Seminário de Equações Diferenciais
IMECC-UNICAMP

Steady Flow for Incompressible Fluids in Domains with Unbounded Curved Channels

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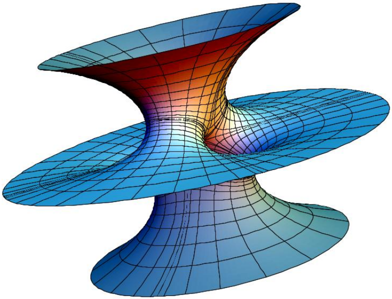
Research Interests
Differential geometry, geometric analysis.

Selected Publications
F. Xavier (with L. P.Jorge), *A complete minimal surface between two parallel planes*, Annals of Mathematics, **112** (1980), 203–206.
F. Xavier, *The Gauss map of complete minimal surface cannot omit 7 points of the sphere*, Annals of Mathematics, **113** (1981) (erratum on v.115, 211—214).

⋮

Costa's surface



3D-XplorMath-J

From Wikipedia:
"In mathematics, Costa's minimal surface is an embedded minimal surface discovered in 1982 by the Brazilian mathematician Celso José da Costa. It is also a surface of finite topology, which means that it can be formed by puncturing a compact surface. Topologically, it is a thrice-punctured torus. Until its discovery, the plane, helicoid and the catenoid were believed to be the

Costa's surface, continued

only embedded minimal surfaces that could be formed by puncturing a compact surface. The Costa surface evolves from a torus, which is deformed until the planar end becomes catenoidal. Defining these surfaces on rectangular tori of arbitrary dimensions yields the Costa surface. Its discovery triggered research and discovery into several new surfaces and open conjectures in topology.
The Costa surface can be described using the Weierstrass zeta and the Weierstrass elliptic functions.

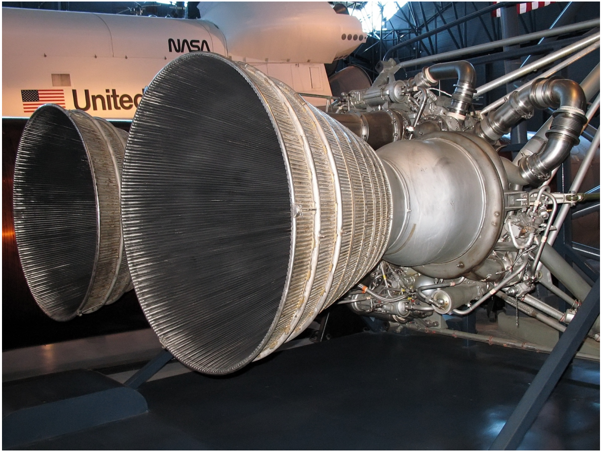
References:

Costa, Celso José da (1982). Imersões mínimas completas em $\mathbb{R}^3$ de gênero um e curvatura total finita. Ph.D. Thesis, IMPA, Rio de Janeiro, Brazil.
Costa, Celso José da (1984). Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends. Bol. Soc. Bras. Mat. 15, 47–54.
Weisstein, Eric W. "Costa Minimal Surface."
Retrieved 2006-11-19. From MathWorld–A Wolfram Web Resource."

WolframMathWorld – Costa Minimal Surface
Imagens de "Costa surface": Google

Costa, C. J. *Classification of complete minimal surfaces in R3 with total curvature 12π*. Invent. Math. **105** (1991), no. 2, 273–303.

Rocket fuel chambers



Titan I XLR-87 Rocket Engine
Imagens Google
Wikipedia

Domains with unbounded channels

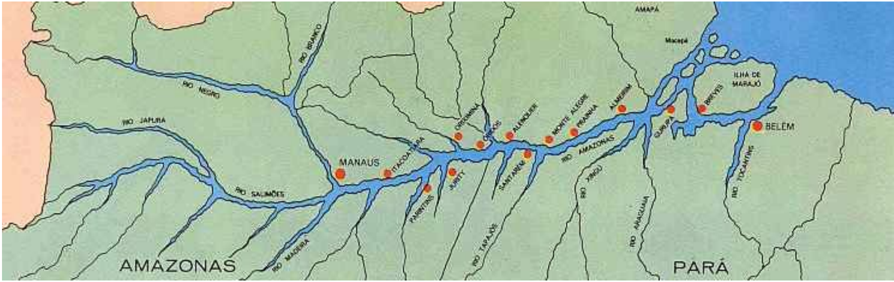
Water distribution

Rhama

Sewage

Domains with unbounded channels, continued

Rivers and lakes



Overview

- Poiseuille flow
- Leray problem and Amick's solution; domains with straight cylindrical ends (channels)
- Ladyzhenskaya-Solonnikov problem; domains with ends containing straight cylinders
- Power law fluids
- Domains with curved ends

The stationary Navier-Stokes equations for incompressible newtonian fluids

$$(NS) \qquad \begin{cases} -\Delta v + v\nabla v + \nabla \mathcal{P} = 0 & \text{Navier-Stokes system} \\ \text{div } v = 0 & \text{incompressibility equation} \end{cases}$$

$\mathcal{P}$: pressure

$v = (v_1, v_2, v_3)$: velocity,

$v\nabla v$: convective term

$(v\nabla v = \sum_{j=1}^{3} v_j \partial_j v)$

Poiseuille flow

Let C be a straight (unbounded) cylinder in $\mathbb{R}^3$, i.e.

