B_{2r_h}(x_h)} \subset \mathcal{M}_+ \cap \mathbf{B_r(x_0)}$, for all $\mathbf{0 < h << 1}$.

Define $\varphi^{\delta,h}$, satisfying the transport equation

$$\varphi^{\delta,\mathbf{h}}_{\mathbf{t}} + \mathrm{div}(\varphi^{\delta,\mathbf{h}}\mathbf{u}^{\delta}) = \mathbf{0}$$

with initial datum

$$\varphi^{\delta,\mathbf{h}}|_{\mathbf{t=0}} = \varphi^{\mathbf{h}}_{\mathbf{0}},$$

where $\varphi^{\mathbf{h}}_{\mathbf{0}}$ is a smooth function with support in $\mathbf{B_{r_h}(x_h)}$, $\int \varphi^{\mathbf{h}}_{\mathbf{0}}(\mathbf{x})\mathbf{dx} = \mathbf{1}$, and $\mathbf{0} \le \varphi^{\mathbf{h}}_{\mathbf{0}} \le \mathbf{C^h}$, for some positive number C^h.

Then $\varphi^{\delta,\mathbf{h}}$ has support in $\mathbf{X}^{\delta}(\mathbf{t},\cdot)\mathbf{B_{r_h}(x_h)}$, $\int \varphi^{\delta,\mathbf{h}}(\mathbf{x},\mathbf{t})\mathbf{dy} = \mathbf{1}$ for every $\mathbf{t \ge 0}$, and $\mathbf{0} \le \varphi^{\delta,\mathbf{h}} \le \mathbf{C^h(T)}$ if $\mathbf{0 \le t \le T}$ for some positive number $C^h(T)$. This is a consequence of the fact that

$$\int_{\mathbf{0}}^{\mathbf{T}} ||\mathbf{F}^{\delta}(\cdot,\mathbf{t})||_{\mathbf{L}^{\infty}\left(\mathbf{X}^{\delta}(\mathbf{t},\cdot)\mathbf{B_r(x_0)}\right)}\ \mathbf{dt} \le \mathbf{C(T + T^{\gamma})},$$

where $\gamma = \gamma(r) > 0$.

Write $\mathbf{L}^\delta \equiv \log \rho^\delta$**,** $\mathbf{L} \equiv \log \rho$ **and the mass equation in the form**

$$\mathbf{L_t^\delta} + \nabla \mathbf{L}^\delta \cdot \mathbf{u}^\delta = -(\lambda + \mu)^{-\mathbf{1}} (\mathbf{F}^\delta + \mathbf{P}(\rho^\delta) - \mathbf{P}(\tilde{\rho}))$$

$$\int \varphi^{\delta,\mathbf{h}} \mathbf{L}^\delta \, \mathbf{dx} \, |_{\mathbf{0}}^{\mathbf{t}} = -(\lambda + \mu)^{-\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{t}} \int \left(\mathbf{F}^\delta + \mathbf{P}(\rho^\delta) - \mathbf{P}(\tilde{\rho})\right) \varphi^{\delta,\mathbf{h}} \, \mathbf{dxds}.$$

We want to take the limits as $\delta \to 0$ **and then** $h \to 0$**. To do that we use that** X^δ **converges to** X **in** $[0, t] \times B_r(x_0)$ **uniformly with respect to** δ **and that for each** $h > 0$ **there is a** $\delta_0(h) > 0$ **such that**

$$X^\delta(s, \cdot) B_{r_h}(x_h) \subset X(s, \cdot) B_{2r_h}(x_h)$$

for all $\delta \leq \delta_0(h)$ **and** $s \in [0, t]$**.**

For fixed t, we write

$$\int \varphi^{\delta,\mathbf{h}} \mathbf{L}^\delta \mathbf{dx} = \int \varphi^{\delta,\mathbf{h}} (\mathbf{L}^\delta - \mathbf{L}) \mathbf{dx} + \int \varphi^{\delta,\mathbf{h}} \mathbf{L} \mathbf{dx} \equiv \mathbf{I} + \mathbf{II}$$

and notice that $\mathbf{I} \to \mathbf{0}$ as $\delta \to \mathbf{0}$ because its integrand tends to zero a.e. and it is bounded by some constant $\mathbf{C^h(t)}$. Thus given h, there is a $\delta_\mathbf{0}(\mathbf{h})$ such that $\mathbf{I} \leq \mathbf{h}$ if $\delta \leq \delta_\mathbf{0}(\mathbf{h})$. Regarding II, we have

$$\mathbf{II} \leq \mathrm{ess\,sup} \mathbf{L}(\cdot, \mathbf{t}) | \mathbf{X}(\mathbf{t}, \cdot) \mathbf{B}_{\mathbf{2r_h}}(\mathbf{x_h})$$

