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Stationary Flow for Power-law Fluids in Domains with Unbounded Boundaries

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Overview

- Description of the problem and statement of the main theorem
- History and some related results
- Some idea on the proof

Power law model

(NS)

$$\begin{cases} -\operatorname{div}(|D(v)|^{p-2}D(v)) + v\nabla v + \nabla\mathcal{P} = 0 & \text{Navier-Stokes system} \\ \operatorname{div} v = 0 & \text{incompress. equation} \end{cases}$$

$\mathcal{P}$: pressure

v: velocity, $D(v) = \nabla v + (\nabla v)^t$

$v\nabla v$: convective term

$(v\nabla v = \sum_{j=1}^{n} v_j \frac{\partial v}{\partial x_j})$

The *viscous stress tensor*, $\mathbb{S}$, is given by

$$|D(v)|^{p-2}D(v).$$

or, $\quad$ *viscosity* $= |D(v)|^{p-2}$

power law or *Ostwald-de Waele law/model*

See e.g. R. Bird, W. Stewart and E. Lightfoof, *Transport Phenomena*, Johh Wiley & Sons, Inc. (2007).

In the classical book by O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, 2nd ed. (1969), after the last chapter, there is a description of some models including power laws.

Transport Phenomena (book)

"Transport Phenomena is the first textbook that is about transport phenomena. It is specifically designed for chemical engineering students. The first edition was published in 1960, two years after having been preliminarily published under the title Notes on Transport Phenomena based on mimeographed notes prepared for a chemical engineering course taught at the University of Wisconsin-Madison during the academic year 1957-1958. The second edition was published in August 2001. A revised second edition was published in 2007. This text is often known simply as BSL after its authors' initials." **Wikipedia**

$$|D(v)|: \text{ shear rate}$$

$p = 2$: Newtonian fluids (e.g. water, oil)

$p < 2$: *shear-thinning* (or plastic and pseudo-plastic, e.g. most polymer melts and solutions)
- the viscosity is decreasing with respect the shear rate (viscosity $= \infty$ when shear rate $= 0$)

$p > 2$: *shear-thickening* (or dilatant, e.g. mud, clay, cement)
- the viscosity is increasing

See e.g. E. Marusic-Paloka, *Steady Flow of a Non-Newtonian Fluid in Unbounded Channels and Pipes*, Mathematical Models and Methods in Applied Sciences, **10**(9) (2000).

We consider $p \geq 2$.

Parallel fluids

The velocity field is of the form

$$\vec{v}(x) \equiv v(\bar{x})\vec{e}$$

where $\vec{e}$ is a constant vector, $x = (\bar{x}, x') \equiv \bar{x} \oplus x'\vec{e}$ and $v(\bar{x})$ is a scalar function.

In this case, the convection term $v\nabla v$ vanishes, and the Navier-Stokes equations become the *p-Laplacian* equation

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) = c$$

for some constant c, related to "*pressure drop*", i.e. $\nabla\mathcal{P} = -c\vec{e}$.

The domain, Ω

$$\Omega = \bigcup_{i=0}^{2} \Omega_i$$

is an open connect set set in $\mathbb{R}^n$, $n = 2, 3$, with a C^∞ boundary, where Ω_0 is a bounded subset of $\mathbb{R}^n$ and, in possibly different cartesian coordinate system,

$$\Omega_1 = \{x = (\bar{x}, x') \in \mathbb{R}^n;\ x' < 0,\ \bar{x} \in \Sigma_1(x')\}$$

and

$$\Omega_2 = \{x = (\bar{x}, x') \in \mathbb{R}^n;\ x' > 0,\ \bar{x} \in \Sigma_2(x')\},$$

with $\Sigma_i(x')$, $i = 1, 2$, the cross sections, being C^∞ simply connected domains in $\mathbb{R}^{n-1}$ such that

$$\sup_{x', i=1,2} \operatorname{diam} \Sigma_i(x') < \infty$$

and Ω_i, $i = 1, 2$, contains some cylinder

$$C_l^i = \{x \in \mathbb{R}^n;\ (-1)^i x' > 0 \text{ e } |\bar{x}| < l\}, \quad (l > 0)$$

(in particular, $\inf_{x', i=1,2} \operatorname{diam} \Sigma_i(x') > 0$).