$$C = \Sigma \times \mathbb{R}$$

where Σ is a bounded C^∞ simply connected open set in $\mathbb{R}^2$.
We denote by $\mathbf{n}$ the orthonormal vector to Σ pointing toward $+\infty$. Then we can write

$$C = \{(\mathbf{x}, z) \equiv \mathbf{x} + z\mathbf{n}\,; \;\; \mathbf{x} \in \Sigma, \;\; -\infty < z < \infty\},$$

The *Poiseuille flow* in C is the "parallel flow" with (velocity, pressure) $\equiv (\mathbf{v}_P, \mathcal{P})$ solving (NS) with the boundary condition $\mathbf{v}_P|\partial C = 0$, i.e. is the flow of a fluid in C having velocity $\mathbf{v}_P$ of the form

$$\mathbf{v}_P = v(\mathbf{x})\mathbf{n}$$

for some scalar function $v(\mathbf{x})$ such that $v|\partial\Sigma = 0$ and $(v, \mathcal{P})$, for some scalar function (pressure) $\mathcal{P}$, solves the system (NS), i.e.

$$(P) \begin{cases} -\Delta v = c & \text{in } \Sigma \\ v = 0 & \text{on } \partial\Sigma\,. \end{cases}$$

for some constant c (the "pressure drop"; $\nabla\mathcal{P} = -c\mathbf{n}$).

Poiseuille flow cont'ed

Remark. The constant c can be determined by the *flux* $\Phi := \int_\Sigma \mathbf{v}_P \cdot \mathbf{n} = \int_\Sigma v$. Indeed,

$$\Phi = c \int_\Sigma |\nabla v_1|^2$$

where v_1 is the solution of (P) when $c = 1$.

As a consequence, we also have that

$$\int_\Sigma |\nabla v|^2 = c^2 \int_\Sigma |\nabla v_1|^2 = \left(\Phi / \int_\Sigma |\nabla v_1|^2\right)^2 \int_\Sigma |\nabla v_1|^2 = \Phi^2 / \int_\Sigma |\nabla v_1|^2,$$

i.e.

$$\int_\Sigma |\nabla v|^2 = \text{const.}\Phi^2$$

Leray problem

Domain with straight cylindrical ends
Let Ω be an open connect set in $\mathbb{R}^3$ with a C^∞ boundary such that

$$\Omega = \Omega_0 \cup C_1 \cup C_2$$

where Ω_0 is a bounded subset of $\mathbb{R}^3$ and C_i, $i = 1, 2$, is a straight cylinder, i.e., in different cartesian coordinate systems,

$$C_i = \Sigma_i \times [0, \infty)$$

where Σ_i is a bounded C^∞ simply connected open set in $\mathbb{R}^2$.
<u>Notation</u>. Σ will denote any cross section of Ω and $\mathbf{n}$ will denote the orthonormal vector field on Σ pointing from C_1 toward C_2.

Leray problem
Let v_P^i be the Poiseuille flow in C_i. Then, the *Leray problem* is the following:

Find a solution $(v, \mathcal{P})$ *of* (NS) *such that* $v\big|\partial\Omega = 0$ *and*

$$v \to v_P^i \quad \text{as} \quad |\mathbf{x}| \to \infty \quad \text{in} \quad C_i.$$

Amick's theorem, 1977

The Leray problem (for newtonian fluids, i.e. (NS)) has a solution if the flux $\int_\Sigma v_P^i \cdot \mathbf{n}$ *is sufficiently small.*

C.J. Amick, *Steady solutions of the Navier-Stokes equations in unbounded channels and pipes*, Ann. Scuola Norm. Sup. Pisa Cl.Sci., **4**(3) (1977).

In this paper Amick wrote
"*This problem* [Leray problem] *was proposed (I believe) by Leray to Ladyzhenskaya, who in [7] attempted an existence proof under no restrictions on the (constant) viscosity* ν. *The problem is also mentioned by Finn in a review paper ([3], p. 150).*"

[7] O. A. Ladyzhenskaya, *Stationary motion of a viscous incompressible fluid in a pipe*, Dokl. Akad. Nauk. SSSR, **124** (1959).
[3] R. Finn, *Stationary solutions of the Navier-Stokes equations*, Amer. Math. Soc., Proc. Symposia Appl. Math., **17** (1965).

Remark. For arbitrary flux, the solution of Leray problen is an open question.

"*Ladyzhenskaya-Solonnikov problem*"

Domain with ends containing straight cylinders

Let Ω be an open connect set in $\mathbb{R}^3$ with a C^∞ boundary such that

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$$

where Ω_0 is a bounded subset of $\mathbb{R}^3$ and, in different cartesian coordinate system,

$$\Omega_1 = \{(\mathbf{x}, z) \in \mathbb{R}^3;\ z < 0,\ \mathbf{x} \in \Sigma_1(z)\}$$

and

$$\Omega_2 = \{(\mathbf{x}, z) \in \mathbb{R}^3;\ z > 0,\ \mathbf{x} \in \Sigma_2(z)\},$$

with $\Sigma_i(z)$, $i = 1, 2$, the cross sections, being a bounded C^∞ simply connected open sets in $\mathbb{R}^2$ such that

$$\sup_{z,i=1,2} \operatorname{diam} \Sigma_i(z) < \infty$$

and Ω_i, $i = 1, 2$, contains some straight cylinder

$$C_l^i = \{x = (\mathbf{x}, z) \in \mathbb{R}^3;\ (-1)^i z > 0 \text{ e } |\mathbf{x}| < l\}, \quad (l > 0)$$

(in particular, $\inf_{z,i=1,2} \operatorname{diam} \Sigma_i(z) > 0$).

We will denote by $\mathbf{n}$ the orthonormal vector to $\Sigma_i(z)$, or to any cross section Σ of Ω, pointing from Ω_1 toward Ω_2.

