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Log-lipschizian Vector Fields in $\mathbb{R}^n$

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1. INTRODUCTION (MOTIVATION)

Let Ω denote an open set in $\mathbb{R}^n$ and I, an open interval in $\mathbb{R}$. As we know, a vector fied v in Ω (i.e. a map $v : \Omega \to \mathbb{R}^n$) is said to be lipschitzian when there is a constant C such that

$$|v(x_1) - v(x_2)| \leq C|x_1 - x_2| \tag{1}$$

for all $x_1, x_2 \in \Omega$. If the field v is bounded then this condition is equivalent to

$$\sup_{\substack{0 < |\mathbf{x_1} - \mathbf{x_2}| \leq 1 \\ x_1, x_2 \in \Omega}} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|} < \infty.$$

Particle trajectory

If the field v depends also on a real variable $t \in I$ (physically, this second variable denotes time), the **Picard theorem** says that if $v = v(x,t) : \Omega \times I \to \mathbb{R}^n$ is a continuous map and lipschitzian with respect to $x \in \Omega$, uniformly with respect to $t \in I$, i.e. if for some constant C,

$$|v(x_1,t) - v(x_2,t)| \leq C|x_1 - x_2| \tag{2}$$

for all $x_1, x_2 \in \Omega$ and for all $t \in I$, then for any pair $(x_0, t_0) \in \Omega \times I$, there exist a unique solution to the problem

$$x' = v(x,t), \quad x(t_0) = x_0, \tag{3}$$

where $x' \equiv \partial x / \partial t$ is defined in some open open interval $I_0 \equiv I(x_0, t_0) \subset I$.

Particle trajectory cont'd

We shall denote the above solution by $X(\cdot\,; x_0, t_0)$.
The map $t \in I_0 \mapsto X(t; x_0, t_0)$ is said to be an integral curve of the field v, passing by the point x_0 at the time t_0 or, physically speaking, in fluid dynamics, the particle trajectory/the trajectory of the particle that at time t_0 is at the point x_0, if $v(x, t)$ represents the velocity of a fluid particle that at time t is at the point x.

The map $(t; x_0, t_0) \mapsto X(t; x_0, t_0)$ is called the the flux of the field v.

Remark: If v is bounded, then $I_0 = I$, for all $(x_0, t_0) \in \Omega \times I$.

Definition

When for all $(x_0, t_0) \in \Omega \times I$, the problem

$$x' = v(x,t), \quad x(t_0) = x_0,$$

has one and only one solution $X(\cdot\,; x_0, t_0)$, defined is some open interval I_0, we say that the vector field v has a lagrangian structure.

First example

Question: Is the Lipschitz condition (2) ($|v(x_1, t) - v(x_2, t)| \leq C|x_1 - x_2|$) necessary in order that the field v has a lagrangian structure?

Answer: NO!

Example

The function (scalar vector field) defined by $v(x, t) = x \log |x|$, *if* $x \neq 0$ *and* $v(0, t) = 0$, *with* $t \in \mathbb{R}$, *has a lagrangian structure but it does not satisfies the Lipschitz condition* (2).

Proof. For $x \neq 0$, we have that

$$\partial v / \partial x = \log |x| + 1 \to -\infty, \quad \text{when} \quad x \to 0,$$

i.e. this partial derivative is not bounded.

Note

The Lipschitzian condition (2) ($|v(x_1, t) - v(x_2, t)| \leq C|x_1 - x_2|$) implies that

$$|\nabla_x v(x, t)| \leq C$$

for every point (x, t) where this derivative exists. In fact, for who knows Measure Theory and Weak Derivatives, we can say that the condition (2) is equivalent to the following conditions: there exists the partial derivatives $\partial v / \partial x_i$, as weak derivatives, they belong to the space $L^\infty(\Omega)$ and $\|\nabla_x v(\cdot, t)\|_{L^\infty(\Omega)} \leq C$ for all $t \in I$. This statement is the Rademacher theorem. See e.g. the book *Partial Differential Equations*, by Lawrence C. Evans.

