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"Três princípios básicos da Análise Funcional:

- **Princípio da Limitação Uniforme**
- **Teorema de Hahn-Banach**
- **Teorema da Aplicação Aberta"**

[Dunford-Schwartz]

"A Santíssima Trindade"
Carlos Isnard

"Colisão alfabética"

"Sopa de letras"

Campos Vetoriais Log-Lipschtzianos no $\mathbb{R}^n$ e Estrutura Lagrangeana de Fluxos Compressíveis*

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Tópicos:

1. **Campos vetoriais log-Lipschtzianos no $\mathbb{R}^n$.**
2. **Fluxos compressíveis.**
3. **Exemplo.**

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1 Campos log-Lipschtzianos

Um campo vetorial $\mathbf{u}$ ***no*** $\mathbb{R}^n$ ***é dito*** ***log-Lipschitziano*** (LL) ***se***

$$\langle \mathbf{u} \rangle_{LL} \equiv \sup_{0<|\mathbf{x}-\mathbf{y}|\leq 1} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{y}| \log |\mathbf{x}-\mathbf{y}|} < \infty.$$

Exemplos básicos:

1) ***Todo campo Lipschtziano é log-Lipschtziano.*** **Com efeito,** $|\mathbf{x}-\mathbf{y}| \leq 1 \ \Rightarrow \ |\mathbf{x}-\mathbf{y}| - |\mathbf{x}-\mathbf{y}| \log |\mathbf{x}-\mathbf{y}| \ \geq \ |\mathbf{x}-\mathbf{y}|$.

2) **Em geral** $\mathbf{u} \in \mathbf{H}^2(\mathbb{R}^2) \ \not\Rightarrow \ \mathbf{u} \in \mathbf{Lip}(\mathbb{R}^2)$ **(i.e.** $\nabla \mathbf{u} \in \mathbf{L}^\infty(\mathbb{R}^2)$**)** **mas**

$$\mathbf{H}^2(\mathbb{R}^2) \ \subset \ \mathbf{LL}(\mathbb{R}^2).$$

3) **Campos do tipo** $\mathbf{u}(\mathbf{x}) = |\mathbf{x}|^\alpha \, \tilde{\mathbf{e}}$, **onde** $\tilde{\mathbf{e}}$ **é um vetor constante do** $\mathbb{R}^n$ **e** α **convenientemente escolhido, não é um campo log-Lipschtziano.**

4) **([Bahouri-Chemin] e.g.? 1994)** ***Sejam*** $\mathbf{w} \in \mathbf{L}^\mathbf{p}(\mathbb{R}^\mathbf{n}) \cap \mathbf{L}^\infty(\mathbb{R}^\mathbf{n})$, ***onde*** $\mathbf{p} \in [\mathbf{1}, \infty)$, ***e*** $\mathbf{\Gamma}$ ***a solução fundamental do Laplaciano no*** $\mathbb{R}^\mathbf{n}$***. Então*** $\nabla \mathbf{\Gamma} * \mathbf{w} \ \in \ \mathbf{LL}(\mathbb{R}^\mathbf{n})$ ***e***

$$\langle \nabla \mathbf{\Gamma} * \mathbf{w} \rangle_{\mathbf{LL}} \leq \mathbf{C}(||\mathbf{w}||_\mathbf{p} + ||\mathbf{w}||_\infty)$$

onde $\mathbf{C} = \mathbf{C}(\mathbf{n}, \mathbf{p})$.

<u>Prova</u>: Sejam $\mathbf{\bar{x} = (x + y)/2}$ e $\epsilon = |\mathbf{x - y}|$.

