Cálculo da integral $\frac{2x_n}{\omega_n}\int_{\mathbb{R}^{n-1}}\frac{dy}{|x-y|^n}=1$, $x=(\bar{x},x_n)\in\mathbb{R}^n_+$, $n\geq 3$
(Núcleo de Poisson no semiespaço)

$$\frac{2x_n}{\omega_n}\int_{\mathbb{R}^{n-1}}\frac{dy}{|x-y|^n}=\frac{2x_n}{\omega_n}\int_{\mathbb{R}^{n-1}}\frac{dy}{[||\bar{x}-y|^2+x_n^2]^{n/2}}=\frac{2x_n}{\omega_n}\int_{\mathbb{R}^{n-1}}\frac{dy}{(|y|^2+x_n^2)^{n/2}}=\frac{2}{\omega_n x_n^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{dy}{(|y/x_n|^2+1)^{n/2}}$$
$$=\frac{2}{\omega_n}\int_{\mathbb{R}^{n-1}}\frac{dy}{(|y|^2+1)^{n/2}}\overset{\text{C.P.}}{=}\frac{2\omega_{n-1}}{\omega_n}\int_0^\infty\frac{r^{n-2}dr}{(r^2+1)^{n/2}}\overset{\text{M.V. } r=\tan\theta}{=}\frac{2\omega_{n-1}}{\omega_n}\int_0^{\pi/2}\operatorname{sen}^{n-2}d\theta$$

Usando as fórmulas
$$\int_0^{\pi/2}\operatorname{sen}^{n-2}\theta d\theta=\frac{n-3}{n-2}\int_0^{\pi/2}\operatorname{sen}^{n-4}\theta d\theta=\frac{n-3}{n-2}\frac{n-5}{n-4}\int_0^{\pi/2}\operatorname{sen}^{n-6}\theta d\theta=\cdots$$
e
$$\frac{\omega_{n-1}}{\omega_n}=\frac{(n-1)\pi^{(n-1)/2}}{\Gamma(1+(n-1)/2)}\frac{\Gamma(1+n/2)}{n\pi^{n/2}}=\pi^{-1/2}\frac{n-1}{n}\frac{\Gamma(1+n/2)}{\Gamma(1+(n-1)/2)}=\pi^{-1/2}\frac{n-1}{n}\frac{\frac{n}{2}\Gamma(\frac{n}{2})}{\frac{n-1}{2}\Gamma(\frac{n-1}{2})}=$$
$$\pi^{-1/2}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}=\pi^{-1/2}\frac{n-2}{n-3}\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-3}{2})}=\pi^{-1/2}\frac{n-2}{n-3}\frac{n-4}{n-5}\frac{\Gamma(\frac{n-4}{2})}{\Gamma(\frac{n-5}{2})}=\cdots,$$
podemos calcular $\frac{2\omega_{n-1}}{\omega_n}\int_0^{\pi/2}\operatorname{sen}^{n-2}d\theta=1$, separando os casos n par e n ímpar, mas podemos também obter esse resultado usando que

$\omega_n=2\int_{S^{n-1}_+}$ onde $S^{n-1}_+:=\{x=(x_1,\cdots,x_n);x_n>0\}$

$=2\int_D\sqrt{1+|\nabla f|^2}dy$ onde $D:=\{y\in\mathbb{R}^{n-1};|y|<1\}$
e f é a função $f(y)=\sqrt{1-|y|^2}$
(estamos usando a fórmula $\int_D\sqrt{1+|\nabla f|^2}$ para calcular a 'área' do gráfico de uma função f sobre um domínio D)

$$=2\int_D\frac{1}{\sqrt{1-|y|^2}}dy\overset{\text{C.P.}}{=}2\omega_{n-1}\int_0^1\frac{r^{n-2}}{\sqrt{1-r^2}}dr\overset{\text{M.V. } r=\operatorname{sen}\theta}{=}2\omega_{n-1}\int_0^{\pi/2}\frac{\operatorname{sen}^{n-2}\theta}{\cos\theta}\cos\theta d\theta,$$
logo,
$$\boxed{\frac{2\omega_{n-1}}{\omega_n}\int_0^{\pi/2}\operatorname{sen}^{n-2}\theta d\theta=1}.$$