The (scalar) field $v(x,t) = x \log|x|$ has a lagrangian structure, i.e.

the problem

$$x' = x \log|x|, \quad x(t_0) = x_0$$

has a unique solution.

Proof. For $x_0 \neq 0$, this is a consequence of Picard theorem. In fact, we can compute this solution explicitly by separation of variables, which is (Exercise/Home Work!):

$$X(t; x_0, t_0) = (\mathrm{sgn} x_0)|x_0|^{e^{(t-t_0)}},$$

where $\mathrm{sgn} x_0 = 1$ if $x_0 > 0$ and $\mathrm{sgn} x_0 = -1$ if $x_0 < 0$.

Proof cont'd: the case $x_0 = 0$

In the case $x_0 = 0$, it is clear that $x(t) \equiv 0$ is a solution. This solution is unique, since if another solution $\varphi(t) \equiv X(\cdot\,; 0, t_0)$ were not null in some point t_1, setting $x_1 = \varphi(t_1)$, we would have

$$\varphi(t) = X(t;\, x_1, t_1), \quad \forall\, t \in \mathbb{R},$$

and this is a contradiction, since $\varphi(t_0) = 0$ but $X(t; x_0, t_0) \neq 0$ for all $t \in \mathbb{R}$ if $x_0 \neq 0$.

Remarks

- There are many non lipschitizian fields with lagrangian structure. Another example is $v(x) = 1 + 2x^{2/3}$. See Example 1.2.1 in the book *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations* by R. P. Agarwal and V. Lakshmikantham.

- The function $-x \log x$ is used in Information Theory to define the so-called *Shannon entropy*. See the seminal paper Shannon, C. E. *A mathematical theory of communication*. Bell System Technical Journal, **27** (1948) or Section 4.1 of the book Yockey, Hubert P. *Information theory, evolution, and the origin of life*.

- The function $x \log x$ is also used in Optimization. See e.g. the works Lopes, Marcos Vinícius. *Trajetória Central Associada à Entropia e o Método do Ponto Proximal em Programação Linear*. Masters dissertation - Federal University of Goiás, Brazil, 2007. Advisor: Orizon Pereira Ferreira

  and

  Ferreira, O. P.; Oliveira, P. R.; Silva, R. C. M. *On the convergence of the entropy-exponential penalty trajectories and generalized proximal point methods in semidefinite optimization*. J. Global Optim. 45 (2009).

The field $v(x) \equiv v(x,t) = x \log|x|$ satisfies the following property:

$$|v(x_1) - v(x_2)| \le C|x_1 - x_2|\,(1 - \log|x_1 - x_2|) \qquad (4)$$

(Lipschitz condition modified)

for all $x_1, x_2 \in (-1, 1)$ such that $0 < |x_1 - x_2| \le 1$.

Notice that $-\log|x_1 - x_2| > 0$!

Proof

For $x_1, x_2 \in (0,1)$, assuming $x_1 < x_2$, without loss of generality, we have

$$
\begin{aligned}
& |v(x_1) - v(x_2)| = \left| \int_0^1 \frac{d}{ds} v\left(x_1 + s(x_2 - x_1)\right) ds \right| \\
= \;& \left| (x_2 - x_1) \int_0^1 v'\left(x_1 + s(x_2 - x_1)\right) ds \right| \\
= \;& \left| (x_2 - x_1) \int_0^1 \left(1 + \log(x_1 + s(x_2 - x_1))\right) ds \right| \\
\leq \;& |x_2 - x_1| \left(1 - \int_0^1 \log(x_1 + s(x_2 - x_1))\, ds\right) \\
\leq \;& |x_2 - x_1| \left(1 - \int_0^1 \log s(x_2 - x_1) ds\right) \\
= \;& |x_2 - x_1| \left(1 - \int_0^1 \left(\log s + \log(x_2 - x_1)\right) ds\right) \\
= \;& |x_2 - x_1| \left(1 - \int_0^1 \log s \, ds - \int_0^1 \log |x_2 - x_1| \, ds\right) \\
= \;& |x_2 - x_1| \left(2 - \log |x_2 - x_1|\right) \leq 2|x_2 - x_1| \left(1 - \log |x_2 - x_1|\right)
\end{aligned}
$$

Proof cont'd

For $x_1, x_2 \in (-1, 0)$, we can conclude the same inequality as above by using that $v(x) = x \log |x|$ is an odd function.