$$
\begin{aligned}
&(\Gamma_{x_j} * w)(x) - (\Gamma_{x_j} * w)(y) \\
= &\int_{\mathbb{R}^n} \left[\Gamma_{x_j}(x - z) - \Gamma_{x_j}(y - z)\right] w(z)dz \\
&= \int_{B_\epsilon(\bar{x}) \cup [B_2(\bar{x})/B_\epsilon(\bar{x})] \cup B_2(\bar{x})^c} \cdots \\
\equiv &\; I + II + III.
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{I}| \leq &\; \mathbf{C}||\mathbf{w}||_\infty \left[\int_{\mathbf{B_{2\epsilon}(x)}} |\mathbf{x - z}|^{\mathbf{1-n}}\mathbf{dz} + \int_{\mathbf{B_{2\epsilon}(y)}} |\mathbf{y - z}|^{\mathbf{1-n}}\mathbf{Dz}\right] \\
= &\; \mathbf{C}||\mathbf{w}||_\infty \epsilon;
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{II}| \leq &\; \mathbf{C}||\mathbf{w}||_\infty \int_{\mathbf{B_2(\bar{x})/B_\epsilon(\bar{x})}} \left|\frac{\mathbf{x_j - z_j}}{|\mathbf{x-z}|^\mathbf{n}} - \frac{\mathbf{y_j - z_j}}{|\mathbf{y-z}|^\mathbf{n}}\right| \mathbf{dz} \\
\leq &\; \mathbf{C}||\mathbf{w}||_\infty \epsilon \int_\mathbf{0}^\mathbf{1} \int_{\mathbf{B_2(\bar{x})/B_\epsilon(\bar{x})}} |\mathbf{x}_\theta - \mathbf{z}|^{-\mathbf{n}} \mathbf{dz}\theta
\end{aligned}
$$

$\mathbf{x}_\theta = \mathbf{x} + \theta(\mathbf{y - x})$, $\theta \in (\mathbf{0}, \mathbf{1})$. **Como** $\mathbf{B_2(\bar{x})/B_\epsilon(\bar{x}) \subset B_3(x_\theta)/B_{\epsilon/2}(x_\theta)}$,

$$|\mathbf{II}| \leq \mathbf{C}||\mathbf{w}||_\infty \epsilon(\log \mathbf{3} + \log \mathbf{2} - \log \epsilon);$$

analogamente,

$$|\mathbf{III}| \leq \mathbf{C}\,\epsilon \int_\mathbf{0}^\mathbf{1} \int_{\mathbf{B_1(x_\theta)^c}} |\mathbf{x}_\theta - \mathbf{z}|^{-\mathbf{n}} |\mathbf{w(z)}| \mathbf{dzd}\theta.$$

Daí, usando a desigualdade de Hölder,

$$|\mathbf{III}| \leq \mathbf{C}\,||\mathbf{w}||_\mathbf{p}\,\epsilon\,. \qquad \blacksquare$$

<u>Observação</u>: **Se** $p < n$, **temos também**

$$||\mathbf{w}||_\infty \leq \mathbf{C}(||\mathbf{w}||_\mathbf{p} + ||\mathbf{w}||_\infty)\,.$$

<u>Observação:</u> $\mathbf{u} \in \mathbf{LL}(\mathbb{R}^{\mathbf{n}}) \Rightarrow \mathbf{u} \in \mathbf{C}(\mathbb{R}^{\mathbf{n}})$
Na verdade, para qualquer $\alpha \in (0,1)$**,**

$$\begin{aligned} \mathbf{u} \in \mathbf{LL} \Rightarrow |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})| &\leq \langle \mathbf{u} \rangle_{\mathbf{LL}} (\, |\mathbf{x} - \mathbf{y}| - |\mathbf{x} - \mathbf{y}| \log |\mathbf{x} - \mathbf{y}| \,) \\ &\leq \langle \mathbf{u} \rangle_{\mathbf{LL}} \mathbf{C} |\mathbf{x} - \mathbf{y}|^{\alpha} \end{aligned}$$

para todo $\mathbf{x}, \mathbf{y}$ **tal que** $|\mathbf{x} - \mathbf{y}| << 1$**. Logo,** $\mathbf{LL} \subset \mathbf{C}^{\alpha}$**. Mais geralmente,**

$$\mathbf{Lip} \subset \mathbf{LL} \subset \mathbf{C}^{\alpha}.$$

Estrutura Lagrangeana – Teorema de Picard generalizado:

Todo campo log-Lipschitziano tem "estrutura Lagrangeana", i.e. se $\mathbf{u} \in \boldsymbol{LL}(\boldsymbol{\Omega})$***, onde*** $\boldsymbol{\Omega}$ ***é um aberto do*** $\mathbb{R}^{\mathbf{n}}$***, então para todo*** $\mathbf{x} \in \Omega$ ***existe uma única aplicação*** $\mathbf{X} \equiv \mathbf{X}(\cdot, \mathbf{x}) \in \mathbf{C}([\mathbf{0}, \mathbf{t_x}); \boldsymbol{\Omega})$***,*** $\mathbf{t_x} > \mathbf{0}$***, tal que***

$$\mathbf{X}(\mathbf{t}) \equiv \mathbf{X}(\mathbf{t}, \mathbf{x}) = \mathbf{x} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{u}(\mathbf{X}(\tau, \mathbf{x}))\, \mathbf{d}\tau, \qquad \mathbf{0} \leq \mathbf{t} < \mathbf{t_x}.$$

Prova: A existência segue-se do Teorema de Peano.