Ladyzhenskaya-Solonnikov problem

Given $\Phi \in \mathbb{R}$, find a solution $(v, \mathcal{P})$ of (NS) such that

$$v = 0 \quad \text{on } \partial\Omega\,,$$
$$\textit{the flux} \equiv \int_\Sigma v \cdot \mathbf{n} = \Phi,$$

and

$$\sup_{t>0} t^{-1} \int_{\Omega^t} |\nabla v|^2 < \infty\,,$$

where $\Omega^t := \Omega_0 \cup \Omega_1^t \cup \Omega_2^t, \quad \Omega_i^t := \{(\mathbf{x}, z) \in \Omega_i\,;\, 0 < (-1)^i z < t\}$, $i = 1, 2$.

This is *Problem 1.1* in

[LS] O.A. Ladyzhenskaya and V.A. Solonnikov, *Determination of the Solutions of Boundary Value Problems for Steady-State Stokes and Navier-Stokes Equations in Domains Having an Unbounded Dirichlet Integral* (1980). English transl. in J. Soviet Math. **21** (1983).

Ladyzhenskaya-Solonnikov's theorem, 1980

The problem above has a solution for any flux Φ.

This is Theorem 5.1 in **[LS]**.

Some related results

Many authors have studied steady flows for newtonian incompressible fluids in domains with unbounded boundaries, including unbounded cross sections, e.g.

K. Pileckas, Nazarov, Kapitanskii, ...

See e.g.
G.P. Galdi, *An Introduction to the Mathematical Theory o f the Navier-Stokes Equations*, Springer-Verlag (1994).

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Others:

—, F. Ammar-Khodja: Leray and Ladyzhenskaya-Solonnikov problems for Newtonian fluids in 2D with non-constant density;
Methods Appl. Anal. **13** (2006)
Progr. Nonlinear Differential Equations Appl. **66** (2006).

Fábio V. Silva: micropolar fluids;
J. Math. Anal. Appl. **306**(2) (2005)
Nonlinear Anal. **64**(4) (2006)

Power law fluid (model)

$$(NS)_p \qquad \begin{cases} -\operatorname{div}(|D(v)|^{p-2}D(v)) + v\nabla v + \nabla\mathcal{P} = 0 \\ \operatorname{div} v = 0 \end{cases}$$

$$D(v) = \frac{1}{2}(\nabla v + (\nabla v)^t)$$

i.e. the *viscous stress tensor*, $\mathbb{S}$, is given by

$$\mathbb{S} = |D(v)|^{p-2}D(v).$$

or,

$$viscosity = |D(v)|^{p-2}$$

power law or *Ostwald-de Waele law/model*

See e.g. R. Bird, W. Stewart and E. Lightfoof, *Transport Phenomena*, Johh Wiley & Sons, Inc. (2007).
In the classical book by O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, 2nd ed. (1969), after the last chapter, there is a description of some models including power laws.

Transport Phenomena (book)

Wikipedia, https://en.wikipedia.org/wiki/Transport_Phenomena_(book):

"Transport Phenomena is the first textbook about transport phenomena. It is specifically designed for chemical engineering students. The first edition was published in 1960, two years after having been preliminarily published under the title Notes on Transport Phenomena based on mimeographed notes prepared for a chemical engineering course taught at the University of Wisconsin-Madison during the academic year 1957-1958. The second edition was published in August 2001. A revised second edition was published in 2007. This text is often known simply as BSL after its authors' initials."

Paper on this book:

Thirty-Five Years of BSL, Gianni Astarita, Julio Ottino. Ind. Eng. Chem. Res., 1995, **34** (10), pp 3177–3184

Abstract: *Few engineering books remain influential for 35 years; even fewer can be said to have affected undergraduate and graduate education.* Transport Phenomena *(BSL) accomplished both and it brought fundamental changes to the way chemical engineers think: BSL can be arguably regarded as the most important book in chemical engineering ever published. In this essay we place BSL in the context of its times and surrounding paradigms, review and comment on the early reception of the book, offer comments on style, and speculate on its possible revision.*

$|D(v)|$: shear rate

$p = 2$: newtonian fluids (e.g. water, oil)

$p < 2$: *shear-thinning* (or plastic and pseudo-plastic, e.g. most polymer melts and solutions)
- the viscosity is decreasing with respect the shear rate (viscosity $= \infty$ when shear rate $= 0$)

$p > 2$: *shear-thickening* (or dilatant, e.g. mud, clay, cement)
- the viscosity is increasing

See e.g. E. Marusic-Paloka, *Steady Flow of a Non-Newtonian Fluid in Unbounded Channels and Pipes*, Mathematical Models and Methods in Applied Sciences, **10**(9) (2000).

Parallel fluids

The velocity field is of the form

$$v(\mathbf{x})\mathbf{n}$$

where $\mathbf{n}$ is a constant vector and $v(\mathbf{x})$ is a scalar function.

In this case, the Navier-Stokes equations $(NS)_p$ become the *p-Laplacian* equation

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) = c$$

for some constant c, related to "*pressure drop*", i.e. $\nabla \mathcal{P} = -c\mathbf{n}$.