Next, let $x_1 \in (-1, 0)$ and $x_2 \in (0, 1)$.

$\boxed{\text{If } |x_1 - x_2| = x_2 - x_1 \leq e^{-1}}$, using that $\psi := -x \log |x|$ is increasing in the interval $[-e^{-1}, e^{-1}]$, we have the following estimates:

$$
\begin{aligned}
& |v(x_1) - v(x_2)| = x_1 \log |x_1| - x_2 \log |x_2| = \psi(-x_1) + \psi(x_2) \\
\leq \; & \psi(-x_1 + x_2) + \psi(x_2 - x_1) = 2\psi(x_2 - x_1) \\
= \; & -2(x_2 - x_1) \log(x_2 - x_1) < 2\,(1 - (x_2 - x_1) \log(x_2 - x_1)) .
\end{aligned}
$$

Proof cont'd

On the other hand, $\boxed{\text{if } |x_1 - x_2| > e^{-1}}$, using that $v(x) = x \log x$ is bounded in the interval $(-1, 1)$, we have

$$
\begin{aligned}
& |v(x_1) - v(x_2)| = \tfrac{|v(x_1) - v(x_2)|}{|x_1 - x_2|} |x_1 - x_2| \\
\leq \quad & 2(\max |v|) e |x_1 - x_2| \\
\leq \quad & 2(\max |v|) e |x_1 - x_2| \left(1 - \log |x_1 - x_2|\right).
\end{aligned}
$$

Finally, we observe that if $x_1 = 0$ or $x_2 = 0$, the condition (4) is trivial, since $v(0) = 0$.

$\square$

Remark

As we can see from the last argument, to show (4), i.e. $|v(x_1) - v(x_2)| \leq C|x_1 - x_2|(1 - \log|x_1 - x_2|)$, it is enough to consider this inequality for $|x_1 - x_2|$ less than a certain constant, since v is bounded.

2. LOG-LIPSCHIZIAN VECTOR FIELDS

Definition

A vector field v in Ω (an open set in $\mathbb{R}^n$) is called log-lipschitzian if it is bounded (for convenience) and satisfies the inequality (4), i.e.

$$|v(x_1) - v(x_2)| \leq C|x_1 - x_2|\,(1 - \log|x_1 - x_2|)$$

for all $x_1, x_2 \in \Omega$ such that $0 < |x_1 - x_2| \leq 1$ i.e.

$$\sup_{\substack{0 < |x_1 - x_2| \leq 1 \\ x_1, x_2 \in \Omega}} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|(1 - \log|x_1 - x_2|)} < \infty. \qquad (5)$$

Remarks

- The set of log-lipschitizian vector fields is a normed vector space with the norm

$$\|v\|_{LL} \equiv \|v\|_{LL(\Omega)} := \|v\|_{L^\infty(\Omega)} + \langle v \rangle_{LL} \tag{6}$$

where $\langle v \rangle_{LL}$ is the seminorm defined in (5) i.e.

$$\langle v \rangle_{LL} := \sup_{\substack{0 < |x_1 - x_2| \leq 1 \\ x_1, x_2 \in \Omega}} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|(1 - \log|x_1 - x_2|)}$$

and $\|v\|_{L^\infty(\Omega)} = \|v\|_{\sup} := \sup_{x \in \Omega} |v(x)|$.

- We denote this space by LL or, more precisely, by $LL(\Omega)$.
- It is not difficult to show that LL is a Banach space.

