Nome: ______________________________ RA: ____________________

Métodos Matemáticos I (F520/MS550) - Prova 2

30 de junho de 2010

1. Considere a equação de Bessel $y''(x) + \frac{1}{x}y'(x) + (1 - \frac{\nu^2}{x^2})y(x) = 0$ e suas soluções de primeira espécie J_ν, $\nu \geq 0$.

   (a) (2 pontos) Mostre que J_0 possui um número infinito de zeros no intervalo $(0, \infty)$ (não vale usar a expressão assintótica de J_0 sem deduzí-la). Dica: transformação de variáveis $y(x) = x^{-1/2}\, u(x)$.

   (b) (2 pontos) Deduza uma fórmula para $J_{1/2}$ em termos de funções elementares. Faça o mesmo para $J_{3/2}$.

2. (2 pontos) Dado n inteiro não-negativo, seja P_n o n-ésimo polinômio de Legendre. Mostre que:

$$\int_{-1}^{1} x^n P_n(x) dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

   Dica: $\int_{-1}^{1}(1-x^2)^n dx = \sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}$.

3. Considere a equação diferencial

$$y''(x) + y'(x) = f(x),$$

   sujeita às condições de contorno $y(0) = y(\infty) = 0$.

   (a) (2 pontos) Ache a função de Green para tal problema.

   (b) (2 pontos) Resolva este problema para $f(x) = e^{-x}$.

Formulário

$$f(z_0)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-z_0}\,dz,\qquad \operatorname{Res}f(z_0)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right],$$

$$y''+\left(\frac{1-\alpha-\alpha'}{t}+\frac{1-\gamma-\gamma'}{t-1}\right)y'+\left[\frac{\alpha\alpha'}{t^2}+\frac{\gamma\gamma'}{(t-1)^2}+\frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{t(t-1)}\right]y=0\quad(\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1),$$

$$z(1-z)y''+[c-(a+b+1)z]y'-aby=0,\qquad zy''+(c-z)y'-ay=0,\qquad (a)_n=a(a+1)\cdots(a+n-1),$$

$${}_2F_1(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\qquad {}_1F_1(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!},\qquad {}_2F_1(a,b,c;z)=(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b,c;z),$$

$${}_2F_1(a,b,c;z)=(1-z)^{-a}\,{}_2F_1(a,c-b,c;z/(z-1)),\qquad {}_2F_1(a,b,c;z)=(1-z)^{-b}\,{}_2F_1(c-a,b,c;z/(z-1)),$$

$${}_2F_1(a,b,c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,\qquad \frac{d}{dz}[\,{}_2F_1(a,b,c;z)]=\frac{ab}{c}\,{}_2F_1(a+1,b+1,c+1;z),$$

$$\frac{d^n}{dz^n}\left[z^{a-1+n}\,{}_2F_1(a,b,c;z)\right]=(a)_n z^{a-1}\,{}_2F_1(a+n,b,c;z),\qquad U(a,c;z)=\frac{\Gamma(1-c)}{\Gamma(1+a-c)}\,{}_1F_1(a,c;z)+\frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\,{}_1F_1(a-c+1,2-c;z),$$

$${}_1F_1(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \mathrm{e}^{zt}t^{a-1}(1-t)^{c-a-1}\,dt,\qquad U(a,c;z)=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{c-a-1}\,dt,\qquad (1-x)^{-\alpha}=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}x^k$$

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\,dt,\qquad \Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)},\qquad \frac{1}{\Gamma(z)}=z\mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\mathrm{e}^{-z/n},\qquad \gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$$

$$\Gamma(1/2)=\sqrt{\pi},\qquad n!=\Gamma(n+1),\qquad (2n+1)!!=(2n+1)(2n-1)(2n-3)\cdots 1,\qquad (2n)!!=(2n)(2n-2)(2n-4)\cdots 2,$$

$$\Gamma(z+1)=z\Gamma(z),\qquad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\qquad B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}=2\int_0^{\pi/2}\cos^{2z-1}\theta\sin^{2w-1}\theta\,d\theta=\int_0^1 t^{z-1}(1-t)^{w-1}\,dt$$