Quanto à unicidade,

$$|\mathbf{X_1}(\mathbf{t}) - \mathbf{X_2}(\mathbf{t})| \leq \int_{\mathbf{0}}^{\mathbf{t}} |\mathbf{u}|_{\mathbf{LL}}\ \mu(|\mathbf{X_1}(\mathbf{s}) - \mathbf{X_2}(\mathbf{s})|)\mathbf{ds}$$

$$\mu(\mathbf{r}) := \begin{cases} \mathbf{r}(\mathbf{1} - \mathbf{r}) \log \mathbf{r}, & \mathbf{0} \leq \mathbf{r} \leq \mathbf{1} \\ \mathbf{r}, & \mathbf{r} \geq \mathbf{1} \end{cases}$$

$$|\mathbf{u}|_{\mathbf{LL}} := \sup_{\substack{\mathbf{x}, \mathbf{y} \in \Omega \\ \mathbf{x} \neq \mathbf{y}}} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|}{\mu(|\mathbf{x} - \mathbf{y}|)}.$$

■

Lema de Osgood (Gronwall generalizado). *Sejam* $\mathbf{I} = [\mathbf{0}, \mathbf{t_1})$, $\rho \geq \mathbf{0}$ *uma função mensurável e localmente limitada em I,* $\mathbf{0} \leq \gamma \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbf{I})$, $\mathbf{a} \geq \mathbf{0}$, $\mu \in \mathrm{C}^0(\mathbb{R}_+)$ *crescente,* $\mu(\mathbf{r}) \neq \mathbf{0}$ *se* $\mathbf{r} \neq \mathbf{0}$, *e* $\int_{\mathbf{0}}^{\mathbf{1}} \frac{\mathbf{dr}}{\mu(\mathbf{r})} = \infty$, *tais que*

$$\rho(\mathbf{t}) \leq \mathbf{a} + \int_{\mathbf{0}}^{\mathbf{t}} \gamma(\mathbf{s})\mu(\rho(\tau))\mathbf{d}\tau, \qquad \mathbf{t} \in \mathrm{I}.$$

Se $\mathbf{a} \neq 0$, *então*

$$-\mathcal{M}(\rho(\mathbf{t})) + \mathcal{M}(\mathbf{a}) \leq \int_{\mathbf{0}}^{\mathbf{t}} \gamma(\tau)\,\mathbf{d}\tau, \qquad \mathcal{M}(\mathbf{r}) := \int_{\mathbf{r}}^{\mathbf{1}} \frac{\mathbf{dr}'}{\mu(\mathbf{r}')};$$

se $\mathbf{a} = \mathbf{0}$, *então* $\rho(\mathbf{t}) \equiv \mathbf{0}$.

Prova:

$$\mathbf{R}(\mathbf{t}) := \mathbf{a} + \int_{\mathbf{0}}^{\mathbf{t}} \gamma(\tau)\mu(\rho(\mathbf{s}\tau))\mathbf{ds}\tau$$

$$\rho(\mathbf{t}) \leq \mathbf{R}(\mathbf{t}), \quad \mathbf{R}'(\mathbf{t}) = \gamma(\mathbf{t})\mu(\rho(\mathbf{t})) \leq \gamma(\mathbf{t})\mu(\mathbf{R}(\mathbf{t}))$$

Caso $\mathbf{a} \neq 0$: $\quad \mathbf{R}(\mathbf{t}) > \mathbf{0}$ e

$$-\frac{\mathbf{d}}{\mathbf{dt}}\mathcal{M}(\mathbf{R}(\mathbf{t})) = \frac{\mathbf{1}}{\mu(\mathbf{R}(\mathbf{t}))}\mathbf{R}'(\mathbf{t}) \leq \gamma(\mathbf{t})$$

Logo, integrando de 0 a t,

$$-\mathcal{M}(\rho(\mathbf{t})) + \mathcal{M}(\mathbf{a}) \leq \int_{\mathbf{0}}^{\mathbf{t}} \gamma(\tau)\mathbf{d}\tau.$$