"Ladyzhenskaya-Solonnikov problem for power law fluids"

Given $\Phi \in \mathbb{R}$, find a solution $(v, \mathcal{P})$ of $(NS)_p$ such that

$$v|\partial\Omega = 0\,, \quad \textit{the flux} \equiv \int_\Sigma v \cdot \mathbf{n} = \Phi \quad \text{and} \quad \sup_{t>0} t^{-1} \int_{\Omega^t} |\nabla v|^p < \infty\,.$$

<u>Theorem</u>. (–, Gilberlandio Dias, J.D.E. 2012)
Let $p \geq 2$. Then, for any flux Φ, the Ladyzhesnkaya-Solonnikov problem for power-law fluids $(NS)_p$ has a weak solution $(v, \mathcal{P})$ in $W^{1,p}_{loc}(\Omega) \times L^{p'}_{loc}(\Omega)$, $p' = p/(p-1)$, i.e. there exist a $(v, \mathcal{P})$ belonging to this space such that

$$\begin{cases} \displaystyle\int_\Omega |D(v)|^{p-2} D(v) : \nabla\Psi = -\int_\Omega (v\nabla v) \cdot \Psi + \int_\Omega \mathcal{P} \operatorname{div}\Psi\,, \ \forall\ \Psi \in C_c^\infty(\Omega; \mathbb{R}^3) \\ \displaystyle\int_\Omega v \cdot \nabla\psi = 0\,, \qquad \forall\ \psi \in C_c^\infty(\Omega; \mathbb{R}) \end{cases}$$

$$v|\partial\Omega = 0, \quad \int_\Sigma v \cdot \mathbf{n} = \Phi \quad \text{and} \quad \sup_{t>0} t^{-1} \int_{\Omega^t} |\nabla v|^p < \infty\,.$$

<u>Remark</u>. The case $p = 2$ (newtonian fluid) is due to Ladyzhenskaya and Solonnikov [LS].

Results for **non** newtonian fluids $(NS)_p$

There are several results for bounded domains - boundary value problem for $(NS)_p$, e.g.
J.L. Lions, *Quelques Méthods de Resolution des Problémes Aux Limites Non Linéaires*, Dunod, Gauthier-Villars (1969), Ch. 2, Remark 5.5:

$$p \geq 3n/(n+2).$$

W. Sadowski, *On the Stationary Flow of the Power Law Fluid in 2D*, J. Appl. Analysis, **8**, 2002: $1 < p < 2$.

In unbounded domains there are few results, e.g.

E. Marusic-Paloka, 2000: Leray problem, $p > 2$.

Proofs

Amick's solution of Leray's problem for newtonian fluids, with small flux:

$$v = u + a; \quad u \in H^1_0(\Omega), \quad \operatorname{div} u = 0,$$
$$a \in H^1_{\text{loc}}(\Omega), \; \operatorname{div} a = 0, \; \boxed{a\big|C_i = v^i_P}, \; a\big|\partial\Omega = 0.$$

Notice that the Poiseuille flows v^i_P are not in $H^1(\Omega)$ (they are constant with respect to z); $v^i_P \in H^1_{\text{loc}}(\Omega)$.

A divergence free vector field u in $H^1_0(\Omega) = \overline{C^\infty_c(\Omega)}^{H^1(\Omega)}$ carries no flux, i.e. $\int_\Sigma u \cdot \mathbf{n} = 0$, for any cross section Σ of Ω. Indeed, if $\psi \in C^\infty_c(\Omega)$ then $\int_{\Sigma(z)} \psi \cdot \mathbf{n} = 0$ for all sufficiently large $|z|$.

NS-equations become

$$-\Delta u + u\nabla u + \mathrm{l}(u) + \nabla \mathcal{P} = 0,$$

where

$$\mathrm{l}(u) = a\nabla u + u\nabla a + a\nabla a - \Delta a\,.$$

Method: compactness method, with Galerkin approximations.

Estimate of the nonlinear term $\int (a\nabla u)u$ by $\int |\nabla u|^2$ (a priori estimate):

$$\int |(a\nabla u)u| \le \left(\int |\nabla u|^2\right)^{1/2} \left(\int |a|^2\, |u|^2\right)^{1/2}$$

$$\begin{aligned}
\int_{\Omega_i} |a|^2\, |u|^2 = & \int_{\Omega_i} |u|^2\, |v_P^i|^2 \\
= & \left| \int_0^{\pm\infty} \int_\Sigma |v_P^i|^2\, |u|^2 \right| \\
\le & \left| \int_0^{\pm\infty} \left(\int_\Sigma |v_P^i|^4\right)^{1/2} \left(\int_\Sigma |u|^4\right)^{1/2} \right| \\
= & \left| \int_0^{\pm\infty} \|v_P^i\|^2_{L^4(\Sigma)}\, \|u\|^2_{L^4(\Sigma)} \right| \\
\le & \; c \left| \int_0^{\pm\infty} \|\nabla v_P^i\|^2_{L^2(\Sigma)}\, \|\nabla u\|^2_{L^2(\Sigma)} \right| \\
= & \; c\, \|\nabla v_P^i\|^2_{L^2(\Sigma)} \left| \int_0^{\pm\infty} \|\nabla u\|^2_{L^2(\Sigma)} \right| \\
= & \; c\, \Phi^2 \int_{\Omega_i} |\nabla u|^2
\end{aligned}$$

Similarly, we can estimate $\int (u\nabla a)u$.