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\Gamma(2z),\qquad \psi_n(z)=\frac{d^{n+1}}{dz^{n+1}}\ln\Gamma(z),\qquad \psi_0(z+1)=-\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{z+k}\right),\qquad -\gamma=\int_0^\infty \mathrm{e}^{-t}\ln t\,dt,$$

$$\psi_n(z+1)=(-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{1}{(z+k)^{n+1}},\qquad \zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s},\qquad \operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,dt,\qquad \operatorname{Ei}(x)=\int_{-\infty}^{x}\frac{\mathrm{e}^t}{t}\,dt=\gamma+\ln x+\sum_{k=1}^{\infty}\frac{x^k}{k!k}$$

$$z^2y''+zy'+(z^2-\nu^2)y=0,\qquad J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu},\qquad Y_\nu(z)=\frac{\cos\nu\pi J_\nu(z)-J_{-\nu}(z)}{\sin\nu\pi}$$

$$\mathrm{e}^{z(t-1/t)/2}=\sum_{n=-\infty}^{+\infty}J_n(z)t^n,\qquad J_{n-1}(z)+J_{n+1}(z)=\frac{2n}{z}J_n(z),\qquad J_{n-1}(z)-J_{n+1}(z)=2J_n'(z)$$

$$J_{n+1}(z)=-z^n\frac{d}{dz}(z^{-n}J_n(z)),\qquad J_{n-1}(z)=x^{-n}\frac{d}{dz}(z^nJ_n(z)),\qquad J_n(z)=(-1)^nz^n\left(\frac{1}{z}\frac{d}{dz}\right)^nJ_0(z)$$

$$J_n(z)=\frac{1}{\pi}\int_0^\pi\cos(n\theta-z\sin\theta)\,d\theta,\qquad I_\nu(z)=i^{-\nu}J_\nu(iz),\qquad j_n(z)=\sqrt{\frac{\pi}{2z}}J_{n+1/2}(z),\qquad y_n(z)=\sqrt{\frac{\pi}{2z}}Y_{n+1/2}(z)$$

$$\int_0^a J_\nu(\alpha_{\nu m}x/a)J_\nu(\alpha_{\nu n}x/a)x\,dx=(a^2/2)[J_{\nu+1}(\alpha_{\nu n})]^2\delta_{nm},\qquad P_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!x^{n-2k}}{2^nk!(n-k)!(n-2k)!},\qquad P_\nu(x)=\,{}_2F_1(-\nu,\nu+1,1;(1-x)/2)$$

$$(1-2xt+t^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n,\qquad P_n(x)=\frac{1}{2^nn!}\frac{d^n}{dx^n}[(x^2-1)^n],\qquad P_n(x)=\frac{1}{2\pi i}\oint_C\frac{(u^2-1)^n\,du}{2^n(x-u)^{n+1}}$$

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{mn},\qquad (2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x),\qquad (2n+1)P_n(x)=P_{n+1}'(x)-P_{n-1}'(x)$$

$$Q_\nu(x)=\frac{\sqrt{\pi}\Gamma(\nu+1)}{\Gamma(\nu+3/2)(2x)^{\nu+1}}\,{}_2F_1(1+\nu/2,(\nu+1)/2,\nu+3/2;1/x^2),\qquad Q_n(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(y)\,dy}{x-y}$$

$$P_\nu^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\nu(x),\qquad Q_\nu^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}Q_\nu(x),\qquad (1-x^2)y''-2xy'+[\nu(\nu+1)-m^2/(1-x^2)]y=0$$

1. (a) Fazendo $y(x) = \dfrac{u(x)}{\sqrt{x}}$ (dica), temos que

$$y'' + \frac{1}{x}y' + y = 0 \Leftrightarrow u'' + \left(1 + \frac{1}{4x^2}\right)u = 0 \qquad (1)$$

Comparando (1) com a eq $v'' + v = 0$, (2)

vemos que $1 + \dfrac{1}{4x^2} \geq 1 \xrightarrow[\text{de Sturm}]{\text{teorema de comparação}}$ u tem pelo menos um zero entre dois zeros consecutivos de uma solução $v(x)$ de (2).

Como $v(x) = \operatorname{sen} x$ é solução de (2) e $\operatorname{sen} x$ tem infinitos zeros em $(0, \infty)$, temos que $J_0(x) = \dfrac{u(x)}{\sqrt{x}}$, u solução de (1), tem infinitos zeros em $(0, \infty)$.

(b) Da expressão em série para $J_\nu(x)$, temos

$$J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m + \frac{1}{2} + 1)} \left(\frac{x}{2}\right)^{2m + \frac{1}{2}}$$

$$= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\frac{1}{2})(m-\frac{1}{2})(m-\frac{3}{2})\cdots(\frac{3}{2})(\frac{1}{2})\underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}}} \left(\frac{x}{2}\right)^{2m+1}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{[(2m)(2m-2)\cdots 2]\,[(2m+1)(2m-1)\cdots 3\cdot 1]}\, x^{2m+1}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\, x^{2m+1} = \sqrt{\frac{2}{\pi}}\, \frac{\operatorname{sen} x}{\sqrt{x}} \quad \rfloor_a$$

Por outro lado, $J_{n+1}(x) = -x^{n}\dfrac{d}{dx}\left(x^{-n}J_n(x)\right)$.