We will denote by $\mathbf{n}$ the ortonormal vector to $\Sigma(x')$, or to any cross section of Ω, pointing from Ω_1 toward Ω_2.

“*Ladyzhenskaya-Solonnikov problem*”

Given any $\Phi \in \mathbb{R}$, find a solution $(v, \mathcal{P})$ of (NS) such that

$$v = 0 \quad \text{on } \partial\Omega\,,$$
$$\textit{the flux} \equiv \int_{\Sigma(x')} v \cdot \mathbf{n} = \Phi$$

and

$$\sup_{t>0} t^{-1} \int_{\Omega^t} |\nabla v|^p < \infty\,,$$

where $\Omega^t := \Omega_0 \cup \Omega_1^t \cup \Omega_2^t$, $\Omega_i^t := \{(\bar{x}, x') \in \Omega_i\,;\, 0 < (-1)^i x' < t\}$, $i = 1, 2$.

Cf. “*Problem 1.1*” in
[LS] O.A. Ladyzhenskaya and V.A. Solonnikov, *Determination of the Solutions of Boundary Value Problems for Steady-State Stokes and Navier-Stokes Equations in Domains Having an Unbounded Dirichlet Integral* (1980). English transl. in J. Soviet Math. **21** (1983).

Theorem

(–, Gilberlandio Dias, J.D.E. 2012)

Let $p \geq 2$ e $n = 2, 3$. Then the Ladyzhesnkaya-Solonnikov problem for power-law fluids has a weak solution $(v, \mathcal{P})$ in $W^{1,p}_{loc}(\Omega) \times L^{p'}_{loc}(\Omega)$, $p' = p/(p-1)$, i.e.

$$\begin{cases} \displaystyle\int_\Omega |D(v)|^{p-2} D(v) : \nabla\psi = -\int_\Omega (v\nabla v)\cdot\psi + \int_\Omega \mathcal{P}\mathrm{div}\psi\,, \ \forall\, \psi \in C_c^\infty(\Omega;\mathbb{R}^n) \\ \displaystyle\int_\Omega v\cdot\nabla\psi = 0\,, \qquad \forall\, \psi\in C_c^\infty(\Omega;\mathbb{R}) \end{cases}$$

(*(NS) equations are satisfied in the sense of distributions*),

$$v = 0 \quad \text{on } \partial\Omega\,,$$
$$\int_\Sigma v\cdot\mathbf{n} = \Phi$$

(*with* $v\big|\partial\Omega$ *and* $v\big|\Sigma$ *in the sense of the trace*)

and

$$\sup_{t>0} t^{-1}\int_{\Omega^t} |\nabla v|^p < \infty\,.$$

Remark: The case $p = 2$ (newtonian fluid) is due to Ladyzhenskaya and Solonnikov [LS].

History and Related Results

Leray Problem

Let $\vec{v}_P(x) = v_P(\bar{x})\vec{e}$ be the parallel velocity field of a Newtonian fluid $(p = 2)$ in a straight (unbounded) cylinder

$$C = \{(\bar{x}, x') \equiv \bar{x} \oplus x'\vec{e}\,;\;\; \bar{x} \in \Sigma,\;\; 0 < |x'| < \infty\},$$

(Σ here is independent of x') such that $v_P\big|\partial\Sigma = 0$, i.e.

$$\begin{cases} -\Delta v_P = c & \text{in } \Sigma \\ \quad\;\; v_P = 0 & \text{on } \partial\Sigma\,. \end{cases}$$

The vector field $\vec{v}_P$ is called *Poiseuille flow*.

The constant c can be determined by the flux $\Phi = \int_\Sigma v_P$:

$$\Phi = c\int_\Sigma |\nabla v_1|^2$$

where v_1 is the solution corresponding to $c = 1$.

As a consequence, we also have that

$\int_\Sigma |\nabla v_P|^2 = c^2\int_\Sigma |\nabla v_1|^2 = \left(\Phi / \int_\Sigma |\nabla v_1|^2\right)^2 \int_\Sigma |\nabla v_1|^2 = \Phi^2 / \int_\Sigma |\nabla v_1|^2,$

i.e.

$$\int_\Sigma |\nabla v_P|^2 = \text{const.}\Phi^2$$

Leray problem

Suppose that Ω_i, $i = 1, 2$, are straight cylinders and let v_P^i be the Poiseuille flow in Ω_i. Then, *Leray problem* is the following:

Find a solution $(v, \mathcal{P})$ *of* (NS) *such that* $v\big|\partial\Omega = 0$ *and*

$$v \to v_P^i \quad \text{as} \quad |x'| \to \infty \quad \text{in} \quad \Omega_i.$$

Amick's theorem, 1977

Leray problem for Newtonian fluids ($p = 2$) has a solution if the flux $\int_\Sigma v_P^i \cdot \mathbf{n}$ is sufficiently small.