3. PICARD THEOREM

Lagrangian structure

The Picard Theorem (Cauchy-Lipschitz theorem) holds for log-lipschtizian vector fields. More precisely, if $v : \Omega \times I \to \mathbb{R}^n$ is a continuous map such that, for some constant C,

$$|v(x_1,t) - v(x_2,t)| \leq C|x_1 - x_2|(1 - \log|x_1 - x_2|) \tag{7}$$

for all $x_1, x_2 \in \Omega$ with $0 < |x_1 - x_2| \leq 1$, i.e. if $\sup_{t\in I}\langle v(\cdot,t)\rangle_{LL} := \sup_{\substack{0 < |x_1 - x_2| \leq 1 \\ x_1, x_2 \in \Omega}} \frac{|v(x_1)-v(x_2)|}{|x_1-x_2|(1-\log|x_1-x_2|)} < \infty$ then the field v has a lagrangian structure, i.e. the problem

$$x' = v(x,t), \quad x(t_0) = x_0$$

has a unique solution $X(\cdot; x_0, t_0)$, defined in some interval $I_0 \equiv I(x_0, t_0)$, for all $(x_0, t_0) \in \Omega \times I$.

Let us discuss a proof for this theorem.

In the paper

Chemin, J. Y.; Lerner, N. *Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes*, J. Differential Equations, **121** (1995), no. 2, 314-328

or in the book

Chemin, J. Y. *Perfect incompressible fluids*. Oxford Lecture Series in Mathematics and its Applications, **14**. The Clarendon Press (1998)

we can find a proof which is also valid in infinite dimensional space, i.e. with Ω being an open set of a Banach space E and $v : \Omega \times I \to E$ as above.

Comparing this proof with the classical proof of Picard theorem, the difference is the way of showing that the Picard iteration is convergent.

Picard iteration

$$\varphi_0 = x_0, \quad \varphi_k(t) = x_0 + \int_{t_0}^{t} v(\varphi_{k-1}(s), s)ds,$$

$$k = 1, 2, \cdots$$

Convergence: Let I_0 be an open interval such that $t_0 \in I_0$, $\overline{I_0} \subset I$ and sufficiently small such that $\varphi_k(t) \in \overline{B_r(x_0)} \subset \Omega$, for some closed ball $\overline{B_r(x_0)}$ and for all $t \in \overline{I_0}$.

To facilitate the notation, here and in the sequel, we introduce the function

$$m(r) = \begin{cases} r(1 - \log r), & \text{if } 0 < r < 1 \\ r, & \text{if } r \geq 1 \,. \end{cases} \tag{8}$$

(Notice that we also can write $r(1 - \log r) = r \log(1/r)$.)

Convergence of Picard iteration cont'd

Observe that, for $k, l \in \{1, 2, \cdots\}$, we have the inequality

$$|\varphi_{k+l}(t) - \varphi_k(t)| \leq |\int_{t_0}^{t} \langle v(\cdot, s)\rangle_{LL} m(|\varphi_{k+l-1}(s) - \varphi_{k-1}(s)|) ds|, \tag{9}$$

for $t \in \overline{I_0}$. Then, setting $\rho_k(t) = \sup_l |\varphi_{k+l}(t) - \varphi_k(t)|$ and using that $m(r)$ is an increasing function, we obtain

$$\rho_k(t) \leq C|\int_{t_0}^{t} m(\rho_{k-1}(s)) ds|, \quad \forall\, t \in \overline{I_0}.$$

and, taking the $\limsup$ with respect to k, it follows that

$$\boxed{\rho(t) \leq C|\int_{t_0}^{t} m(\rho(s)) ds|}, \quad \forall\, t \in \overline{I_0}, \tag{10}$$

where $\rho(t) := \limsup_k \rho_k(t)$.

Remark

To pass the above $\limsup$ over the integral we used that $\limsup_k \rho_k$ is the limit of the sequence $\zeta_k := \sup_k\{\varphi_k, \varphi_{k+1}, \cdots\}$, then, since m is increasing, we have $\rho_k(t) \leq C|\int_{t_0}^t m(\zeta_{k-1}(s))ds|$, and therefore, by the "reverse Fatou lemma"*, it follows that $\limsup_k \rho_k(t) \leq C|\int_{t_0}^t \limsup_k m(\zeta_{k-1}(s))ds| = C|\int_{t_0}^t \lim_k m(\zeta_{k-1}(s))ds| = C|\int_{t_0}^t \lim_k m(\rho(s))ds|$.

