→ Exercícios retirados do Capítulo 2 do livro:
Sturm-Liouville theory and its applications
M.A. Al-Gwaiz
Springer, 2008.

EXERCISES

2.8 Prove that any nontrivial solution of $y'' + r(x)y = 0$ on a finite interval has at most a finite number of zeros. Suponha aqui que o intervalo é fechado (e que r é contínua). Dê um contra-exemplo para um intervalo não-fechado.

2.9 Prove that any nontrivial solution of the equation

$$y'' + \frac{k}{x^2}y = 0$$

on $(0, \infty)$ is oscillatory if, and only if, $k > 1/4$. Hint: Use the substitution $x = e^t$.

2.10 Use the result of Exercise 2.9 to conclude that any nontrivial solution of the equation $y'' + r(x)y = 0$ on $(0, \infty)$ has an infinite number of zeros if $r(x) \geq k/x^2$ for some $k > 1/4$, and only a finite number if $r(x) < 1/4x^2$.

2.11 Let φ be a nontrivial solution of $y'' + r(x)y = 0$ on $(0, \infty)$, where $r(x) > 0$. If $\varphi(x) > 0$ on $(0, a)$ for some positive number a, and if there is a point $x_0 \in (0, a)$ where $\varphi'(x_0) < 0$, prove that φ vanishes at some point $x_1 > x_0$.

2.12 Determine which equations have oscillatory solutions on $(0, \infty)$:

(a) $y'' + (\sin^2 x + 1)y = 0$.

(b) $y'' - x^2 y = 0$.

(c) $y'' + \dfrac{1}{x}y = 0$.

2.13 Find the general solution of Bessel's equation of order 1/2, and determine the zeros of each independent solution.

2.14 If $\lim_{x\to\infty} f(x) = 0$, prove that the solutions of $y'' + (1 + f(x))y = 0$ are oscillatory.

2.15 Prove that any solution of Airy's equation $y'' + xy = 0$ has an infinite number of zeros on $(0, \infty)$, and at most one zero on $(-\infty, 0)$.

2.16 Given $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + r$, prove the Lagrange identity

$$uLv - vLu = [p(uv' - vu')]'.$$

The integral of this identity,

$$\int_a^b (uLv - vLu)dx = [p(uv' - vu')]\Big|_a^b ,$$

is known as Green's formula.

2.17 Determine the eigenfunctions and eigenvalues of the differential operators

(a) $\frac{d^2}{dx^2} : C^2(0,\infty) \to C(0,\infty)$.

(b) $\frac{d^2}{dx^2} : \mathcal{L}^2(0,\infty) \cap C^2(0,\infty) \to \mathcal{L}^2(0,\infty)$.

2.18 Prove that the differential operator $-(d^2/dx^2) : \mathcal{L}^2(0,\pi) \cap C^2(0,\pi) \to \mathcal{L}^2(0,\pi)$ is self-adjoint under the boundary conditions $u(0) = u'(\pi) = 0$. Verify that its eigenvalues and eigenfunctions satisfy Theorem 2.14(iii).

2.19 Put each of the following differential operators in the form $p(d^2/dx^2) + p'(d/dx) + r$, with $p > 0$, by multiplying by a suitable weight function:

(a) $x^2\frac{d^2}{dx^2}, \quad x > 0,$

(b) $\frac{d^2}{dx^2} - x\frac{d}{dx}, \quad x \in \mathbb{R},$

(c) $\frac{d^2}{dx^2} - x^2\frac{d}{dx}, \quad x > 0,$

(d) $x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + (x^2 - \lambda), \quad x > 0.$

2.20 Determine the eigenvalues and eigenfunctions of $u'' + u = 0$ on $(0, l)$ subject to the boundary conditions $u(0) = 0$, $hu(l) + u'(l) = 0$, where $h \leq 0$.