Caso a=0: Sejam $\tilde{\rho}(\mathbf{t}) = \sup_{\tau\in[\mathbf{0},\mathbf{t}]} \rho(\tau)$ **e** $\mathbf{t_2} > \mathbf{0}$ **tal que** $\tilde{\rho}(\mathbf{t_2}) > \mathbf{0}$**.**
Então

$$\tilde{\rho}(\mathbf{t}) \leq \int_{\mathbf{0}}^{\mathbf{t}} \gamma(\tau)\mu(\tilde{\rho}(\tau))\mathbf{d}\tau$$

$$\tilde{\rho}(\mathbf{t}) \leq \mathbf{a}' + \int_{\mathbf{0}}^{\mathbf{t}} \gamma(\tau)\mu(\tilde{\rho}(\tau))\mathbf{d}\tau$$

qualquer que seja a$' > \mathbf{0}$**, logo, pelo caso anterior,**

$$\mathcal{M}(\mathbf{a}') \leq \int_{\mathbf{0}}^{\mathbf{t_2}} \gamma(\tau)\,\mathbf{d}\tau + \mathcal{M}(\tilde{\rho}(\mathbf{t_2})) < \infty, \qquad \forall\ \mathbf{a}' > \mathbf{0}\ :$$

contradição, pois $\mathcal{M}(\mathbf{0}) = \infty$ **por hipótese.** ■

Observação – **Vale resultado similar para campos não-autônomos com semi-norma LL localmente integrável no tempo:**

Se para cada $t \geq 0$***,*** $\mathbf{u}(\mathbf{x}, t)$ ***é um campo em*** Ω ***tal que*** $\mathbf{u}(\cdot, t) \in \mathrm{LL}$ ***e*** $\langle \mathbf{u}(\cdot, t) \rangle_{LL} \in L^1_{\boldsymbol{loc}}([0, \infty))$ ***então para todo*** $\mathbf{x} \in \Omega$ ***existe uma única aplicação*** $\mathbf{X}(\cdot, \mathbf{x}) \in \mathbf{C}([\mathbf{0}, \mathbf{t_x}); \boldsymbol{\Omega})$***,*** $\mathbf{t_x} > \mathbf{0}$***, tal que***

$$\mathbf{X}(\mathbf{t}, \mathbf{x}) = \mathbf{x} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{u}(\mathbf{X}(\tau, \mathbf{x}), \tau)\, \mathbf{d}\tau, \qquad \mathbf{0} \leq \mathbf{t} < \mathbf{t_x}.$$

2 Fluxos Compressíveis

Equações de Navier-Stokes:

$$\begin{cases} \rho_t + \mathbf{div}(\rho \mathbf{u}) = 0 \\ \\ (\rho u^j)_t + \mathbf{div}(\rho u^j \mathbf{u}) + P(\rho)_{x_j} \\ \qquad = \mu \Delta u^j + \lambda \mathbf{div} \mathbf{u}_{x_j} + \rho f^j \\ \\ (\rho, \mathbf{u})\big|_{t=0} = (\rho_0, \mathbf{u}_0), \\ \\ \mathbf{x} \in \mathbb{R}^n, \quad n = 2, 3. \end{cases}$$

Hipóteses sobre a pressão P:

$\mathbf{P} \in \mathbf{C^2}([\mathbf{0}, \bar\rho])$, $\bar\rho > \mathbf{0}$, $\mathbf{P}(\mathbf{0}) = \mathbf{0}$, **existe** $\tilde\rho \in (\mathbf{0}, \bar\rho)$ **tal que** $\mathbf{P}'(\tilde\rho) > \mathbf{0}$
e $(\rho - \tilde\rho)[\mathbf{P}(\rho) - \mathbf{P}(\tilde\rho)] > \mathbf{0}$ **se** $\rho \neq \tilde\rho$, $\rho \in [\mathbf{0}, \bar\rho]$.

Força externa ("energia pequena"):

$$\sup_{\mathbf{t} \geq \mathbf{0}} ||\mathbf{f}(\cdot, \mathbf{t})||_{\mathbf{p}} + \int_{\mathbf{0}}^{\infty} \big(||\mathbf{f}(\cdot, \mathbf{t})||_{\mathbf{2}} + ||\mathbf{f}(\cdot, \mathbf{t})||_{\mathbf{2}}^{\mathbf{2}} + \sigma(\mathbf{t})^\gamma ||\mathbf{f_t}(\cdot, \mathbf{t})||_{\mathbf{2}}^{\mathbf{2}} \big)\, \mathbf{dt} \ \leq \ \mathbf{C_f} \ << \ \mathbf{1}$$

onde

$$\boxed{\mathbf{n} < \mathbf{p} \approx \mathbf{n}}, \qquad \boxed{\sigma(\mathbf{t}) := \min\{\mathbf{1}, \mathbf{t}\}} \qquad \mathbf{e} \qquad \gamma = \begin{cases} 3, & \mathbf{n} = \mathbf{2} \\ 5, & \mathbf{n} = \mathbf{3}. \end{cases}$$