*V. e.g. Wikipedia. Notice that $\{m(\zeta_{k-1})\}$ is uniformly bounded in I_0, since $\varphi_k(t) \in \overline{B_r(x_0)}$ for all $k = 1, 2, \cdots$ and $t \in \overline{I_0}$

- The inequality (10) implies that ρ is null (next slide).
- Let $\varphi(t) = \lim_k \varphi_k(t), \quad t \in I_0$.
- To conclude that $\varphi(t)$ is a solution to the problem (3), we can use the Lebesgue Dominated Convergence Theorem and take the $\lim_{k\to\infty}$ in the Picard iteration $\varphi_k(t) = x_0 + \int_{t_0}^{t} v(\varphi_{k-1}(s), s)ds$.

4. OSGOOD LEMMA

Osgood lemma

Let $\gamma : \mathbb{R} \to \mathbb{R}_+ := [0, \infty)$ a locally integrable function and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, continuous non decreasing and such that $\omega(r) > 0$ if $r > 0$. Assume that a continuous non negative function ρ satisfies the inequality

$$\rho(t) \leq a + \int_{t_0}^{t} \gamma(s)\omega(\rho(s))ds,$$

for a real number $a \geq 0$, $t_0 \in \mathbb{R}$, and all $t \geq t_0$ in an interval J containing t_0. If $a > 0$ then

$$-M(\rho(t)) + M(a) \leq \int_{t_0}^{t} \gamma(s)ds, \quad t \in J \cap [t_0, \infty)$$

where $M(x) := \int_x^1 \frac{dr}{\omega(r)}, \; x > 0$. If $a = 0$ and $\int_0^1 \frac{1}{\omega(r)} dr = \infty$, then $\rho(t) = 0$ for all $t \in J \cap [t_0, \infty)$.

Proof

Letting $R(t) = a + \int_{t_0}^{t} \gamma(s)\omega(\rho(s))ds$, $t \in J \cap [t_0, \infty)$, we have $\rho(t) \le R(t)$ and $R'(t) = \gamma(t)\omega(\rho(t)) \le \gamma(t)\omega(R(t))$, a.e.
In the case $a > 0$, $R(t) > 0$, then

$$-\frac{d}{dt}M(R(t)) = \frac{1}{\omega(R(t))}R'(t) \le \gamma(t).$$

Therefore, integrating from t_0 to t, we obtain the inequality $-M(\rho(t)) + M(a) \le \int_{t_0}^{t} \gamma(s)ds$.
In the case $a = 0$, we have $\rho(t) \le a' + \int_{t_0}^{t} \gamma(s)\omega(\rho(s))ds$, for all $a' > 0$, then, by the case $a > 0$, we obtain

$$M(a') \le \int_{t_0}^{t} \gamma(s)\, ds + M(\rho(t))$$

for all $a' > 0$. Therefore, droping a' to zero, it follows that $\int_0^1 \frac{1}{\omega(r)} dr < \infty$ if $\rho(t) > 0$ for some $t \in J \cap [t_0, \infty)$. $\square$

Remarks

1. If $\omega = \text{id.}$ (i.e. $\omega(r) = r, \ \forall\, r \in \mathbb{R}_+$), Osgood lemma = Gronwall lemma.
2. M is a strictly decreasing function ($M'(x) = -1/\omega(x) < 0$). Then it has an inverse M^{-1}, with domain $(-\int_1^\infty \frac{dr}{\omega(r)}, \ \int_0^1 \frac{dr}{\omega(r)})$.
3. In the case that $M(a) - \int_{t_0}^t \gamma(s)ds \in \text{Dom.}M^{-1}$, the above inequality when $a > 0$ is equivalent to

$$\rho(t) \leq M^{-1}\left(M(a) - \int_{t_0}^t \gamma(s)ds\right).$$

Remars cont'd

4. For the function $\omega = m$ given in (8),

$$M(x) = \begin{cases} \ln(1 - \ln x), & \text{if } 0 < x \leq 1 \\ -\ln x, & \text{if } x \geq 1 \end{cases}$$

$$M^{-1}(y) = \begin{cases} e^{-y}, & \text{if } y \leq 0 \\ e^{(1-\exp(y))}, & \text{if } y \geq 0 \end{cases}.$$