Logo, $J_{3/2}(x) = -x^{1/2}\dfrac{d}{dx}\left(x^{-1/2}J_{1/2}(x)\right) =$

$$= -\sqrt{\frac{2}{\pi}}\,\sqrt{x}\,\frac{d}{dx}\left(\frac{\operatorname{sen} x}{x}\right) =$$

$$= -\sqrt{\frac{2}{\pi}}\,\sqrt{x}\left[\frac{\cos x}{x} - \frac{\operatorname{sen} x}{x^2}\right)$$

Assim, $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\dfrac{\operatorname{sen} x}{x} - \cos x\right)$

$\rfloor_L$.

(2) Da fórmula de Rodrigues,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n,$$

temos

integração por partes

$$\int_{-1}^{1} x^n P_n(x)\,dx = \frac{1}{2^n n!} \int_{-1}^{1} x^n \frac{d^n}{dx^n}(x^2-1)^n =$$

0 (sempre sobra um termo em (x^2-1).

$$= \frac{1}{2^n n!}\left\{ \cancel{\left[x^n \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \right]_{-1}^{1}} - n\int_{-1}^{1} x^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx \right\}$$

$$= \cdots = \frac{1}{2^n n!}(-1)^n n! \int_{-1}^{1} (x^2-1)^n dx$$

$$= \frac{1}{2^n} \int_{-1}^{1} (1-x^2)^n dx \overset{\text{dica}}{=}$$

$$= \frac{\sqrt{\pi}}{2^n} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}$$

$$= \frac{\sqrt{\pi}}{2^n} \frac{n!}{(n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2})\cdots(\frac{1}{2})\Gamma(\frac{1}{2})} =$$

$$= \frac{\cancel{\sqrt{\pi}}}{\cancel{\Gamma(\frac{1}{2})}} \frac{2\,n!}{(2n+1)(2n-1)(2n-3)\cdots 3\cdot 1} =$$

$$= 2\,n!\,\frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n+1)!} = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

(3) $y''(x) + y'(x) = f(x)$, $0 < x < \infty$

$y(0) = y(\infty) = 0$

Eq. homogênea: $y'' + y' = 0 \Rightarrow v' + v = 0$, $v = y'$

$\Rightarrow v(x) = \text{cte}\, e^{-x}$

$\Rightarrow y(x) = a e^{-x} + b$, a, b constantes

Assim, dado $\xi \in (0, \infty)$: $G(x,\xi) = \begin{cases} a e^{-x} + b, & 0 < x < \xi \\ c e^{-x} + d, & \xi < x < \infty \end{cases}$

$$\left.\begin{array}{l} G(0,\xi) = 0 \Rightarrow a + b = 0 \\ G(\infty,\xi) = 0 \Rightarrow d = 0 \end{array}\right\} \Rightarrow G(x,\xi) = \begin{cases} a(e^{-x} - 1), & 0 < x < \xi \\ c e^{-x}, & \xi < x \end{cases}$$

continuidade em $x = \xi$: $a(e^{-\xi} - 1) = c e^{-\xi}$

pulo de G' em $x = \xi$ $\quad -c e^{-\xi} + a e^{-\xi} = 1$

$\Rightarrow a = 1$ e $c = 1 - e^{\xi}$

Logo: $G(x,\xi) = \begin{cases} e^{-x} - 1, & 0 < x < \xi \\ (1 - e^{\xi}) e^{-x}, & x > \xi \end{cases}$

(b) A solução é $y(x) = \int_0^\infty G(x,\xi) f(\xi)\, d\xi = \int_0^x (1 - e^{\xi}) e^{-x} e^{-\xi}\, d\xi +$

$+ \int_x^\infty (e^{-x} - 1) e^{-\xi}\, d\xi = e^{-x} \int_0^x (e^{-\xi} - 1)\, d\xi + (e^{-x} - 1) \int_x^\infty e^{-\xi}\, d\xi =$

$= e^{-x} \left[-e^{-\xi} - \xi \right]_0^x + (e^{-x} - 1) \left[-e^{-\xi} \right]_x^\infty = e^{-x} \left[-e^{-x} - x + 1 \right] + (e^{-x} - 1) e^{-x} =$

$\Rightarrow \boxed{y(x) = -x e^{-x}}$