Exercícios do livro do Spivak – Calculus Capítulo 1

1. Prove the following:

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.
(ii) $x^2 - y^2 = (x - y)(x + y)$.
(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.
(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
(v) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.
(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. (There is a particularly easy way to do this, using (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.)

2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned} x^2 &= xy, \\ x^2 - y^2 &= xy - y^2, \\ (x + y)(x - y) &= y(x - y), \\ x + y &= y, \\ 2y &= y, \\ 2 &= 1. \end{aligned}$$

3. Prove the following:

(i) $\dfrac{a}{b} = \dfrac{ac}{bc}$, if $b, c \neq 0$.

6. (a) Prove that if $0 \leq x < y$, then $x^n < y^n$, $n = 1, 2, 3, \ldots$.
(b) Prove that if $x < y$ and n is odd, then $x^n < y^n$.
(c) Prove that if $x^n = y^n$ and n is odd, then $x = y$.
(d) Prove that if $x^n = y^n$ and n is even, then $x = y$ or $x = -y$.

7. Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a + b}{2} < b.$$

Notice that the inequality $\sqrt{ab} \leq (a + b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem 2-22.

***8.** Although the basic properties of inequalities were stated in terms of the collection P of all positive numbers, and $<$ was defined in terms of P, this procedure can be reversed. Suppose that P10–P12 are replaced by

(P′10) For any numbers a and b one, and only one, of the following holds:

(i) $a = b$,
(ii) $a < b$,
(iii) $b < a$.

(P′11) For any numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$.
(P′12) For any numbers a, b, and c, if $a < b$, then $a + c < b + c$.
(P′13) For any numbers a, b, and c, if $a < b$ and $0 < c$, then $ac < bc$.

Show that P10–P12 can then be deduced as theorems.

***15.** Prove that if x and y are not both 0, then

$$x^2 + xy + y^2 > 0,$$
$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0.$$

Hint: Use Problem 1.

18. (a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b + \sqrt{b^2 - 4c}}{2}, \qquad \frac{-b - \sqrt{b^2 - 4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.

(b) Suppose that $b^2 - 4c < 0$. Show that there are no numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.

(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.

(d) For which numbers α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?

(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

19. The fact that $a^2 \geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\,\sqrt{y_1{}^2 + y_2{}^2}.$$

(A more general form occurs in Problem 2-21.) The three proofs of the Schwarz inequality outlined below have only one thing in common—their reliance on the fact that $a^2 \geq 0$ for all a.

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number λ, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$\begin{aligned} 0 &< (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2(y_1{}^2 + y_2{}^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1{}^2 + x_2{}^2). \end{aligned}$$

Using Problem 18, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy \leq x^2+y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{x_1{}^2 + x_2{}^2}}, \qquad y = \frac{y_i}{\sqrt{y_1{}^2 + y_2{}^2}},$$

first for $i = 1$ and then for $i = 2$.

(c) Prove the Schwarz inequality by first proving that

$$(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

25. Suppose that we interpret "number" to mean either 0 or 1, and $+$ and $\cdot$ to be the operations defined by the following two tables.

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Check that properties P1–P9 all hold, even though $1 + 1 = 0$.

Exercícios do livro do Spivak – Calculus Capítulo 2

1. Prove the following formulas by induction.

(i) $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

3. If $0 \le k \le n$, the "binomial coefficient" $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{if } k \neq 0, n$$

$$\binom{n}{0} = \binom{n}{n} = 1.$$ (This becomes a special case of the first formula if we define $0! = 1$.)

(a) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(The proof does not require an induction argument.)

This relation gives rise to the following configuration, known as "Pascal's triangle"—a number not on one of the sides is the sum of the two numbers above it; the binomial coefficient $\binom{n}{k}$ is the $(k+1)$st number in the $(n+1)$st row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(b) Notice that all the numbers in Pascal's triangle are natural numbers. Use part (a) to prove by induction that $\binom{n}{k}$ is always a natural number. (Your entire proof by induction will, in a sense, be summed up in a glance by Pascal's triangle.)

(c) Give another proof that $\binom{n}{k}$ is a natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers each chosen from $1, \ldots, n$.