C.J. Amick, *Steady solutions of the Navier-Stokes equations in unbounded channels and pipes*, Ann. Scuola Norm. Sup. Pisa Cl.Sci., **4**(3) (1977).

In this paper Amick wrote
"This problem was proposed (I believe) by Leray to Ladyzhenskaya, who in [7] attempted an existence proof under no restrictions on the [constant] viscosity ν. The problem is also mentioned by Finn in a review paper ([3], p. 150)."

[7] O. A. Ladyzhenskaya, *Stationary motion of a viscous incompressible fluid in a pipe*, Dokl. Akad. Nauk. SSSR, **124** (1959).
[3] R. Finn, *Stationary solutions of the Navier-Stokes equations*, Amer. Math. Soc., Proc. Symposia Appl. Math., **17** (1965).

Remark: For arbitrary flux, the solution of Leray problen is an open question.

Ladyzhenskaya-Solonnikov's theorem, 1980

For arbitrary flux, "Ladyzhenskaya-Solonnikov problem" has a solution, in the case of a newtonian fluid ($p = 2$).

Some related results

Many authors have studied steady flows for newtonian fluids in domains with unbounded boundaries, including unbounded cross sections, e.g. K. Pileckas, Nazarov, Kapitanskii, ...

See e.g.
G.P. Galdi, *An Introduction to the Mathematical Theory o f the Navier-Stokes Equations*, Springer-Verlag (1994).

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Some others:
—, F. Ammar-Khodja: Leray and Ladyzhenskaya-Solonnikov problems for Newtonian fluids in 2D with non-constant density;
Methods Appl. Anal. **13** (2006)
Progr. Nonlinear Differential Equations Appl. **66** (2006).

Fábio V. Silva: micropolar fluids;
J. Math. Anal. Appl. **306**(2) (2005)
Nonlinear Anal. **64**(4) (2006)

Results for non newtonian fluids

Several results for bounded domains - Boundary Value Problem for (NS), e.g.
J.L. Lions, *Quelques Méthods de Resolution des Problémes Aux Limites Non Linéaires*, Dunod, Gauthier-Villars (1969), Ch. 2, Remark 5.5:

$$p \geq 3n/(n+2).$$

W. Sadowski, *On the Stationary Flow of the Power Law Fluid in 2D*, J. Appl. Analysis, **8**, 2002: $1 < p < 2$.

<u>In unbounded domains</u> there are few results, e.g.

E. Marusic-Paloka, 2000: Leray problem, $p > 2$.

Proofs

Amick's solution of Leray's problem for newtonian fluids, with small flux:

$$v = u + a; \quad u \in H^1_0(\Omega), \quad \mathrm{div} u = 0,$$
$$a \in H^1_{\mathrm{loc}}(\Omega),\ \mathrm{div} a = 0,\ \boxed{a\big|\Omega_i = v^i_P},\ a\big|\partial\Omega = 0.$$

Notice that the Poiseuille flows v^i_P are not in $H^1(\Omega)$ (they are constant with respect to x'); $v^i_P \in H^1_{\mathrm{loc}}(\Omega)$.

A divergence free vector field u in $H^1_0(\Omega) = \overline{C^\infty_c(\Omega)}^{H^1(\Omega)}$ carries no flux, i.e. $\int_\Sigma u \cdot \mathbf{n} = 0$, for any cross section Σ of Ω. Indeed, if $\psi \in C^\infty_c(\Omega)$ then $\int_{\Sigma(x')} \psi \cdot \mathbf{n} = 0$ for all sufficiently large $|x'|$.

NS-equations become

$$-\Delta u + u\nabla u + \mathrm{l}(u) + \nabla\mathcal{P} = 0,$$

where

$$\mathrm{l}(u) = a\nabla u + u\nabla a + a\nabla a - \Delta a\,.$$

Method: compactness method, with Galerkin approximations.