The terms $\int_{\Omega_i} (-\Delta a)u$ and $\int_{\Omega_i} (a\nabla a)u$ vanish, since $a\nabla a = 0$, because $a = v_P^i$ in Ω_i is "parallel", and $-\Delta a = (-\Delta v_P^i)\mathbf{n} = c\mathbf{n}$ in Ω_i, so

$$\int_{\Omega_i} (-\Delta a)u = \left| c \int_0^{\pm\infty} \int_\Sigma u \cdot \mathbf{n} \right| = 0.$$

□

Ladyzhenskaya-Solonnikov's solution, for newtonian fluids (NS) with arbitrary flux

$$v = u + a; \quad u \in H^1_{loc}(\Omega), \ \mathrm{div} u = 0, \ u\big|\partial\Omega = 0$$

and a is given by the following lemma:

Lemma [LS]. *For any $\delta > 0$ there exists a vector field a such that*

a$_1$) $a \in H^1_{loc}(\overline{\Omega})$, $\mathrm{div} a = 0$, $a|\partial\Omega = 0$,

a$_2$) $\int_\Sigma a \cdot \mathbf{n} = \Phi$ *for any cross section* Σ *of* Ω,

a$_3$) $\int_{\Omega_i^{t-1,t}} |\nabla a|^2 \le c\Phi^2$ *for* $i = 1, 2$ *and all* $t \ge 1$, *where* $\Omega_i^{t-1,t} = \{(\mathbf{x}, z) \in \Omega_i \, ; \, t - 1 < |z| < t\}$,

and

a$_4$) $\int_{\Omega^t} |a|^2 |u|^2 \le c\delta\Phi^2 \int_{\Omega^t} |\nabla u|^2$ for all $t > 0$ and $u \in C_c^\infty(\Omega)$,

where, in a$_3$) and a$_4$), c is a constant depending only on Ω.

Now $a\big|\Omega_i$ might not be the Poiseuille v_P^i, but the compactness method still works, by truncating the domain and long computations:

Let u^t be a solution of the NS-equations

$$-\Delta u^t + u^t \nabla u^t + \mathrm{l}(u^t) + \nabla \mathcal{P}^t = 0$$

in $H_0^1(\Omega^t)$ (joint with some pressure function $\mathcal{P}^t \in L^2_{\mathrm{loc}}(\Omega^t)$).

Now, let $\boxed{t' > t}$. Multiplying the equation $-\Delta u^{t'} + u^{t'} \nabla u^{t'} + \mathrm{l}(u^{t'}) + \nabla \mathcal{P}^{t'} = 0$ by $u^{t'}$ and integrating by parts in Ω^t, we obtain

$$\int_{\Omega^t} |\nabla u^{t'}|^2 \leq ct + \int_{\Sigma(t)} (\text{bound. terms}),$$

for all $t < t'$. Integrating in t, from $\eta - 1$ to $\eta \leq t'$, we get

$$z(\eta) := \int_{\eta-1}^{\eta} \left(\int_{\Omega^t} |\nabla u^{t'}|^2 \right) dt \leq c\eta - \frac{1}{2} + \int_{\Omega^{\eta-1,\eta}} (\text{bound. terms}).$$

Using the equation, is possible to estimate $\int_{\Omega^{\eta-1,\eta}} (\text{bound. terms})$ by a linear combinations of powers of $\int_{\Omega^{\eta-1,\eta}} |\nabla u^{t'}|^2$. But

$$\int_{\Omega^{\eta-1,\eta}} |\nabla u^{t'}|^2 = z'(\eta)\,!$$

Thus,

$$z(\eta) := \int_{\eta-1}^{\eta} \left(\int_{\Omega^t} |\nabla u^{t'}|^2 \right) dt \leq c\eta + g(z'(\eta)), \quad \forall\ \eta \leq t',$$

for some function $g : \mathbb{R} \to \mathbb{R}$. Besides,

$$z(t') \leq \int_{\Omega^{t'}} |\nabla u^{t'}|^2 \leq ct'.$$

Then, by a kind of "*reverse Gronwall lemma*" [LS], we have

$$z(\eta) \leq c\eta,$$

which implies

$$\int_{\Omega^{\eta-1}} |\nabla u^{t'}|^2 \leq c\eta, \quad \forall\ \eta \leq t'.$$

So, fixing t (arbitrary), $\{u^{t'}\}_{t'>t}$ is bounded in $H^1(\Omega^t)$, by $c(t+1)$.

Construction of a in [LS]

In Ω_i, the field a is given by

$$a = \frac{1}{2\pi}\nabla \times (\zeta b) = \frac{1}{2\pi}\nabla\zeta \times b$$

where

$$b(x) = \left(-\frac{x_2}{|\mathbf{x}|^2}, \frac{x_1}{|\mathbf{x}|^2}, 0\right), \quad \mathbf{x} = (x_1, x_2),$$

and ζ is the "truncating E. Hopf's function":

$$\zeta(x) = \psi\left(\varepsilon \log \frac{\sigma(|\mathbf{x}|)}{\rho(x)}\right);$$

$\rho(x)$:	the regularized distance to $\partial\Omega$
$\sigma, \psi : \mathbb{R} \to \mathbb{R}$:	smooth nondecreasing functions,

$$\sigma(s) = \begin{cases} \frac{l}{4}, & s \le \frac{l}{4} \\ t, & s > \frac{l}{2} \end{cases}$$

$$\psi(s) = \begin{cases} 0, & s \le 0 \\ 1, & s > 1 \end{cases}$$

$\varepsilon = \varepsilon(\delta)$.

Construction of a for non newtonian fluids $(NS)_p$, $p > 2$

Let **a** be **a smooth divergence free vector field, which is bounded and has bounded derivatives in $\overline{\Omega}$, vanishes on $\partial\Omega$, and has flux Φ, i.e. $\int_\Sigma \mathbf{a} = \Phi$ over any cross section Σ of Ω.** Then, for some constant c depending only on **a**, p and Ω:

i) $\int_{\Omega_t} |\mathbf{a}|^{p'} |\varphi|^{p'} \le c|\Phi|^{p'} \, t^{(p-2)/(p-1)} \|\nabla\varphi\|^{p'}_{L^p(\Omega_t)}$,
$\qquad \forall\ t > 0, \quad \forall\ \varphi \in \mathcal{D}(\Omega)$;

ii) $\int_{\Omega_{i,t-1,t}} |\nabla \mathbf{a}|^p \le c|\Phi|^p, \quad \forall\ t \ge 1, \quad i = 1, 2$;

iii) $\int_{\Omega_t} |\nabla \mathbf{a}|^p \le c|\Phi|^p (t+1)\ , \quad \forall\ t \ge 1.$

Estimate of the nonlinear terms

We want to estimate all the nonlinear terms by $\int |\nabla u|^p$.

