IMA Journal of Mathematical Control and Information (2020) **00**, 1–27
doi:10.1093/imamci/dnz036

A weak maximum principle for optimal control problems with mixed constraints under a constant rank condition

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[Received on 8 March 2019; revised on 9 September 2019; accepted on 21 October 2019]

Necessary optimality conditions for optimal control problems with mixed state-control equality constraints are obtained. The necessary conditions are given in the form of a weak maximum principle and are obtained under (i) a new regularity condition for problems with mixed linear equality constraints and (ii) a constant rank type condition for the general non-linear case. Some instances of problems with equality and inequality constraints are also covered. Illustrative examples are presented.

Keywords: mixed constraints, weak maximum principle, constant rank.

1. Introduction

In general, first order necessary optimality conditions for optimal control problems are stated in the form of the so-called Pontryagin maximum principle (PMP). Nevertheless, it is well known that it may not be valid in the presence of mixed constraints. Regularity conditions need to be imposed on the mixed constraints in order that the maximum principle holds true. One of the main approaches is by means of implicit function theorems, which involves the assumption of full rankness of the Jacobian matrix related to the constraints (with respect to the control variables). The weak maximum principle was established under such an assumption, for example in Hestenes (1966), Milyutin & Osmolovskii (1998), de Pinho & Ilchmann (2002) and de Pinho (2003).

Regularity conditions of the Mangasarian–Fromovitz type (also known as positive linear independence) were used in Dmitruk (1993), Arutyunov (2000), Clarke (2005) and de Pinho & Rosenblueth

(2008), where in the three first references, strong versions of the PMP were obtained. In de Pinho & Rosenblueth (2008), a weak, but non-smooth, PMP is derived. This kind of regularity condition was also used to establish first and second order optimality conditions by Páles & Zeidan (1994).

The strong PMP was likewise obtained in Devdariani & Ledyaev (1999) and de Pinho *et al.* (2001) under a regularity condition that requires convexity assumptions.

In Clarke & Pinho (2010) a new regularity condition, called bounded slope condition, is introduced. Necessary optimality conditions including the transversality condition, the Euler adjoint inclusion as well as the Weierstrass condition are obtained. The optimality conditions derived under the bounded slope condition subsume those that involve full rank assumptions or Mangasarian–Fromovitz type conditions. In Clarke *et al.* (2011), the authors showed that the Schwarzkopf multiplier rule can be obtained from the necessary conditions given in Clarke & Pinho (2010). The non-smooth maximum principle for optimal control problems with mixed constraints stated in Clarke & Pinho (2010) is also used in Biswas & Pinho (2015) and Boccia *et al.* (2016) to establish a maximum principle for problems in which the pathwise constraints, in addition to the mixed constraints, include pure state constraints. The results in Boccia *et al.* (2016) extend those in Biswas & Pinho (2015) and apply to problems with more general mixed constraints when compared to those treated in the aforementioned paper. Optimal control problems with differential and algebraic equations are studied in de Pinho (2016) in which the necessary optimality conditions are obtained through the results from Clarke & Pinho (2010).

Li & Ye (2016) introduced the weak basic constraint qualification. They showed the validity of the maximum principle for optimal control problems with mixed constraints by assuming the weak basic constraint qualification along with the calmness of a certain set-valued mapping (that is defined in terms of the data of the problem). This assumption is weaker than the calibrated constraint qualification, given in Clarke & Pinho (2010) as a specialization of the bounded slope condition. The weak basic constraint qualification combined with the calmness conditions were also used in Li & Ye (2018) to obtain necessary optimality conditions to optimal control problems with implicit control systems.

A non-degenerated maximum principle is provided in Arutyunov *et al.* (2016) under a weak regularity condition. This regularity condition has a different nature in comparison to those cited above.

