also, by virtue of eqn (7.66), and p itself is therefore an odd function of θ. There is therefore no 'horizontal' force on the inner cylinder in Fig. 7.13; the 'upward' force is in fact

$$F = \frac{12\pi\mu U\lambda}{\varepsilon^2(1-\lambda^2)^{\frac{1}{2}}(2+\lambda^2)}. \tag{7.68}$$

The factor $(1-\lambda^2)^{\frac{1}{2}}$ in the denominator implies that the eccentricity parameter λ can, in principle, adjust itself so as to permit any external load on the inner cylinder, however large.

Returning to the flow itself, it is easy to show from eqns (7.66) and (7.67) that $\mathrm{d}p/\mathrm{d}\theta$ is positive at $\theta = \pi$, so that there is an adverse pressure gradient in some neighbourhood of that angle, and consequently the possibility of *reversed flow* near the stationary outer cylinder. That this does, indeed, occur can be seen by using eqns (7.65), (7.66), and (7.67) to calculate the velocity gradient $\partial u_\theta/\partial z$ on the outer cylinder:

$$\left.\frac{\partial u_\theta}{\partial z}\right|_{z=h} = \frac{2Ua\varepsilon\lambda}{h^2}\left[\frac{4\lambda^2-1}{\lambda(2+\lambda^2)} - \cos\theta\right]. \tag{7.69}$$

It is then a simple matter to show that if

$$\lambda > \tfrac{1}{2}(\sqrt{13}-3) \doteqdot 0.30 \tag{7.70}$$

there is a range of θ for which $\partial u_\theta/\partial z$ is positive at $z = h$, corresponding to reversed flow in the neighbourhood of the outer cylinder (see Fig. 7.13).

In practical lubrication theory this particular feature is overshadowed by other complications, but it is of some relevance to the arguments at the beginning of §8.6, and it has been clearly observed in experiments with very viscous fluids between offset rotating cylinders (see Chaiken *et al.* 1986, Fig. 3; also Aref 1986, Fig. 5 and Ottino 1989b, Fig. 7.4).

Exercises

7.1. Viscous fluid is contained between two rigid boundaries, $z = 0$ and $z = h$. The lower plane is at rest, the upper plane rotates about a vertical axis with constant angular velocity Ω. The Reynolds number $R = \Omega h^2/\nu$ is small, so that the slow flow equations (7.3) provide a good approximation to the resulting flow. Use §2.4 to write these equations in cylindrical polar coordinates, and show that they admit a purely rotary flow solution $\boldsymbol{u} = u_\theta(r, z)\boldsymbol{e}_\theta$ provided that

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)u_\theta = 0. \tag{7.71}$$

Write down the boundary conditions which u_θ must satisfy at $z = 0$ and $z = h$. Hence seek a solution of the form $u_\theta = rf(z)$. Show that the θ-component of stress, t_θ, on the upper plane is $-\mu\Omega r/h$.

Suppose instead that both upper and lower boundaries are horizontal discs of radius a. If end effects are neglected, show that the external torque on the upper disc needed to sustain the flow is

$$\tau = \tfrac{1}{2}\pi\mu\Omega\frac{a^4}{h}.$$

7.2. A rigid sphere of radius a is immersed in an infinite expanse of viscous fluid. The sphere rotates with constant angular velocity Ω. The Reynolds number $R = \Omega a^2/\nu$ is small, so that the slow flow equations

$$\nabla p = -\mu\nabla \wedge (\nabla \wedge \boldsymbol{u}), \qquad \nabla \cdot \boldsymbol{u} = 0$$

apply (see eqns (7.3) and (6.12)). Using spherical polar coordinates (r, θ, ϕ) with $\theta = 0$ as the rotation axis, show that a purely rotary flow $\boldsymbol{u} = u_\phi(r, \theta)\boldsymbol{e}_\phi$ is possible provided that

$$\frac{\partial^2}{\partial r^2}(ru_\phi) + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\phi \sin\theta)\right] = 0. \tag{7.72}$$

(This is, of course, just eqn (7.71) written in terms of different coordinates; u_ϕ here means the same thing as u_θ in Exercise 7.1.)

Write down the boundary conditions which u_ϕ must satisfy at $r = a$ and as $r \to \infty$, and hence seek an appropriate solution to eqn (7.72), thus finding

$$u_\phi = \frac{\Omega a^3}{r^2}\sin\theta.$$

Show that the ϕ-component of stress on $r = a$ is $t_\phi = -3\mu\Omega\sin\theta$, and deduce that the torque needed to maintain the rotation of the sphere is

$$\tau = 8\pi\mu\Omega a^3.$$

[In practice, in both the above situations there will be a small *secondary circulation*, of order R, in addition to the rotary flow. In the case of Exercise 7.1 we have already seen that the full Navier–Stokes equations do not admit a purely rotary flow solution (Exercise 2.11).]

7.3. Consider uniform slow flow past a spherical bubble of radius a by modifying the analysis of §7.2 accordingly, i.e. by replacing the no-slip condition on $r = a$ by the condition of no tangential stress ($t_\theta = 0$) on $r = a$. Show, in particular, that

$$\Psi = \tfrac{1}{2}U(r^2 - ar)\sin^2\theta$$

and that the normal component of stress on $r = a$ is $t_r = 3\mu U a^{-1}\cos\theta$. Hence show that the drag on the bubble is

$$D = 4\pi\mu U a$$

in the direction of the free stream (cf. eqn (7.9)).

[A similar but rather more involved problem is the uniform slow flow past a spherical drop of different fluid, of viscosity $\bar{\mu}$, say. This involves solving the slow flow equations separately outside and inside $r = a$, with $u_r = 0$ at $r = a$, and tangential stresses continuous at $r = a$. The drag on the drop is

$$D = 4\pi\mu U a\left(\frac{\mu + \frac{3}{2}\bar{\mu}}{\mu + \bar{\mu}}\right);$$

the limit $\bar{\mu}/\mu \to 0$ gives the 'bubble' result, while the limit $\bar{\mu}/\mu \to \infty$ gives a drag identical to that for a rigid sphere (see eqn (7.9)).]

7.4. Consider uniform slow flow past a circular cylinder, and show that the problem reduces to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0,$$

with $\partial\psi/\partial r = \partial\psi/\partial\theta = 0$ on $r = a$ and

$$\psi \sim Ur\sin\theta \qquad \text{as } r \to \infty.$$

Show that seeking a solution of the form $\psi = f(r)\sin\theta$ leads to

$$\psi = \left[Ar^3 + Br\log r + Cr + \frac{D}{r}\right]\sin\theta \tag{7.73}$$

and thus fails, in that for no choice of the arbitrary constants can all the boundary conditions be satisfied.

[There is, as stated in §7.2, no solution of *any* form to the problem as posed, but this takes rather more proving. Proudman and Pearson (1957) show that the equivalent dimensionless expression to eqn (7.13) in the neighbourhood of the cylinder $r = 1$ is

$$\psi \doteqdot \left[\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right)\sin\theta\right] \Big/ \left[\log\left(\frac{8}{R}\right) - \gamma + \tfrac{1}{2}\right],$$