Now we have two main nonlinear terms:

$$\int (a\nabla u)u \quad \text{and} \quad \int |D(v)|^{p-2}D(v) : D(u), \quad v = u + a\,.$$

Known inequalities:

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c|x - y|^p, \ \forall\, x, y \in \mathbb{R}^n, \quad (p > 2)$$

$$\int |\nabla u|^p \leq c \int |D(u)|^p \qquad \text{(Korn's inequality}^1\text{)}.$$

The argument in the truncated (bounded) domain Ω^t:

Taking $x = D(v^{t'}) = D(u^{t'}) + D(a)$, $t' > t$, and $y = D(a)$ ($\Rightarrow\ x - y = D(u^{t'})$) in the first inequality and using Korn's inequality, we get – writing $u = u^{t'}, v = v^{t'}$,

$$\int_{\Omega^t} |D(v)|^{p-2}D(v) : D(u) \geq c\int_{\Omega_t} |\nabla u|^p + \int_{\Omega_t} |D(a)|^{p-2}D(a) : D(u)\,.$$

[1] Patrizio Neff, Proc. Royal Soc. Edinb. **132A** (2002);
V.A. Kondrat'ev and O.A. Oleinik, Russian Math. Surveys **43** (5) (1988)

By Young inequality and a_3),

$$\begin{aligned} \left|\int_{\Omega_t} |D(a)|^{p-2} D(a) : D(u)\right| &\leq \int_{\Omega_t} |D(a)|^{p-1} |D(u)| \\ &\leq \int_{\Omega_t} (\epsilon |D(u)|^p + c_\epsilon |D(a)|^p) \\ &\leq \epsilon \int_{\Omega_t} |\nabla u|^p + c_\epsilon ct\,. \end{aligned}$$

Regarding the term $\int (a\nabla u)u$, by Hölder inequality, a_4) and Young inequality, we have

$$
\begin{aligned}
|\int_{\Omega_t}(a\nabla u)u\,| &\le \left(\int_{\Omega_t}|\nabla u|^p\right)^{1/p}\left(\int_{\Omega_t}|a|^{p'}|u|^{p'}\right)^{1/p'} \\
&\le \left(\int_{\Omega_t}|\nabla u|^p\right)^{1/p}\left(c\,t^{(p-2)/(p-1)}\left(\int_{\Omega_t}|\nabla u|^p\right)^{1/(p-1)}\right)^{1/p'} \\
&= \left(\int_{\Omega_t}|\nabla u|^p\right)^{2/p}(c)^{1/p'}\,t^{(p-2)/p} \\
&\le \epsilon\int_{\Omega_t}|\nabla u|^p + ct\,.
\end{aligned}
$$

To pass to the limit from approximate solutions, the compactness method is not enough due to the nonlinear term

$$A(u) := -\mathrm{div}\left(|D(u) + D(a)|^{p-2}(D(u) + D(a))\right) .$$

But the inequality

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c|x - y|^p$$

implies that the operator $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))'$ is monotone and the method of Browder and Minty enables us to pass to the limit. (See e.g. § 9.1 of L.C. Evans, *Partial Differential Equations*.)

Some important features in the case of non newtonian fluids $(NS)_p$, $p > 2$

• The construction of the vector field a can be simplified.
It is enough that a be a bounded vector field of divergence zero and vanishing on $\partial\Omega$!

• Extra non linear term

$$|D(v)|^{p-2}D(v)$$

Monotonicity, **Browder-Minty method**

• Inequalities:

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geq c|x-y|^p, \ \forall\ x, y \in \mathbb{R}^n, \quad (p > 2)$$

$$\int_{\Omega^t} |\nabla u^{t'}|^p \leq c \int_{\Omega^t} |D(u^{t'})|^p$$

Korn inequality, with $u^{t'}$ vanishing only on a part of $\partial\Omega^t$:

- Patrizio Neff, Proc. Royal Soc. Edinb. A **132** (2002).

Some important features in the case of non newtonian fluids, continued

• There is no regularity for the generalized solution of the system $(NS)_p$.
To get regularity we needed to modify $|D(\mathbf{v})|^{p-2}D(\mathbf{v})$ to

$$(\varepsilon + |D(\mathbf{v})|)^{p-2}\, D(\mathbf{v}), \quad \varepsilon > 0$$

and adapt the proof of

- Beirão da Veiga, Kaplický and Růžička, *Boundary regularity of shear thickening flows*. J. Math. Fluid Mech. (2011). Abridged version: C. R. Math. Acad. Sci. Paris (2010).
- 2D: Kaplický, Málek and Stará $C^{1,\alpha}$*-solutions to a class of nonlinear fluids in two dimensions — stationary Dirichlet problem*, J. Math. Sci. (2002).