Estimate of the nonlinear term $\int (a\nabla u)u$ by $\int |\nabla u|^2$ (a priori estimate):

$$\int |(a\nabla u)u| \leq \left(\int |\nabla u|^2\right)^{1/2} \left(\int |a|^2\,|u|^2\right)^{1/2}$$

$$\begin{aligned}
\int_{\Omega_i} |a|^2\,|u|^2 = &\quad \int_{\Omega_i} |u|^2\,|v_P^i|^2 \\
= &\quad |\int_0^{\pm\infty} \int_\Sigma |v_P^i|^2\,|u|^2| \\
\leq &\quad |\int_0^{\pm\infty} \left(\int_\Sigma |v_P^i|^4\right)^{1/2} \left(\int_\Sigma |u|^4\right)^{1/2}| \\
= &\quad |\int_0^{\pm\infty} \|v_P^i\|^2_{L^4(\Sigma)}\,\|u\|^2_{L^4(\Sigma)}| \\
\leq &\quad c\,|\int_0^{\pm\infty} \|\nabla v_P^i\|^2_{L^2(\Sigma)}\,\|\nabla u\|^2_{L^2(\Sigma)}| \\
= &\quad c\,\|\nabla v_P^i\|^2_{L^2(\Sigma)}|\int_0^{\pm\infty} \|\nabla u\|^2_{L^2(\Sigma)}| \\
= &\quad c\,\Phi^2 \int_{\Omega_i} |\nabla u|^2
\end{aligned}$$

Similarly, we can estimate $\int (u\nabla a)u$.

The terms $\int_{\Omega_i}(-\Delta a)u$ and $\int_{\Omega_i}(a\nabla a)u$ vanish, since $a\nabla a = 0$, because $a = v_P^i$ in Ω_i is "parallel", and $-\Delta a = (-\Delta v_P^i)\mathbf{n} = c\mathbf{n}$ in Ω_i, so

$$\int_{\Omega_i}(-\Delta a)u = |c\int_0^{\pm\infty}\int_\Sigma u\cdot\mathbf{n}| = 0.$$

□

Ladyzhenskaya-Solonnikov's solution, for newtonian fluids with arbitrary flux

$$v = u + a; \quad u \in H^1_{loc}(\Omega), \; \mathrm{div} u = 0, \; u\big|\partial\Omega = 0$$

and a is given by the following lemma:

Lemma [LS]. *For any $\delta > 0$ there exists a vector field a such that*

a$_1$) $a \in H^1_{loc}(\overline{\Omega})$, div$a = 0$, $a|\partial\Omega = 0$,

a$_2$) $\int_\Sigma a \cdot \mathbf{n} = 1$ *for any cross section Σ of Ω,*

a$_3$) $\int_{\Omega_i^{t-1,t}} |\nabla a|^2 \le c$ *for $i = 1, 2$ and all $t \ge 1$, where* $\Omega_i^{t-1,t} = \{(\bar{x}, x') \in \Omega_i\,;\; t-1 < |x'| < t\}$,

and

a$_4$) $\int_{\Omega^t} |a|^2|u|^2 \le c\delta \int_{\Omega^t} |\nabla u|^2$ for all $t > 0$ and $u \in C^\infty_c(\Omega)$,

where, in a$_3$) and a$_4$), c is a constant depending only on Ω.

Remark: Given any $\Phi \in \mathbb{R}$, multiplying a by Φ, we obtain a vector field having flux Φ.

Now $a\big|\Omega_i$ might not be the Poiseuille v^i_P, but the compactness method still works, by truncating the domain and long computations:

Let u^t be a solution of the NS-equations

$$-\Delta u^t + u^t \nabla u^t + \mathrm{l}(u^t) + \nabla \mathcal{P}^t = 0$$

in $H_0^1(\Omega^t)$ (joint with some pressure function $\mathcal{P}^t \in L^2_{\mathrm{loc}}(\Omega^t)$).