if $\delta \leq \delta_\mathbf{0}(\mathbf{h})$ for some $\delta_\mathbf{0}(\mathbf{h}) > \mathbf{0}$. Then, there is a $\delta_\mathbf{0}(\mathbf{h}) > \mathbf{0}$ such that $\delta \leq \delta_\mathbf{0}(\mathbf{h})$ implies

$$\int \varphi^{\delta,\mathbf{h}} \mathbf{L}^\delta \mathbf{dx} \leq \mathrm{ess\,sup} \mathbf{L}(\cdot, \mathbf{t}) | \mathbf{X}(\mathbf{t}, \cdot) \mathbf{B}_{\mathbf{2r_h}}(\mathbf{x_h}) + \mathbf{h}.$$

Similarly,

$$\mathrm{ess\,inf}\, \mathbf{L}(\cdot, \mathbf{t}) | \mathbf{X}(\mathbf{t}, \cdot) \mathbf{B}_{\mathbf{2r_h}}(\mathbf{x_h}) - \mathbf{h} \leq \int \varphi^{\delta,\mathbf{h}} \mathbf{L}^\delta \mathbf{dx}$$

$$\leq \mathrm{ess\,sup}\, \mathbf{L}(\cdot, \mathbf{t}) | \mathbf{X}(\mathbf{t}, \cdot) \mathbf{B}_{\mathbf{2r_h}}(\mathbf{x_h}) + \mathbf{h}.$$

for all $\delta \leq \delta_\mathbf{0}(\mathbf{h})$.

Also, we write

$$\begin{aligned}\int_0^{\mathbf{t}}\!\!\int \varphi^{\delta,\mathbf{h}}\mathbf{F}^\delta \mathbf{dxds} &= \int_0^{\mathbf{t}}\!\!\int \varphi^{\delta,\mathbf{h}}(\mathbf{F}^\delta - \mathbf{F})\mathbf{dsdx} + \int_0^{\mathbf{t}}\!\!\int \varphi^{\delta,\mathbf{h}}\mathbf{F}\mathbf{dxds} \\ &\equiv \mathbf{I} + \mathbf{II}.\end{aligned}$$

I tends to zero as $\delta \to \mathbf{0}$:

$$\mathbf{supp}\,\varphi^{\delta,\mathbf{h}}(\cdot,\mathbf{s}) \subset \mathbf{X}^\delta(\mathbf{s},\cdot)\mathbf{B}_{\mathbf{r_h}}(\mathbf{x_h}) \subset \mathbf{X}^\delta(\mathbf{s},\cdot)\mathbf{B}_{\mathbf{2r_h}}(\mathbf{x_0}) \subset \mathbf{X}(\mathbf{s},\cdot)\mathbf{B}_{\mathbf{r}}(\mathbf{x_0})$$

and

$$\begin{aligned}&|\int_0^{\tau}\!\!\int \varphi^{\delta,\mathbf{h}}(\mathbf{F}^\delta - \mathbf{F})\mathbf{dxds}| \\ \le\ & \mathbf{C^h}(\mathbf{t})|\mathbf{K}|\int_0^{\tau} \big(||\mathbf{F}^\delta|| + ||\mathbf{F}||\big)_{\mathbf{L}^\infty(\,\mathbf{X}(\mathbf{s},\cdot)\mathbf{B_r}(\mathbf{x_0})\,)}\,\mathbf{ds} \le \mathbf{C^h}(\mathbf{t})|\mathbf{K}|\tilde{\mathbf{C}}\,\tau^\gamma,\end{aligned}$$

for some positive constants $\tilde{C}, \gamma$, and any sufficiently small $\tau > 0$, where $|K|$ is the Lebesgue measure of the compact set $K = X([0,t] \times \overline{B_r(x_0)})$. On the other hand,

$$|\varphi^{\delta,\mathbf{h}}(\mathbf{F}^\delta - \mathbf{F})| \le \mathbf{C^h}(\mathbf{t})|\mathbf{F}^\delta - \mathbf{F}|$$

converges to zero a.e. and, for $\mathbf{s} \in [\tau, \mathbf{t}]$ and χ being the charateristic function of the set K, we have

$$\begin{aligned}|\varphi^{\delta,\mathbf{h}}(\mathbf{F}^\delta - \mathbf{F})| &\le \mathbf{C^h}(\mathbf{t})\chi(\mathbf{x})(||\mathbf{F}^\delta||_\infty + \liminf_{\delta\to\mathbf{0}} ||\mathbf{F}^\delta||_\infty) \\ &\le \mathbf{2C^h}(\mathbf{t})\chi(\mathbf{x})\sup_\delta ||\mathbf{F}^\delta||_\infty \le \mathbf{C}_\tau\mathbf{C^h}(\mathbf{t})\chi(\mathbf{x}).\end{aligned}$$