Domain with nozzles

https://en.wikipedia.org/wiki/Nozzle

Domain with curved ends

In this section we propose a definition for domains with unbounded channels not necessarily containing straight cylinders and give an idea how to show the existence of steady flow for incompressible fluids with arbitrary fluxes in such domains. More precisely, using some concepts from Geometry, we argue below that the following statement is true:

Let $\overline{\Omega}$ be a smooth 3-manifold with boundary in $\mathbb{R}^3$diffeomorphic to a compact smooth 3-manifold with boundary in $\mathbb{R}^3$ with k "holes" removed from its boundary. Suppose that the volumes of the cut domains Ω_t (defined below) are of order t. Then, given any set of real numbers Φ_i, $i = 1, \cdots, k$, such that $\Phi_1 + \cdots \Phi_k = 0$, the Navier-Stokes equations $(NS)_p$, with $p > 2$, and $\Omega = \overline{\Omega} - \partial\overline{\Omega}$, have a weak solution $\mathbf{v}$ in $W^{1,p}_{\rm loc}(\Omega)$ having flux Φ_i in each end $\Omega_{(i)}$ (defined below), for each $i = 1, \cdots, k$, and satisfying the Dirichlet homogeneous boundary condition $\mathbf{v}|\partial\Omega = 0$.

Next we give more details about this statement and then give an idea for its proof.

Definition of $\overline{\Omega}$ and ends

Let $\overline{\Omega}$ be a smooth 3-manifold with boundary such that there is a compact smooth 3-manifold with boundary $\overline{\mathcal{B}}$ in $\mathbb{R}^3$ and a diffeomorphism $H : \overline{\mathcal{B}} - \{\tilde{p_1}, \cdots, \tilde{p_k}\} \to \overline{\Omega}$, where $\tilde{p_1}, \cdots, \tilde{p_k}$ are neighborhoods in $\partial\overline{\mathcal{B}}$ of given points $p_1, \cdots, p_k$ in $\partial\overline{\mathcal{B}}$ $(k < \infty)$. Denote $\mathcal{B} = \overline{\mathcal{B}} - \partial\overline{\mathcal{B}}$, $\mathcal{M} = \partial\mathcal{B} - \{\tilde{p_1}, \cdots, \tilde{p_k}\}$ and $\mathcal{S} = \partial\Omega = \partial\overline{\Omega}$, where $\Omega = \overline{\Omega} - \partial\overline{\Omega}$. Then $h := H|\mathcal{M}$ is a diffeormorphism from $\mathcal{M}$ onto $\mathcal{S}$, so $\mathcal{S}$ is a punctured surface (punctured 2-manifold), or, a 2-manifold with a finite number of ends. We define an end $\Omega_{(i)}$ of Ω as follows: $\Omega_{(i)}$ is the image by H of the intersection of $\mathcal{B}$ with an open ball $B_\varepsilon(p_i)$ in $\mathbb{R}^3$ centered at p_i with radius ε_i, sufficiently small such that $\mathcal{B} \cap B_\varepsilon(p_i)$ is a simply connected set. We denote this intersection by $V_{\varepsilon_i}(p_i)$. Thus, $\Omega_{(i)} := H(V_{\varepsilon_i}(p_i)) = H(\mathcal{B} \cap B_\varepsilon(p_i))$. In particular, $\Omega_{(i)}$ is an open and simply connected set in $\mathbb{R}^3$. Similarly, we define an end $\mathcal{S}_{(i)}$ of $\mathcal{S}$ as the image by h of $\mathcal{M} \cap \partial V_{\varepsilon_i}(p_i)$. $\mathcal{S}_{(i)}$ is a connected smooth surface (possibly unbounded).

Definition of cross sections and cut domains Ω_t

Now we define cross sections of $\Omega_{(i)}$ and the cut domains Ω_t of Ω, for $t \geq 1$. We define a cross section $\Sigma(t) \equiv \Sigma_i(t)$ of $\Omega_{(i)}$, as the image of $\mathcal{B} \cap \partial V_{t^{-1}\varepsilon_i}(p_i)$ by H. Notice that $V_{t^{-1}\varepsilon_i}(p_i) \subset V_{\varepsilon_i}(p_i)$, since $t \geq 1$, and $\Sigma(t)$ is a simply connected smooth $(n-1)$-manifold in $\Omega_{(i)}$ (without boundary). The boundary of a cross section $\Sigma(t)$ is a smooth simple closed curve in $\mathcal{S}_{(i)} = \partial\Omega_{(i)}$ which turns around $\Omega_{(i)}$. In particular, it is not homotopic to a point, as it is not its preimage by h in $\mathcal{M}$. Indeed, this preimage is a loop (i.e. a smooth simple closed curve) in $\mathcal{M}$ around p_i, i.e. with p_i in its interior.

Finally, regarding the *cut domain* Ω_t we define it as being the following set: $\Omega_t = H(\mathcal{B} - \cup_{i=1}^k V_{t^{-1}\varepsilon_i}(p_i))$. Notice that the sets Ω_t are bounded and smooth open sets in $\mathbb{R}^3$ (i.e. with smooth boundaries), they satisfy $\Omega_{t_1} \subset \Omega_{t_2}$ if $t_1 < t_2$, and $\Omega = \cup_{t \geq 1}\Omega_t$.