Viscosidades $\lambda, \quad \mu$**:**

$$\begin{cases} \lambda,\, \mu > \mathbf{0}\,, & \mathbf{n} = \mathbf{2} \\ \mathbf{0} < \lambda < \frac{\mathbf{5}}{\mathbf{4}}\mu\,, & \mathbf{n} = \mathbf{3} \end{cases}$$

Dados iniciais **("energia pequena"):**

$$\int_{\mathbb{R}^{\mathbf{n}}} \left[\ \rho_{\mathbf{0}}|\mathbf{u_0}|^{\mathbf{2}} + |\rho_{\mathbf{0}} - \tilde{\rho}|^{\mathbf{2}}\ \right] \mathbf{dx} \ \leq \mathbf{C_0} \ << \ \mathbf{1}$$

$$\mathbf{0} \ \leq \ \rho_{\mathbf{0}} \ \leq \ ||\rho - \mathbf{0}||_{\infty} \ < \ \bar{\rho} \qquad \textbf{a.e.}$$

Theorem 1. EXISTÊNCIA DE SOLUÇÃO (D. HOFF, 1995, 1997, 2005).

Sob as hipóteses acima e mais algumas (técnicas), existe uma solução fraca global $(\rho, \mathbf{u})$ ***satisfazendo***

- ***(estimativa de energia)***

$$\sup_{\mathbf{t}>0}\int_{\mathbb{R}^\mathbf{n}} \left[\ \rho(\mathbf{x},\mathbf{t})|\mathbf{u}(\mathbf{x},\mathbf{t})|^2 + |\rho(\mathbf{x},\mathbf{t}) - \tilde{\rho}|^2 + \sigma(\mathbf{t})|\nabla\mathbf{u}(\mathbf{x},\mathbf{t})|^2\right] \mathbf{dx} + \int_0^\infty\int_{\mathbb{R}^\mathbf{n}} \left[\ |\nabla\mathbf{u}|^2 + \sigma(\mathbf{t})^\mathbf{n}|\nabla\dot{\mathbf{u}}|^2\ \right]\mathbf{dxdt} \ \leq\ \mathbf{C}\,(\mathbf{C_0} + \mathbf{C_f})^\theta < \infty$$

$\dot{\mathbf{u}}$ ***é a 'derivada convectiva' ('material'):*** $\dot{\mathbf{u}}^\mathbf{j} := \mathbf{u}^\mathbf{j}_\mathbf{t} + \mathbf{u}\cdot\nabla\mathbf{u}^\mathbf{j}$.

- $$\int_0^\infty\int_{\mathbb{R}^\mathbf{n}} \sigma(\mathbf{t})\rho|\dot{\mathbf{u}}|^2\,\mathbf{dxdt} \leq \mathbf{C}\,(\mathbf{C_0} + \mathbf{C_f})^\theta \qquad \underline{\textit{se}\quad \inf\rho_0 > 0}$$

- $$\boxed{\mathbf{C}^{-1}\inf\rho_0\ \leq\ \rho\ \leq\ \bar{\rho}} \qquad \textit{a.e.} \qquad (\ \mathbf{C} > 0\ )$$

- ***<u>Hölder continuidade</u>: Para qualquer*** $\tau > 0$, ***temos que*** $\mathbf{u}$,

$$\boxed{\mathbf{F} := (\lambda + \mu)\,div\,\mathbf{u}\ -\ (\,\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho})\,)}$$

('effective viscous flux')

e $\omega^{\mathbf{j},\mathbf{k}} = \mathbf{u}^\mathbf{j}_{\mathbf{x_k}} - \mathbf{u}^\mathbf{k}_{\mathbf{x_j}}$ ***(matriz de vorticidade) são Hölder contínuas em*** $\mathbb{R}^\mathbf{n}\times[\tau,\infty)$.

- ***A solução*** $(\rho, \mathbf{u})$ ***é obtida como limite de funções*** $(\rho^\delta, \mathbf{u}^\delta)$ ***satisfazendo as estimativas acima com constantes independentes de*** δ, $\rho^\delta_0 = \mathbf{j}_\delta * \rho_0 + \delta$ $(\Rightarrow \inf\rho^\delta_0 > 0)$.