Thus, similarly to above, there is a $\delta_0(\mathbf{h})$ such that for $\delta \le \delta_0(\mathbf{h})$,

$$\int_0^t \operatorname{ess\,inf} \mathbf{F}(\cdot,\mathbf{s})|\mathbf{X}(\mathbf{s},\cdot)\mathbf{B}_{2r_h}(\mathbf{x_h})\,\mathbf{ds} \;-\; \mathbf{h} \;\le\; \int_0^t\!\!\int \varphi^{\delta,\mathbf{h}}\mathbf{F}^{\delta}\mathbf{dxds}$$
$$\le \int_0^t \operatorname{ess\,sup} \mathbf{F}(\cdot,\mathbf{s})|\mathbf{X}(\mathbf{s},\cdot)\mathbf{B}_{2r_h}(\mathbf{x_h})\,\mathbf{ds} \;+\; \mathbf{h}.$$

Analogously, we have a similar estimate w.r.t. $\mathbf{P}-\mathbf{P}(\tilde{\rho})$:

$$\int_0^t \operatorname{ess\,inf}\,[\mathbf{P}(\rho(\cdot,\mathbf{s}))-\mathbf{P}(\tilde{\rho})]\big|\mathbf{X}(\mathbf{s},\cdot)\mathbf{B}_{2r_h}(\mathbf{x_h})\,\mathbf{ds}-\mathbf{h}$$
$$\le \int_0^t\!\!\int \varphi^{\delta,\mathbf{h}}[\mathbf{P}(\rho^{\delta})-\mathbf{P}(\tilde{\rho})]\mathbf{dxds}$$
$$\le \int_0^t \operatorname{ess\,sup}\,[\mathbf{P}(\rho(\cdot,\mathbf{s}))-\mathbf{P}(\tilde{\rho})]\big|\mathbf{X}(\mathbf{s},\cdot)\mathbf{B}_{2r_h}(\mathbf{x_h})\,\mathbf{ds}+\mathbf{h}.$$

Now,

$$\begin{aligned}\mathbf{L}(\mathbf{x_0^t}+,\mathbf{t}) &= \lim_{\mathbf{r'}\to\mathbf{0}} \operatorname{ess\,sup} \mathbf{L}(\cdot,\mathbf{t})|(\mathbf{B_{r'}}(\mathbf{x_0^t})\cap\mathcal{M}_+^{\mathbf{t}}) \\ &= \lim_{\mathbf{r'}\to\mathbf{0}} \operatorname{ess\,inf} \mathbf{L}(\cdot,\mathbf{t})|(\mathbf{B_{r'}}(\mathbf{x_0^t})\cap\mathcal{M}_+^{\mathbf{t}}),\end{aligned}$$

and for each $\mathbf{r'} > \mathbf{0}$ there is a $\mathbf{h_{r'}} > \mathbf{0}$ such that

$$\mathbf{X}(\mathbf{t},\cdot)\mathbf{B_{2r_h}}(\mathbf{x_h}) \subset \mathbf{B_{r'}}(\mathbf{x_0^t})\cap\mathcal{M}_+^{\mathbf{t}}$$

for all $\mathbf{h} \leq \mathbf{h_{r'}}$.

Thus, for each $\mathbf{r'} > \mathbf{0}$ and all $\mathbf{h} \leq \mathbf{h_{r'}}$,

$$\begin{aligned}&\operatorname{ess\,inf} \mathbf{L}(\cdot,\mathbf{t})|(\mathbf{B_{r'}}(\mathbf{x_0})\cap\mathcal{M}_+^{\mathbf{t}}) \leq \operatorname{ess\,inf} \mathbf{L}(\cdot,\mathbf{t})|\mathbf{X}(\mathbf{t},\cdot)\mathbf{B_{2r_h}}(\mathbf{x_h}) \\ &\leq \operatorname{ess\,sup} \mathbf{L}(\cdot,\mathbf{t})|(\mathbf{B_{r'}}(\mathbf{x_0})\cap\mathcal{M}_+^{\mathbf{t}}).\end{aligned}$$

Then, taking here first the $\liminf$ **and** $\limsup$ **as $\mathbf{h} \to \mathbf{0}$, and then the limit as $\mathbf{r'} \to \mathbf{0}$, we see that there exists the**

$$\lim_{\mathbf{h}\to\mathbf{0}} \operatorname{ess\,inf} \mathbf{L}(\cdot,\mathbf{t})|\mathbf{X}(\mathbf{t},\cdot)\mathbf{B_{2r_h}}(\mathbf{x_h}) = \mathbf{L}(\mathbf{x_0^t}+,\mathbf{t}).$$