Construction of **a**; de Rham theorem

Now that we have set terminologies, we give the idea for a proof on the existence of steady flow in the described set Ω. Analogously to **[LS]** (see above), we search a velocity $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{u} + \mathbf{a}$, where $\mathbf{a}$ is a given vector field defined in Ω such that it is divergence free, $\mathbf{a}|\partial\Omega = 0$, it is bounded and has bounded derivatives in $\overline{\Omega}$, and has flux Φ_i in each end $\Omega_{(i)}$, i.e. $\int_{\Sigma_i} \mathbf{a} = \Phi_i$, for $i = 1, \cdots, k$. The construction of such vector field $\mathbf{a}$, as we have seen, is an important step. Let $\mathcal{M}$ be oriented by a normal vector field $\widetilde{\mathbf{N}}$ pointing to the exterior of $\mathcal{B}$. Considering the class of homotopic loops around the point p_i, $i = 1, \cdots, k$, which we denote by $[\gamma_i]$, and assuming that any loop in $\mathcal{M}$ is positively oriented with respect to $\widetilde{\mathbf{N}}$, let l_i be a linear functional (defined on the space of singular 1-chains om $\mathcal{M}$) such that $l_i([\gamma_j]) = \Phi_i\delta_{ij}$ (where δ_{ij} is the Kronecker delta), $i, j = 1, \cdots, k$. Then by the de Rham theorem (see e.g. [8, §4.17]) there exists a closed vector field (i.e. a closed 1-form) $\mathbf{b}_i$ on $\mathcal{M}$ such that l_i can be identified to $\mathbf{b_i}$ through the formula $l_i([\gamma]) = \int_\gamma \mathbf{b}_i$, for any class $[\gamma]$ of a loop γ in $\mathcal{M}$.

Construction of **a** cont'ed

Then if we take $\widetilde{\mathbf{b}} := \sum_{i=1}^{k-1} \mathbf{b}_i$ and let $\mathbf{b}$ be the pullback of $\widetilde{\mathbf{b}}$ by h^{-1}, we obtain a tangent vector field $\mathbf{b}$ on $\partial\Omega$ such that its integral on the boundary of any cross section of the outlet $\Omega_{(i)}$ is equal to Φ_i, for $i = 1, \cdots, k$. Next, we can extend $\mathbf{b}$ to Ω, first by extending it to a tubular neighborhood V of $\partial\Omega$ inside Ω, by setting $\mathbf{b}(y, s) = \mathbf{b}(y) + s\mathbf{N}(y)$, for $(y, s) \in V$ (i.e. $y \in \partial\Omega$ and s in some interval $(-\epsilon_y, 0)$), where $\mathbf{N}$ is the unit normal vector field to $\partial\Omega$ pointing to the exterior of Ω. Then we extend $\mathbf{b}$ to the entire set Ω by multiplying it by a smooth bounded function $\zeta : \mathbb{R}^n \to \mathbb{R}$ such that it is equal to 1 on V. Finally, we define $\mathbf{a}$ to be the curl of the vector $\zeta\mathbf{b}$.

Then **a** is divergence free and if $\Sigma_i(t)$ is a cross section of the outlet $\Omega_{(i)}$ with a normal vector field $\mathbf{n}_i$ pointing to infinity, by Stokes theorem and the construction of **a**, we have

$$\begin{aligned}\int_{\Sigma_i(t)} \mathbf{a}\cdot\mathbf{n}_i = & \int_{\partial\Sigma_i(t)} \zeta\mathbf{b} = \int_{\partial\Sigma_i(t)} \mathbf{b} = \int_{\partial V_{t^{-1}\varepsilon_i}(p_i)} \widetilde{\mathbf{b}} \\ = & \sum_{j=1}^{k-1} \int_{\partial V_{t^{-1}\varepsilon_i}(p_i)} \mathbf{b}_j = \sum_{j=1}^{k-1} l_j([\partial V_{t^{-1}\varepsilon_i}(p_i)]) \\ = & \Phi_i\end{aligned}$$

for $i = 1, \cdots, k-1$. For $i = k$ this also holds true, due to the divergence theorem, the condition $\sum_{i=1}^{k} \Phi_i = 0$ and the fact that **a** is divergence free.

Besides, since, by hypothesis, the volumes of the cut domains Ω_t are of order t, i.e. $\int_{\Omega_t} \leq ct$ for some constant c, and the vector field $\mathbf{a}$ is bounded, the estimate i) in Section **??** holds true. Indeed, for new constants c, we have

$$\begin{aligned}
\int_{\Omega_t} |\varphi|^{p'} |\mathbf{a}|^{p'} &\leq c \int_{\Omega_t} |\varphi|^{p'} \leq c \int_{\Omega_t} |\nabla \varphi|^{p'} \\
&\leq c \ t^{1-p'/p} \left(\int_{\Omega_t} |\nabla \varphi|^{p} \right)^{p'/p} \\
&= c \ t^{(p-2)/(p-1)} \|\nabla \varphi\|^{p'}_{L^p(\Omega_t)}
\end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$. Thus, the proof for our statement stated at the beginning of this section can be done by following all steps in the proof of [3, Theorem 2.2].

The case of genus zero; stereographic projection and angle forms

Remark. *In the case that the compact surface $\partial\mathcal{B}$ is of genus zero, the construction of the vector field $\widetilde{\mathbf{b}}$ above can be simplified.* Indeed, in this case we can assume, without loss of generality, that $\mathcal{B}$ is the unit ball in $\mathbb{R}^3$, i.e. $\partial\mathcal{B}$ is the sphere S^2, and we can take $\widetilde{\mathbf{b}}$ as the pullback by a stereographic projection of a linear combinations of angle forms in the plane. More precisely, let $\Pi : S^2 - \{p_k\} \to \mathbb{R}^2$ be the stereographic projection with projection point ("north pole") p_k (we can take any point $p_1, \cdots, p_k$ as the projection point) and ω_i be the 1-form

$$\omega_i(x,y) = \frac{\Phi_i/2\pi}{(x-a_i)^2+(y-b_i)^2}(-(y-b_i)dy+(x-a_i)dx)$$

in $\mathbb{R}^2 - \{\Pi(p_i)\}$, $i = 1, \cdots, k-1$, where $(a_i, b_i) = \Pi(p_i)$. Then $\widetilde{\mathbf{b}} = \sum_{i=1}^{k-1} \Pi^*\omega_i$ has the required properties.

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