(d) Prove the "binomial theorem": If a and b are any numbers and n is a natural number, then

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$
$$= \sum_{j=0}^{n} \binom{n}{j} a^{n-j}b^j.$$

(e) Prove that

(i) $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \cdots + \binom{n}{n} = 2^n.$

(ii) $\displaystyle\sum_{j=0}^{n} (-1)^j \binom{n}{j} = \binom{n}{0} - \binom{n}{1} + \cdots \pm \binom{n}{n} = 0.$

(iii) $\displaystyle\sum_{l \text{ odd}} \binom{n}{l} = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$

(iv) $\displaystyle\sum_{l \text{ even}} \binom{n}{l} = \binom{n}{0} + \binom{n}{2} + \cdots = 2^{n-1}.$

5. (a) Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$ (if $r = 1$, evaluating the sum certainly presents no problem).

(b) Derive this result by setting $S = 1 + r + \cdots + r^n$, multiplying this equation by r, and solving the two equations for S.

12. (a) If a is rational and b is irrational, is $a + b$ necessarily irrational? What if a and b are both irrational?

(b) If a is rational and b is irrational, is ab necessarily irrational? (Careful!)

(c) Is there a number a such that a^2 is irrational, but a^4 is rational?

(d) Are there two irrational numbers whose sum and product are both rational?

13. (a) Prove that $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$ are irrational. Hint: To treat $\sqrt{3}$, for example, use the fact that every integer is of the form $3n$ or $3n+1$ or $3n+2$. Why doesn't this proof work for $\sqrt{4}$?

(b) Prove that $\sqrt[3]{2}$ and $\sqrt[3]{3}$ are irrational.

20. The Fibonacci sequence $a_1, a_2, a_3, \ldots$ is defined as follows:

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 1, \\ a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3. \end{aligned}$$

This sequence, which begins 1, 1, 2, 3, 5, 8, ... , was discovered by Fibonacci (circa 1175–1250), in connection with a problem about rabbits. Fibonacci assumed that an initial pair of rabbits gave birth to one new pair of rabbits per month, and that after two months each new pair behaved similarly. The number a_n of pairs born in the nth month is $a_{n-1} + a_{n-2}$, because a pair of rabbits is born for each pair born the previous month, and moreover each pair born two months ago now gives birth to another pair. The number of interesting results about this sequence is truly amazing—there is even a Fibonacci Association which publishes a journal, *The Fibonacci Quarterly*. Prove that

$$a_n = \frac{\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

One way of deriving this astonishing formula is presented in Problem 24-15.

26. There is a puzzle consisting of three spindles, with n concentric rings of decreasing diameter stacked on the first (Figure 1). A ring at the top of a stack may be moved from one spindle to another spindle, provided that it is not placed on top of a smaller ring. For example, if the smallest ring is moved to spindle 2 and the next-smallest ring is moved to spindle 3, then the smallest ring may be moved to spindle 3 also, on top of the next-smallest. Prove that the entire stack of n rings can be moved onto spindle 3 in $2^n - 1$ moves, and that this cannot be done in fewer than $2^n - 1$ moves.

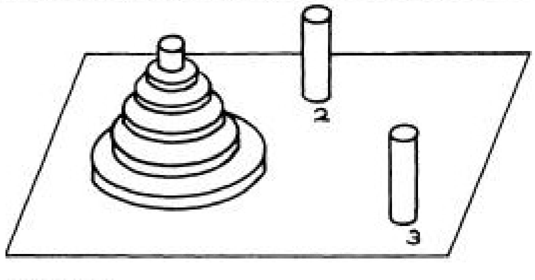

FIGURE 1

***27.** University B. once boasted 17 tenured professors of mathematics. Tradition prescribed that at their weekly luncheon meeting, faithfully attended by all 17, any members who had discovered an error in their published work should make an announcement of this fact, and promptly resign. Such an announcement had never actually been made, because no professor was aware of any errors in her or his work. This is not to say that no errors existed, however. In fact, over the years, in the work of every member of the department at least one error had been found, by some other member of the department. This error had been mentioned to all other members of the department, but the actual author of the error had been kept ignorant of the fact, to forestall any resignations.

One fateful year, the department was augmented by a visitor from another university, one Prof. X, who had come with hopes of being offered a permanent position at the end of the academic year. Naturally, he was apprised, by various members of the department, of the published errors which had been discovered. When the hoped-for appointment failed to materialize, Prof. X obtained his revenge at the last luncheon of the year. "I have enjoyed my visit here very much," he said, "but I feel that there is one thing that I have to tell you. At least one of you has published an incorrect result, which has been discovered by others in the department." What happened the next year?

****28.** After figuring out, or looking up, the answer to Problem 27, consider the following: Each member of the department already knew what Prof. X asserted, so how could his saying it change anything?