Observação: ***Vale a identidade***

$$\mathbf{\Delta u^j} = (\lambda+\mu)^{-1}\,\mathbf{F_{x_j}} + (\omega^{\mathbf{j},\mathbf{k}})_{\mathbf{x_k}} + (\lambda+\mu)^{-1}\,(\,\mathbf{P}(\rho)-\mathbf{P}(\tilde\rho)\,)_{\mathbf{x_j}}$$

e então podemos escrever

$$\boxed{\mathbf{u} = \mathbf{u_{F,\omega}} + \mathbf{u_P}}$$

onde $\mathbf{u_{F,\omega}}$, $\mathbf{u_P}$ ***são definidos de forma que***

$$\mathbf{\Delta u^j_{F,\omega}} = (\lambda+\mu)^{-1}\,\mathbf{F_{x_j}} + (\omega^{\mathbf{j},\mathbf{k}})_{\mathbf{x_k}}, \qquad \mathbf{\Delta u^j_P} = (\lambda+\mu)^{-1}\,(\,\mathbf{P}(\rho)-\mathbf{P}(\tilde\rho)\,)_{\mathbf{x_j}}$$

$$\mathbf{u_p} = (\lambda+\mu)^{-1}\,\nabla\mathbf{\Gamma} * (\mathbf{P}(\rho)-\mathbf{P}(\tilde\rho))$$

Daí segue-se que para cada $\mathbf{t} > \mathbf{0}$

$$\mathbf{u_P}(\cdot,\mathbf{t}) \in \mathbf{LL} \qquad (\ \mathbf{P}(\rho(\cdot,\mathbf{t})) - \mathbf{P}(\tilde\rho) \ \in\ \mathbf{L^2}\cap\mathbf{L^\infty}\ )$$

$$\mathbf{u_{F,\omega}}(\cdot,\mathbf{t}) \in \mathbf{Lip} \qquad (\ \mathbf{F}(\cdot,\mathbf{t}) \text{ e } \omega(\cdot,\mathbf{t}) \text{ são Hölder contínuas})$$

Além disso,

$$\langle \mathbf{u_P}(\cdot,\mathbf{t})\rangle_{\mathbf{LL}} \ \le\ \mathbf{C}\,||\mathbf{P}(\rho(\cdot,\mathbf{t})) - \mathbf{P}(\tilde\rho)||_{\mathbf{L^2\cap L^\infty}} \ \in\ \mathbf{L^1_{loc}}([\mathbf{0},\infty)\,).$$

Questão: $\langle \mathbf{u_{F,\omega}}(\cdot,\mathbf{t})\rangle_{\mathbf{Lip}} \ \in\ \mathbf{L^1_{loc}}([\mathbf{0},\infty))$ **?**

$$\left\langle\, \mathbf{u_{F,\,\omega}}\,\right\rangle_{\mathbf{Lip}} \le \mathbf{C}\,||\nabla\mathbf{F} + \nabla\omega||_{\mathbf{p}}$$

Em termos de F

$$\boxed{\mathbf{F} := (\lambda + \mu)\mathbf{div}\,\mathbf{u} \; - \; (\,\mathbf{P}(\rho) - \mathbf{P}(\tilde{\rho})\,),}$$

da 'vorticidade' $\omega^{\mathbf{j},\mathbf{k}} = \mathbf{u}^{\mathbf{j}}_{\mathbf{x_k}} - \mathbf{u}^{\mathbf{k}}_{\mathbf{x_j}}$ **e da derivada convectiva** $\dot{\mathbf{u}}^{\mathbf{j}} = \mathbf{u}^{\mathbf{j}}_{\mathbf{t}} + \mathbf{u}\cdot\nabla\mathbf{u}^{\mathbf{j}}$, **a equação de Navier-Stokes (equação de momentum)**

$$\boxed{(\rho\mathbf{u}^{\mathbf{j}})_{\mathbf{t}} + \mathbf{div}(\rho\mathbf{u}^{\mathbf{j}}\mathbf{u}) + \mathbf{P}(\rho)_{\mathbf{x_j}} = \mu\boldsymbol{\Delta}\mathbf{u}^{\mathbf{j}} + \lambda\mathbf{div}\mathbf{u}_{\mathbf{x_j}} + \rho\mathbf{f}^{\mathbf{j}}}$$

se escreve como

$$\boxed{\rho\dot{\mathbf{u}}^{\mathbf{j}} = \mathbf{F}_{\mathbf{x_j}} + \mu\omega^{\mathbf{j},\mathbf{k}}_{\mathbf{x_k}} + \rho\mathbf{f}^{\mathbf{j}}}\,.$$