Then, if $M(a) - \int_{t_0}^{t} \gamma(s)ds \geq 0$ and $a \leq 1$, i.e. $a \leq \min\{1, \mathrm{e}^{(1-\exp(\int_{t_0}^{t} \gamma(s)ds))}\}$, we have the inequality

$$\rho(t) \leq e^{1-\exp(M(a)-\int_{t_0}^{t} \gamma(s)ds)} = e^{1-(1-\ln a)\exp(-\int_{t_0}^{t} \gamma(s)ds)},$$

i.e.

$$\rho(t) \leq a^{\exp(-\int_{t_0}^{t} \gamma(s)ds)} e^{1-\exp(-\int_{t_0}^{t} \gamma(s)ds)}. \qquad (11)$$

Uniqueness of trajectories

Let ψ_1 and ψ_2 be solutions to the problem (3). Similarly to (9), we have

$$|\psi_1(t) - \psi_2(t)| \leq \int_{t_0}^{t} \langle v(\cdot, s)\rangle_{LL} m(|\psi_1(s) - \psi_2(s))|)ds,$$

then by Osgood lemma, it follows that $\psi_1 = \psi_2$.

Continuous dependence on the initial point

$$\begin{array}{rl} & |X(t;x_1,t_0)-X(t;x_2,t_0)| \\ \leq & |x_1-x_2| + \int_{t_0}^{t} \langle v(\cdot,s)\rangle_{LL} m(|X(s;x_1,t_0)-X(s;x_2,t_0)|)ds, \end{array}$$

thus, from (11), we have

$$|X(t;x_1,t_0)-X(t;x_2,t_0)| \leq |x_1-x_2|^{\exp(-\int_{t_0}^{t}\gamma(s)ds)} e^{1-\exp(-\int_{t_0}^{t}\gamma(s)ds)}, \tag{12}$$

for $0 < |x_1-x_2| \leq \min\{1, e^{1-\exp(\int_{t_0}^{t}\gamma(s)ds)}\}$, if $\gamma(s) := \langle v(\cdot,s)\rangle_{LL}$ is locally integrable, e.g. if v is uniformly log-lipschitzian, as in (7).

Theorem

The constant C in (7) can be replaced by a locally integrable function in t, i.e. a vector field v in $\Omega \times I$ has a lagrangian structure if the seminorm $\langle v(\cdot, s)\rangle_{LL(\Omega)}$ is locally integrable in I. More precisely, we have the following **theorem**:

Let $v : \Omega \times I \to \mathbb{R}^n$ be a map in $L^1_{\mathrm{loc}}(I; LL(\Omega))$. Then, for any $t_0 \in I$, there exists a unique continuous map $X(\cdot\,; \cdot, t_0) : I \times \Omega \to \Omega$ such that

$$X(t; x, t_0) = x + \int_{t_0}^{t} v(X((s; x, t_0), s)ds, \quad (t, x) \in I \times \Omega. \quad (13)$$

In addition, we have the estimate (12)*, with $\gamma(s) = \langle v(\cdot, s)\rangle_{LL}$, for any $x_1, x_2 \in \Omega$ such that $0 < |x_1 - x_2| \leq \min\{1, e^{1-\exp(\int_{t_0}^{t} \gamma(s)ds)}\}$.*

Remark

The estimate (12) means that for each $t \in I$, the map $x \in \Omega \mapsto X(t; x, t_0)$ is Hölder continuous, with exponent given by $\alpha = \exp(-\int_{t_0}^{t} \langle v(\cdot, s)\rangle_{LL} ds)$.