2.21 Put the eigenvalue equation $u'' + 2u' + \lambda u = 0$ in the standard form $Lu + \lambda\rho u = 0$, where L is self-adjoint, then find its eigenvalues and eigenfunctions on $[0, 1]$ subject to the boundary conditions $u(0) = u(1) = 0$. Verify that the result agrees with Corollary 2.19.

2.22 Determine the eigenfunctions and eigenvalues of the equation $x^2u'' + \lambda u = 0$ on $[1, e]$, subject to the boundary conditions $u(1) = u(e) = 0$.

EXERCISES

2.23 Determine the eigenvalues and eigenfunctions of the boundary-value problem

$$u'' + \lambda u = 0, \qquad a \le x \le b,$$
$$u(a) = u(b) = 0.$$

2.24 Determine the eigenvalues and eigenfunctions of the equation $x^2u'' - xu' + \lambda u = 0$ on $(1, e)$ under the boundary conditions $u(1) = u(e) = 0$. Write the form of the orthogonality relation between the eigenfunctions.

2.25 Verify that $p(f'\bar{g} - f\bar{g}')|_a^b = 0$ if f and g satisfy the separated homogeneous boundary conditions (2.35) and (2.36).

2.26 Verify that $p(f'\bar{g} - f\bar{g}')|_a^b = 0$ if f and g satisfy the periodic conditions $u(a) = u(b)$, $u'(a) = u'(b)$, provided $p(a) = p(b)$.

2.27 Which of the following boundary conditions make $p(f'g - fg')|_a^b = 0$?

(a) $p(x) = 1,\ a \le x \le b,\ u(a) = u(b),\ u'(a) = u'(b)$.

(b) $p(x) = x,\ 0 < a \le x \le b,\ u(a) = u'(b) = 0$.

(c) $p(x) = \sin x,\ 0 \le x \le \pi/2,\ u(0) = 1,\ u(\pi/2) = 0$.

(d) $p(x) = e^{-x},\ 0 < x < 1,\ u(0) = u(1),\ u'(0) = u'(1)$.

(e) $p(x) = x^2,\ 0 < x < b,\ u(0) = u'(b),\ u(b) = u'(0)$.

(f) $p(x) = x^2,\ -1 < x < 1,\ u(-1) = u(1),\ u'(-1) = u'(1)$.

2.28 Which boundary conditions in Exercise 2.27 define a singular SL problem?

2.29 Determine the eigenvalues and eigenfunctions of the problem

$$[(x+3)^2 y']' + \lambda y = 0, \qquad -2 \le x \le 1,$$
$$y(-2) = y(1) = 0.$$

2.30 Let

$$u'' + \lambda u = 0, \qquad 0 \le x \le \pi,$$
$$u(0) = 0, \qquad 2u(\pi) - u'(\pi) = 0.$$

(a) Show how the positive eigenvalues λ may be determined. Are there any eigenvalues in $(-\infty, 0]$?

(b) What are the corresponding eigenfunctions?

2.31 Discuss the sequence of eigenvalues and eigenfunctions of the problem

$$\begin{aligned} u'' + \lambda u &= 0, \qquad 0 \leq x \leq l, \\ u'(0) &= u(0), \qquad u'(l) = 0. \end{aligned}$$

2.32 Let

$$\begin{aligned} (pu')' + ru + \lambda u = 0, \qquad a < x < b, \\ u(a) = u(b) = 0. \end{aligned}$$

If $p(x) \geq 0$ and $r(x) \leq c$, prove that $\lambda \geq -c$.

2.33 Using Equation (2.42), prove that the norm of the operator T is also given by

$$\|T\| = \sup_{\|u\|=1} |\langle Tu, u\rangle|,$$

where $u \in C([a,b])$. Hint: $|\langle Tu, u\rangle| \leq \|T\|$ by the CBS inequality and Equation (2.42). To prove the reverse inequality, first show that $2\,\mathrm{Re}\,\langle Tu, v\rangle \leq (\|u\|^2 + \|v\|^2) \sup |\langle Tu, u\rangle|$, then set $v = Tu/\|Tu\|$.