Daí tomando div e rot obtemos as equações

$$\boxed{\boldsymbol{\Delta}\mathbf{F} = \mathbf{div}(\rho\dot{\mathbf{u}} - \rho\mathbf{f})} \qquad \boxed{\mu\boldsymbol{\Delta}\omega^{\mathbf{j},\mathbf{k}} = \mathbf{rot}(\rho\dot{\mathbf{u}} - \rho\mathbf{f})^{\;\mathbf{j},\mathbf{k}}}$$

Então

$$||\nabla\mathbf{F}||_{\mathbf{p}}\,, \quad ||\mathbf{D}\omega||_{\mathbf{p}} \;\leq\; \mathbf{C}\,(||\rho\dot{\mathbf{u}}||_{\mathbf{p}} + ||\rho\mathbf{f}||_{\mathbf{p}})\,.$$

$$||\,(\rho\dot{\mathbf{u}})(\cdot,\mathbf{t})\,||_{\mathbf{p}} \;\in\; \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\,[\mathbf{0},\infty)\,) \;\; ??$$

Suponhamos que $\inf \rho_0 > 0$.

$$||\dot{\mathbf{u}}||_p \leq C\, ||\dot{\mathbf{u}}||_2^{1-\kappa} ||\nabla \dot{\mathbf{u}}||_2^{\kappa}$$

$$\boxed{\kappa = n(\tfrac{1}{2} - \tfrac{1}{p})}$$

$$\int_0^1 ||\dot{\mathbf{u}}||_2^{1-\kappa} ||\nabla \dot{\mathbf{u}}||_2^{\kappa} dt$$

$$= \int_0^1 \left(t \int |\dot{\mathbf{u}}|^2 dx \right)^{(1-\kappa)/2} \left(t^2 \int |\nabla \dot{\mathbf{u}}|^2 dx \right)^{\kappa/2} t^{(-1-\kappa)/2}\, dt$$

$$\leq \left(\int_0^1 \left(t \int |\dot{\mathbf{u}}|^2 dx \right)^{1-\kappa} \left(t^2 \int |\nabla \dot{\mathbf{u}}|^2 dx \right)^{\kappa} dt \right)^{1/2} \left(\int_0^1 t^{-1-\kappa} dt \right)^{1/2}$$

$$\leq \; C\,(C_0 + C_f)^{\theta} \underbrace{\left(\int_0^1 t^{-1-\kappa} dt \right)^{1/2}}_{\textbf{infinita!!}}$$

CORREÇÃO:

$$\int_0^1 ||\dot{\mathbf{u}}||_2^{1-\kappa} ||\nabla \dot{\mathbf{u}}||_2^{\kappa} dt$$

$$= \int_0^1 \left(t^{1-s} \int |\dot{\mathbf{u}}|^2 dx \right)^{(1-\kappa)/2} \left(t^{2-s} \int |\nabla \dot{\mathbf{u}}|^2 dx \right)^{\kappa/2} t^{(s-1-\kappa)/2dt}$$

$$\leq \left(\int_0^1 \left(t^{1-s} \int |\dot{\mathbf{u}}|^2 dx \right)^{1-\kappa} \left(t^{2-s} \int |\nabla \dot{\mathbf{u}}|^2 dx \right)^{\kappa} dt \right)^{1/2} \left(\int_0^1 t^{s-1-\kappa} dt \right)^{1/2}$$

$$\leq \; c\,(C_0 + C_f)^{\theta} \quad 1^{(s-\kappa)/2} :$$

<u>**finito,**</u> **se** $s > \begin{cases} 0, & n = 2 \\ 1/2, & n = 3 \end{cases}$ **e** $u_0 \in H^s(\mathbb{R}^n)$, **devido a [Hoff] e**

<u>**Observação:**</u> $\inf\limits_{p>n} \kappa = \kappa\big|_{p=n} = n(\frac{1}{2} - \frac{1}{n}) = \begin{cases} 0, & n = 2 \\ 1/2, & n = 3. \end{cases}$