Analogously,

$$\lim_{\mathbf{h}\to\mathbf{0}} \operatorname{ess\,sup} \mathbf{L}(\cdot,\mathbf{t})|\mathbf{X}(\mathbf{t},\cdot)\mathbf{B_{2r_h}}(\mathbf{x_h}) = \mathbf{L}(\mathbf{x_0^t}+,\mathbf{t}).$$

Hence,

$$\lim_{\mathbf{h}\to\mathbf{0}}\lim_{\delta\to\mathbf{0}}\int(\varphi^{\delta,\mathbf{h}}\mathbf{L}^{\delta})(\mathbf{x},\mathbf{t})\mathbf{dx}=\mathbf{L}(\mathbf{x}_{\mathbf{0}}^{\mathbf{t}}+,\mathbf{t}).$$

Similarly,

$$\lim_{\mathbf{h}\to\mathbf{0}}\lim_{\delta\to\mathbf{0}}\int(\varphi^{\delta,\mathbf{h}}\mathbf{L}^{\delta})(\mathbf{x},\mathbf{0})\mathbf{dx}=\mathbf{L}_{\mathbf{0}}(\mathbf{x}_{\mathbf{0}}+),$$

$$\lim_{\mathbf{h}\to\mathbf{0}}\lim_{\delta\to\mathbf{0}}\int_{\mathbf{0}}^{\mathbf{t}}\int\varphi^{\delta,\mathbf{h}}\mathbf{F}^{\delta}\mathbf{dxds}=\int_{\mathbf{0}}^{\mathbf{t}}\mathbf{F}(\mathbf{X}(\mathbf{s},\mathbf{x}_{\mathbf{0}})+,\mathbf{s})\mathbf{ds}$$

and

$$\begin{aligned}&\lim\nolimits_{\mathbf{h}\to\mathbf{0}}\lim\nolimits_{\delta\to\mathbf{0}}\int_{\mathbf{0}}^{\mathbf{t}}\int\varphi^{\delta,\mathbf{h}}[\mathbf{P}(\rho^{\delta})-\mathbf{P}(\tilde{\rho})]\mathbf{dxds}\\&=\int_{\mathbf{0}}^{\mathbf{t}}[\mathbf{P}(\rho(\mathbf{X}(\mathbf{s},\mathbf{x}_{\mathbf{0}})+,\mathbf{s}))-\mathbf{P}(\tilde{\rho})]\mathbf{ds}.\end{aligned}$$

From all the above, it follows that

$$\begin{aligned} &\mathbf{L}(\mathbf{x_0^t}+,\mathbf{t}) - \mathbf{L}(\mathbf{x_0}+) \\ = &-(\lambda+\mu)^{-1}\int_0^{\mathbf{t}} [\,\mathbf{F}(\mathbf{X}(\mathbf{s},\mathbf{x_0})+,\mathbf{s}) + \mathbf{P}\,(\,\rho(\mathbf{X}(\mathbf{s},\mathbf{x_0})+,\mathbf{s})\,) - \mathbf{P}(\tilde{\rho})\,]\,\mathbf{ds}. \end{aligned}$$

This shows that the map $\mathbf{t}\in[\mathbf{0},\infty)\mapsto\mathbf{L}(\mathbf{x_0^t}+,\mathbf{t})$ **is in** $\mathbf{C}([\mathbf{0},\infty))\cap\mathbf{C^1}((\mathbf{0},\infty))$, **hence so it is** $\mathbf{t}\in[\mathbf{0},\infty)\mapsto\rho(\mathbf{x_0^t}+,\mathbf{t})$**. Next, write the same relation for** $\mathbf{L}(\mathbf{x_0^t}-,\mathbf{t})$ **and subtract to get**

$$[\mathbf{L}(\mathbf{x_0^t},\mathbf{t})] - [\mathbf{L}(\mathbf{x_0})] = -(\lambda+\mu)^{-1}\int_0^{\mathbf{t}} [\mathbf{P}\,(\,\rho(\mathbf{X}(\mathbf{s},\mathbf{x_0}),\mathbf{s})\,)]\,\mathbf{ds}.$$

Then,

$$\frac{\mathbf{d}}{\mathbf{dt}}[\mathbf{L}(\mathbf{x_0^t},\mathbf{t})] = -(\lambda+\mu)^{-1}[\mathbf{P}\,\Big(\,\rho(\mathbf{X}(\mathbf{x_0^t},\mathbf{t}),\mathbf{t})\,\Big)] \equiv \mathbf{a}(\mathbf{t})[\mathbf{L}(\mathbf{x_0^t},\mathbf{t})],$$

so integranting this equation we finish the proof.

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