Now, let $\boxed{t' > t}$. Multiplying the equation $-\Delta u^{t'} + u^{t'} \nabla u^{t'} + \mathrm{l}(u^{t'}) + \nabla \mathcal{P}^{t'} = 0$ by $u^{t'}$ and integrating by parts in Ω^t, we obtain

$$\int_{\Omega^t} |\nabla u^{t'}|^2 \leq ct + \int_{\Sigma(t)} (\text{bound. terms}),$$

for all $t < t'$. Integrating in t, from $\eta - 1$ to $\eta \leq t'$, we get

$$z(\eta) := \int_{\eta-1}^{\eta} \left(\int_{\Omega^t} |\nabla u^{t'}|^2 \right) dt \leq c\eta - \frac{1}{2} + \int_{\Omega^{\eta-1,\eta}} (\text{bound. terms})\,.$$

Using the equation, is possible to estimate $\int_{\Omega^{\eta-1,\eta}} (\text{bound. terms})$ by a linear combinations of powers of $\int_{\Omega^{\eta-1,\eta}} |\nabla u^{t'}|^2$. But

$$\int_{\Omega^{\eta-1,\eta}} |\nabla u^{t'}|^2 = z'(\eta)\,!$$

Thus,

$$z(\eta) := \int_{\eta-1}^{\eta} \left(\int_{\Omega^t} |\nabla u^{t'}|^2 \right) dt \leq c\eta + g(z'(\eta)), \ \ \forall\ \eta \leq t',$$

for some function $g : \mathbb{R} \to \mathbb{R}$. Besides,

$$z(t') \leq \int_{\Omega^{t'}} |\nabla u^{t'}|^2 \leq ct'.$$

Then, by a kind of "*reverse Gronwall lemma*" [LS], we have

$$z(\eta) \leq c\eta,$$

which implies

$$\int_{\Omega^{\eta-1}} |\nabla u^{t'}|^2 \leq c\eta, \ \ \forall\ \eta \leq t'.$$

So, fixing t (arbitrary), $\{u^{t'}\}_{t'>t}$ is bounded in $H^1(\Omega^t)$, by $c(t+1)$.

Construction of a

Consider $n = 3$. In Ω_i, the field a is given by

$$a = \frac{1}{2\pi}\nabla \times (\zeta b) = \frac{1}{2\pi}\nabla\zeta \times b$$

where

$$b(x) = \left(-\frac{x_2}{|\bar{x}|^2}, \frac{x_1}{|\bar{x}|^2}, 0\right), \quad \bar{x} = (x_1, x_2),$$

and ζ is the "truncating E. Hopf's function":

$$\zeta(x) = \psi\left(\varepsilon \log \frac{\sigma(|\bar{x}|)}{\rho(x)}\right);$$

$\rho(x)$:	the regularized distance to $\partial\Omega$
$\sigma, \psi : \mathbb{R} \to \mathbb{R}$:	smooth nondecreasing functions,

$$\sigma(s) = \begin{cases} \frac{l}{4}, & s \le \frac{l}{4} \\ t, & s > \frac{l}{2} \end{cases}$$

$$\psi(s) = \begin{cases} 0, & s \le 0 \\ 1, & s > 1 \end{cases}$$

$\varepsilon = \varepsilon(\delta)$.

Non newtonian fluids, $p > 2$

The above vector field a satisfies

Lemma.

a$_1$) $a \in W^{1,p}_{loc}(\overline{\Omega})$, $\operatorname{div} a = 0$, $a|\partial\Omega = 0$,

a$_2$) $\int_\Sigma a \cdot \mathbf{n} = 1$,

a$_3$) $\int_{\Omega_i^{t-1,t}} |\nabla a|^p \leq c$

and

a$_4$) $\int_{\Omega^t} |a|^{p'} |u|^{p'} \leq c\delta t^{(p-2)/(p-1)} \left(\int_{\Omega^t} |\nabla u|^p\right)^{1/(p-1)}$.

Estimate of the nonlinear terms

We want to estimate all the nonlinear terms by $\int |\nabla u|^p$.