Teorema [Hoff-Santos]. ***Seja*** V ***um aberto do*** $\mathbb{R}^n$ ***e suponhamos que*** $\inf \rho_0|_{\mathbf{V}} \geq \underline{\rho} > \mathbf{0}$***. Então, sob as hipóteses acima, temos que:***

a) ***Para todo*** $\mathbf{x_0} \in \mathbf{V}$ ***existe uma única aplicação*** $\mathbf{X}(\cdot, \mathbf{x_0}) \in \mathbf{C}([\mathbf{0}, \infty); \mathbb{R}^{\mathbf{n}}) \cap \mathbf{C^1}((\mathbf{0}, \infty); \mathbb{R}^{\mathbf{n}})$ ***satisfazendo***

$$\mathbf{X}(\mathbf{t}, \mathbf{x_0}) = \mathbf{x_0} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{u}(\mathbf{X}(\tau, \mathbf{x_0}), \tau) \mathbf{d}\tau;$$

b) ***Para cada*** $\mathbf{t} > \mathbf{0}$***,*** $\mathbf{V^t} \equiv \mathbf{X}(\mathbf{t}, \cdot)\mathbf{V}$ ***é um aberto e a aplicação*** $\mathbf{x_0} \mapsto \mathbf{X}(\mathbf{t}, \mathbf{x_0})$ ***é um homeomorfismo de*** $\mathbf{V}$ ***sobre*** $\mathbf{V^t}$***;***

c) ***Se*** $\mathbf{T} > \mathbf{0}$ ***e*** $\mathbf{K}$ ***é um compacto contido em*** $\mathbf{V}$***, então para*** $\mathbf{0} \leq \mathbf{t_1}$***,*** $\mathbf{t_2} \leq \mathbf{T}$***, a aplicação*** $\mathbf{X}(\mathbf{t_1}, \mathbf{y}) \to \mathbf{X}(\mathbf{t_2}, \mathbf{y})$ ***é Hölder contínua de*** $\mathbf{K^{t_1}} \equiv \mathbf{X}(\mathbf{t_1}, \cdot)\mathbf{K}$ ***sobre*** $\mathbf{K^{t_2}}$ ***com expoente*** $\mathbf{e}^{-\mathbf{LT}^{\gamma}}$***, onde*** γ ***depende de*** $\mathbf{n}$ ***e*** $\mathbf{s}$***, e*** $\mathbf{L}$ ***depende de*** $\underline{\rho}$***, da distância de*** $\mathbf{K}$ ***a*** $\partial\mathbf{V}$***, de*** $\mathbf{s}$***, ... ;***

d) ***Seja*** $\mathcal{M} \subseteq \mathbf{K} \subseteq \mathbf{V}$ ***uma variedade de classe*** $\mathbf{C}^{\alpha}$ ***e dimensão*** $\mathbf{d}$***,*** $\alpha \in [\mathbf{0}, \mathbf{1})$***,*** $\mathbf{1} \leq \mathbf{d} \leq \mathbf{n} - \mathbf{1}$***. Então para cada*** $\mathbf{t} > 0$***,*** $\mathcal{M}^{\mathbf{t}} \equiv \mathbf{X}(\mathbf{t}, \cdot)\mathcal{M}$ ***é uma variedade de classe*** $\mathbf{C}^{\beta}$ ***e dimensão*** $\mathbf{d}$***, onde*** $\beta = \beta \mathbf{e}^{-\mathbf{Lt}^{\gamma}}$***;***

e) ***Existe*** $\underset{\sim}{\rho} > 0$ ***tal que para todo*** $\mathbf{t} > \mathbf{0}$***,***

$$\inf \rho(\cdot, \mathbf{t})|_{\mathbf{V^t}} \geq \underset{\sim}{\rho}.$$

3 Exemplo

$$\mathbf{u} = K(t) * \mathbf{u}_0$$

em $\mathbb{R}^3$**, onde** $\mathbf{u}_0(\mathbf{x}) \equiv \varphi(|\mathbf{x}|)\mathbf{x}|\mathbf{x}|^{-p}$**,** $2 < p < 5/2$**,** $\varphi \in C_0^\infty(\mathbb{R})$**,** $0 \le \varphi \le 1$**,** $\varphi(r) = 1$ **se** $|r| \le 1$**.**

$\mathbf{u}_0 \in H^s(\mathbb{R}^3)$***, qualquer que seja*** $s \in (0, \frac{5}{2} - p)$***, e existe uma quantidade infinita de trajetórias*** $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}(t), t)$ ***que tendem a zero quando*** $t \to 0^+$***.***

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