In this work, necessary optimality conditions for optimal control problems with mixed state-control equality constraints are given in the form of a weak maximum principle, where a regularity condition of constant rank type is imposed on the mixed constraints. This regularity condition is weaker than full rank assumptions found in the literature. Although the weak basic constraint qualification is implied by the constant rank condition given here, the derivation of the maximum principle under the weak basic constraint qualification by Li & Ye (2016) needs additional assumptions, such as a calmness condition and the compactness of a certain set defined along the optimal trajectory, and the PMP is stated for autonomous systems only. So their results cannot be directly compared to the necessary conditions obtained here. It should be mentioned that the PMP in Li & Ye (2016) includes the Weierstrass condition while the PMP developed here does not and the optimality concepts are different. Li & Ye (2016) work with local solutions of radius R, where R is a given measurable function of t. Here we work with the classical definition of weak local solutions.

Regarding the calibrated constraint qualification, de Pinho (2016) points out that in the smooth context it is equivalent to the full rank condition. In this paper, the mixed constraints are assumed to be continuously differentiable with respect to the state and control variables.

An important approach to obtaining necessary optimality conditions for optimal control problems is via the Dubovitskii–Milyutin formalism, particularly, for mixed-constrained problems. Such a formalism is based on a unified functional-analytic approach. See Girsanov (1972) for more details. The Dubovitskii–Milyutin formalism is a powerful tool because several instances of optimization as well as optimal control problems can be treated. The formalism was used, for example, in Dmitruk & Osmolovskii (2014) and Ledzewicz (1993) for optimal control problems with mixed constraints, and in Gayte *et al.* (2010) and Sun (2017) for optimal control problems in which the dynamics is governed by partial differential equations.

In this paper, a different technique is used however. It is shown that, under the constant rank condition, some ‘redundant’ equality constraints can be discarded, resulting in a set of mixed equality constraints for which a full rank condition holds and known results from the literature can be applied. Problems with equality and inequality constraints are also covered by transforming the inequality constraints into equality ones through the use of slackness variables. In this case, due to the proof technique, only some instances of problems may satisfy the constant rank condition. For example, it is assumed that there is at least one equality constraint along with the inequality ones. Nevertheless, the constant rank condition is an alternative regularity condition regarding the Mangasarian–Fromovitz type constraint qualification.

One relevant feature of the constant rank condition is that it may be satisfied even in the presence of redundant constraints. Generally speaking, redundant constraints naturally appear during the modelling process and may be difficult to detect, especially in the non-linear case. Moreover, some important numerical methods of optimization are proved to converge under the constant rank condition. For example, in Andreani *et al.* (2007), it is shown that an augmented Lagrangian method is globally convergent down the constant positive linear dependence constraint qualification (which is weaker than the constant rank condition). In von Heusinger *et al.* (2012), the constant rank constraint qualification is used to prove the local quadratic convergence of a Newton type method. In Xu *et al.* (2004), the convergence of a non-monotone trust-region algorithm is analysed, where a constant rank assumption on the gradients of the active constraints is assumed. Then it is natural to expect that robust computational methods for optimal control problems with mixed constraints may be proposed and their convergence established under the constant rank condition. Some of these methods can be found the literature, but their convergence is only guaranteed under full rank conditions. See Dontchev *et al.* (2000), for instance.

Finally, let us comment on the linear case. The linearity of the constraints itself constitutes a constraint qualification when we are dealing with classical mathematical programming problems. One way to show this is via the constant rank constraint qualification, since it is naturally satisfied in this case. Necessary optimality conditions of the Karush–Kuhn–Tucker type are always valid for mathematical programming with linear constraints. As regards optimal control problems, the linearity of the mixed constraints alone does not guarantee the validity of the maximum principle. Besides, due to a technical detail, the constant rank condition introduced here is not automatically satisfied for problems in which the mixed constraints are linear. We, then, decided to investigate which further characteristics the linear constraints should have to the maximum principle holds true. A new suitable regularity condition for optimal control problems with linear equality constraints is, therefore, developed. This new regularity condition is weaker than the full rank condition.

The paper is organized in the following way. Preliminaries are given in the next section. Section 3 is devoted to some auxiliary technical results. Necessary optimality conditions for equality constrained problems are developed in Section 4, where problems with linear equality constraints are treated separately. Necessary optimality conditions for problems involving both equality and inequality constraints are furnished in the last section.