Now we have two main nonlinear terms:

$$\int (a\nabla u)u \quad \text{and} \quad \int |D(v)|^{p-2}D(v) : D(u), \quad v = u + a\,.$$

Known inequalities:

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c|x-y|^p, \ \forall\ x, y \in \mathbb{R}^n, \quad (p > 2)$$

$$\int |\nabla u|^p \leq c \int |D(u)|^p \qquad \text{(Korn's inequality}^1\text{)}.$$

The argument in the truncated (bounded) domain Ω^t:

Taking $x = D(v^{t'}) = D(u^{t'}) + D(a)$, $t' > t$, and $y = D(a)$ ($\Rightarrow\ x - y = D(u^{t'})$) in the first inequality and using Korn's inequality, we get – writing $u = u^{t'}, v = v^{t'}$,

$$\int_{\Omega^t} |D(v)|^{p-2}D(v) : D(u) \geq c\int_{\Omega_t} |\nabla u|^p + \int_{\Omega_t} |D(a)|^{p-2}D(a) : D(u)\,.$$

[1] Patrizio Neff, Proc. Royal Soc. Edinb. **132A** (2002);
V.A. Kondrat'ev and O.A. Oleinik, Russian Math. Surveys **43** (5) (1988)

By Young inequality and a_3),

$$
\begin{aligned}
\left| \int_{\Omega_t} |D(a)|^{p-2} D(a) : D(u) \right| &\leq \int_{\Omega_t} |D(a)|^{p-1} |D(u)| \\
&\leq \int_{\Omega_t} (\epsilon |D(u)|^p + c_\epsilon |D(a)|^p) \\
&\leq \epsilon \int_{\Omega_t} |\nabla u|^p + c_\epsilon ct\,.
\end{aligned}
$$

Regarding the term $\int (a\nabla u)u$, by Hölder inequality, a_4) and Young inequality, we have

$$
\begin{aligned}
& |\int_{\Omega_t} (a\nabla u)u\,| \leq \left(\int_{\Omega_t} |\nabla u|^p\right)^{1/p} \left(\int_{\Omega_t} |a|^{p'} |u|^{p'}\right)^{1/p'} \\
\leq\ & \left(\int_{\Omega_t} |\nabla u|^p\right)^{1/p} \left(c\delta\, t^{(p-2)/(p-1)} \left(\int_{\Omega_t} |\nabla u|^p\right)^{1/(p-1)}\right)^{1/p'} \\
=\ & \left(\int_{\Omega_t} |\nabla u|^p\right)^{2/p} (c\delta)^{1/p'}\, t^{(p-2)/p} \\
\leq\ & \epsilon \int_{\Omega_t} |\nabla u|^p + ct\,.
\end{aligned}
$$

We do not need δ small!

To pass to the limit from approximate solutions, the compactness method is not enough due to the nonlinear term

$$A(u) := -\mathrm{div}\left(|D(u) + D(a)|^{p-2}(D(u) + D(a))\right) .$$

But the inequality

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c|x - y|^p$$

implies that the operator $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))'$ is monotone and the method of Browder and Minty enables us to pass to the limit. (See e.g. § 9.1 of L.C. Evans, *Partial Differential Equations*.)

Some important features in the case of non newtonian fluids i.e. $p > 2$

• The construction of the vector field a can be simplified.
It is enough that a be a bounded vector field of divergence zero and vanishing on $\partial\Omega$!

• Extra non linear term

$$|D(v)|^{p-2}D(v)$$

Monotonicity, **Browder-Minty method**

• Inequalities:

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c|x-y|^p, \ \forall\ x, y \in \mathbb{R}^n, \quad (p > 2)$$

$$\int_{\Omega^t} |\nabla u^{t'}|^p \leq c \int_{\Omega^t} |D(u^{t'})|^p$$

Korn inequality, with $u^{t'}$ vanishing only on a part of $\partial\Omega^t$:

- Patrizio Neff, Proc. Royal Soc. Edinb. A **132** (2002).

Some important features in the case of non newtonian fluids, continued

• There is not regularity for the generalized solution of the system $(NS)_p$
.

To get regularity we needed to modify $|D(\boldsymbol{v})|^{p-2} D(\boldsymbol{v})$ to

$$(\varepsilon + |D(\boldsymbol{v})|)^{p-2} D(\boldsymbol{v}), \quad \varepsilon > 0$$

and adapt the proof of

- Beirão da Veiga, Kaplický and Růžička, *Boundary regularity of shear thickening flows*. J. Math. Fluid Mech. (2011). Abridged version: C. R. Math. Acad. Sci. Paris (2010).
- 2D: Kaplický, Málek and Stará $C^{1,\alpha}$*-solutions to a class of nonlinear fluids in two dimensions — stationary Dirichlet problem*, J. Math. Sci. (2002).