2 comments

- The relation between the space of lipschitzian functions, log-lipschitzian functions and Hölder continuous functions:
$$Lip \subset LL \subset C^\alpha$$
- The inclusion $LL \subset C^\alpha$ has been used to obtain better regularity than C^α in problems of free boundaries or fully nonlinear equations, cf. the papers
Leitão, Raimundo; de Queiroz, Olivaine S.; Teixeira, Eduardo. *Regularity for degenerate two-phase free boundary problems*, Ann. l'Inst. H. Poincare (C) Non Linear An., **32** (2015), no. 4, 741-762;

Teixeira, Eduardo. *Universal moduli of continuity for solutions to fully nonlinear elliptic equations*. Arch. Ration. Mech. Anal. **211** (2014), no. 3, 911-927.

An interesting remark

The Sobolev space $H^{\frac{n}{2}+1}(\mathbb{R}^n)$ is immerse in the space $LL(\mathbb{R}^n)$.

AN USEFUL EXAMPLE

Let Γ be the fundamental solution of the laplacian in $\mathbb{R}^n$, i.e.

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & \text{if } n = 2 \\ \frac{1}{(2-n)\omega_n} |x|^{2-n}, & \text{if } n \geq 3, \end{cases}$$

where ω_n is the area of the unit sphere unitária in $\mathbb{R}^n$, and g be a function in $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $1 \leq p < n$. Then it is possible to show that the vector field

$$v = \nabla\Gamma * g = \int_{\mathbb{R}^n} \nabla\Gamma(x-y) g(y) dy$$

is a log-lipschitzian vector field in $\mathbb{R}^n$ and satisfies the inequality

$$\|\nabla\Gamma * f\|_{LL(\mathbb{R}^n)} \leq C(\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)})$$

where $C = C(n,p)$.

5. APPLICATION TO COMPRESSIBLE FLUIDS

Compressible fluids (Navier-Stokes) equations:

$$\rho_t + \text{div}(\rho v) = 0$$
$$(\rho v^j)_t + \text{div}(\rho v^j v) + P(\rho)_{x_j} = \mu \Delta v^j + \lambda \text{ div } v_{x_j} + \rho f^j$$

Solution obtained by David Hoff:

- Hoff, David. *Global solutions of the Navier-Stokes equations for multidimensional, compressible flow with discontinuous initial data*. J. Diff. Eqns. **120**, no. 1 (1995), 215-254.
- Hoff, David. *Discontinuous solutions of the Navier-Stokes equations for multidimensional, heat-conducting flow*. Archive Rational Mech. Ana. **139** (1997), 303–354.
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Distinguished properties

- $P(\rho) \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n = 2, 3$
- The quantities $\omega^{j,k} = v^j_{x_k} - v^k_{x_j}$ and $F = (\mu + \lambda)\mathrm{div}\, v - P(\rho)$ are Hölder continuous.

Decomposition of the velocity field

$$\begin{aligned} \Delta v^j = v^j_{x_k x_k} = \quad & v^k_{x_k x_j} + (v^j_{x_k} - v^k_{x_j})_{x_k} \\ = \quad & \operatorname{div} v_{x_j} + \omega^{j,k}_{x_k} \\ = \quad & (\mu + \lambda)^{-1} F_{x_j} + \omega^{j,k}_{x_k} + (\mu + \lambda)^{-1} P(\rho)_{x_j} \end{aligned} \tag{14}$$

thus, we can write

$$\boxed{v = v_{F,\omega} + v_P}$$

where

$$\boxed{v_{F,\omega} = (\mu + \lambda)^{-1} \nabla \Gamma * F + \Gamma_{x_k} * \omega^{\cdot,k}}$$

$$\boxed{v_P = \nabla \Gamma * P(\rho)}.$$

Last slide

- From the "useful example", we can conclude that $v_P \in LL$.
- Since F and $\omega^{j,k}$ are Hölder continuous, we can conclude that $v_{F,\omega} \in \text{Lip}$.
- **Theorem** (Hoff- –, ARMA, 2008). $\langle v(\cdot, t) \rangle_{LL} \in L^1_{loc}$.

Thank you!!