2. Preliminaries

This preliminary section is devoted to setting the notation and giving some basic definitions. Moreover, here we state both the basic assumptions and the optimal control problem that the paper addresses. The section is finalized with some auxiliary results from the literature.

2.1. *Notation*

Given $v, w \in \mathbb{R}^k$, the usual inner product between v and w is denoted as $v \cdot w$; $|v|$ denotes the Euclidean norm of v; by $v \leqslant w$ we mean $v_i \leqslant w_i$ for all $i \in \{1, 2, \ldots, k\}$; by $v < w$ we mean $v_i < w_i$ for all $i \in \{1, 2, \ldots, k\}$.

Given a matrix $A \in \mathbb{R}^{m\times n}$, the largest and the smallest singular value of A are denoted respectively as $\sigma_1(A)$ and $\sigma_r(A)$, where $r = \text{rank}(A)$. The induced norm of A is denoted as $|A|$.

B denotes the open unit ball centred at the origin, regardless of the dimension of the space. $\mathbb{R}^k_+$ denotes the non-negative orthant, that is, $\mathbb{R}^k_+ = \{v \in \mathbb{R}^k : v_i \geqslant 0, i = 1, \ldots, k\}$.

$\mathscr{L}$ denotes the Lebesgue subsets of the interval $[0, 1]$; $\mathscr{B}^n$ denotes the Borel sets of $\mathbb{R}^n$; and $\mathscr{L}\times\mathscr{B}^n$ denotes the product σ-algebra.

Given a multifunction $\Gamma : [S, T] \to \mathbb{R}^n$, $\text{Gr}(\Gamma)$ means the graph of Γ, i.e.,

$$\text{Gr}(\Gamma) := \{(t, \gamma) \in [S, T] \times \mathbb{R}^n : \gamma \in \Gamma(t)\}.$$

The set of all absolutely continuous functions $x : [0, 1] \to \mathbb{R}^n$ is denoted by $W^{1,1}([0, 1]; \mathbb{R}^n)$. The set of all essentially bounded functions $u : [0, 1] \to \mathbb{R}^k$ is denoted by $L^\infty([0, 1]; \mathbb{R}^k)$. The set of all integrable functions $q : [0, 1] \to \mathbb{R}^m$ is denoted by $L^1([0, 1]; \mathbb{R}^m)$.

2.2. *Basic definitions*

Given a closed set $S \subset \mathbb{R}^n$ and a point $x \in S$, the set of all directions $v \in \mathbb{R}^n$ such that there exists $M > 0$ satisfying

$$v \cdot (y - x) \leqslant M\|y - x\|^2 \; \forall\, y \in S,$$

is said to be the proximal normal cone to S at x, denoted by $N^P_S(x)$. The set of all directions $v \in \mathbb{R}^n$ such that there exist sequences $x_i \xrightarrow{S} x$ and $v_i \to v$ satisfying $v_i \in N^P_S(x_i)$ for all i, is said to be the limiting normal cone to S at x, denoted by $N_S(x)$.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $x \in \text{dom} f$. By $\partial f(x)$ we mean the limiting Mordukhovich subdifferential of f at x defined as the set

$$\partial f(x) = \{\zeta : (\zeta, -1) \in N_{\text{epi} f}(x, f(x))\}.$$

For more details on non-smooth analysis, we refer the reader to one of the classical books on the subject, such as Clarke (1983), Mordukhovich (2006) and Vinter (2000).

Given a matrix $A \in \mathbb{R}^{m\times n}$, if $A = U\Sigma V^T$ is its singular values decomposition, the matrix $A^+ \in \mathbb{R}^{n\times m}$ defined as $A^+ = V\Sigma^+U^T$ is called the Moore–Penrose pseudo-inverse of A, where

$$\Sigma^+ = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times m}, E = \text{diag}(\sigma_i^{-1})_{i=1}^r, r = \text{rank}(A),$$

and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$ are the singular values of A. The following important properties of A^+ will be used later in the paper.

(i) $AA^+A = A$ and $A^+AA^+ = A^+$;
(ii) $(AA^+)^T = AA^+$ and $(A^+A)^T = A^+A$;
(iii) $(A^+)^T = (A^T)^+$ and $(A^+)^+ = A$;
(iv) For all $b \in \mathbb{R}^m$, AA^+b is the orthogonal projection of b into range(A);
(v) $\|A\|_2 = \sigma_1$ and $\|A^+\|_2 = \sigma_r^{-1}$;
(vi) The singular values of A coincide with the square roots of the eigenvalues of AA^T or A^TA;
(vii) If $n = m$ and A is symmetric, the singular values of A coincide with the absolute values of the eigenvalues of A;
(viii) If $n = m$, then $|\det(A)| = \Pi_{i=1}^r \sigma_i$.

For more properties of the pseudo-inverse matrix, the reader is referred to Noble & Daniel (1977).

2.3. *Problem statement*

The optimal control problem with mixed constraints is posed as follows:

$$\begin{array}{ll} \text{minimize} & l(x(0), x(1)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t), v(t)) \text{ a.e. in } [0, 1], \\ & g(t, x(t), u(t), v(t)) \leqslant 0 \text{ a.e. in } [0, 1], \\ & h(t, x(t), u(t), v(t)) = 0 \text{ a.e. in } [0, 1], \\ & v(t) \in V(t) \text{ a.e. in } [0, 1], \\ & (x(0), x(1)) \in S, \end{array} \tag{P}$$

where $l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(f, g, h) : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^{k_u} \times \mathbb{R}^{k_v} \to \mathbb{R}^n \times \mathbb{R}^{m_g} \times \mathbb{R}^{m_h}$ are given functions, $V(t) \subset \mathbb{R}^{k_v}$ for all $t \in [0, 1]$ and $S \subset \mathbb{R}^n \times \mathbb{R}^n$. Throughout the paper it is assumed that $k_u + k_v \geqslant m_g + m_h$.

We follow the classical nomenclature (see Vinter (2000), for example), which is set below.

A pair of measurable functions $(u, v) : [0, 1] \to \mathbb{R}^{k_u} \times \mathbb{R}^{k_v}$ such that $v(t) \in V(t)$ a.e. in $[0, 1]$ is said to be a *control function*.

A triple (x, u, v) consisting of a control function (u, v) and an arc $x \in W^{1,1}([0, 1]; \mathbb{R}^n)$ obeying the differential equation above is called a *process*.

Given a process (x, u, v), its first component is said to be a *state trajectory*.

A triple (x, u, v) is said to be a *feasible process* if x is a trajectory corresponding to the control (u, v), which satisfies all the constraints of (P).

A feasible process $(\bar{x}, \bar{u}, \bar{v})$ is said to be a *weak local optimal process* if there exists $\varepsilon > 0$ such that $l(\bar{x}(0), \bar{x}(1)) \leqslant l(x(0), x(1))$ for all feasible processes (x, u, v), which satisfy $(x(t), u(t), v(t)) \in T_\varepsilon(t)$ a.e. in $[0, 1]$, where

$$T_\varepsilon(t) := (\bar{x}(t) + \varepsilon\bar{B}) \times (\bar{u}(t) + \varepsilon\bar{B}) \times ((\bar{v}(t) + \varepsilon\bar{B}) \cap V(t)) \text{ a.e. in } [0.1].$$

The unmaximized Hamiltonian function $H : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m_g} \times \mathbb{R}^{m_h} \times \mathbb{R}^{k_u} \times \mathbb{R}^{k_v} \to \mathbb{R}$ related to (P) is defined as

$$H(t, x, p, \lambda, \mu, u, v) := p \cdot f(t, x, u, v) + \lambda \cdot g(t, x, u, v) + \mu \cdot h(t, x, u, v).$$

2.4. *Basic hypotheses*

Let $(\bar{x}, \bar{u}, \bar{v})$ be a feasible process of (P). The basic hypotheses are said to be satisfied at $(\bar{x}, \bar{u}, \bar{v})$ if there exists $\varepsilon > 0$ such that the following conditions are valid:

(H1) l is Lipschitz continuous on $(\bar{x}(0), \bar{x}(1)) + \varepsilon\bar{B}$;

(H2) Function $f(\cdot, x, u, v)$ is $\mathscr{L}$ measurable for each (x, u, v); for almost every $t \in [0, 1]$, $f(t, \cdot, \cdot, \cdot)$ is Lipschitz continuous on $(\bar{x}(t), \bar{u}(t), \bar{v}(t)) + \varepsilon\bar{B}$ with constant $k_f \in L^1([0, 1]; \mathbb{R})$;

(H3) Functions $g(\cdot, x, u, v)$ and $h(\cdot, x, u, v)$ are $\mathscr{L}$ measurable for each (x, u, v); for almost every $t \in [0, 1]$, $g(t, \cdot, \cdot, \cdot)$ and $h(t, \cdot, \cdot, \cdot)$ are continuously differentiable on $(\bar{x}(t), \bar{u}(t), \bar{v}(t)) + \varepsilon\bar{B}$; function $g(\cdot, \bar{x}(\cdot), \bar{u}(\cdot), \bar{v}(\cdot))$ is essentially bounded in $[0, 1]$; there exists an increasing function $\theta : (0, \infty) \to (0, \infty)$ with $\theta(s) \downarrow 0$ as $s \downarrow 0$ such that for almost every $t \in [0, 1]$,

$$|\nabla_{x,u,v}(g, h)(t, x, u, v) - \nabla_{x,u,v}(g, h)(t, x', u', v')| \leqslant \theta(|(x, u, v) - (x', u', v')|)$$

for all $(x, u, v), (x', u', v') \in (\bar{x}(t), \bar{u}(t), \bar{v}(t)) + \varepsilon\bar{B}$; there exists $k_{g,h} > 0$ such that for almost every $t \in [0, 1]$,

$$|\nabla_x(g, h)(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))| + |\nabla_u(g, h)(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))| + |\nabla_v(g, h)(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))| \leqslant k_{g,h};$$

(H4) $\mathrm{Gr}(V)$ is $\mathscr{L} \times \mathscr{B}^{k_v}$ measurable and $(\bar{v}(t) + \varepsilon\bar{B}) \cap V(t)$ is closed almost everywhere in $[0, 1]$;

(H5) S is closed.

2.5. *Auxiliary results from the literature*

For the readers' convenience, we will state some results from the literature that will be used later in the paper.

Let us consider the optimal control problem below, where the mixed constraints were dropped and $k_u = 0$:

$$\begin{array}{ll} \text{minimize} & l(x(0), x(1)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), v(t)) \text{ a.e. in } [0, 1], \\ & v(t) \in V(t) \text{ a.e. in } [0, 1], \\ & (x(0), x(1)) \in S. \end{array} \tag{SP}$$

Note that in this case the Hamiltonian function takes the form

$$H(t, x, p, v) := p \cdot f(t, x, u, v).$$

The following maximum principle is valid.

THEOREM 2.1 (de Pinho & Vinter, 1995). Let $(\bar{x}, \bar{v})$ be a weak local optimal process of (SP). Assume that the basic hypotheses (H1), (H2), (H4) and (H5) are valid at $(\bar{x}, \bar{v})$. Then there exist $\eta \geqslant 0$, $p \in W^{1,1}([0, 1]; \mathbb{R}^n)$ and $\zeta \in L^1([0, 1]; \mathbb{R}^{k_v})$ such that, for almost all $t \in [0, 1]$,

(i) $\eta + \|p\|_\infty \neq 0$;

(ii) $(-\dot{p}(t), \zeta(t)) \in \mathrm{co}\, \partial_{x,v} H(t, \bar{x}(t), p(t), \bar{v}(t))$;

(iii) $\zeta(t) \in \operatorname{co} N_{V(t)}(\bar{v}(t))$;

(iv) $(p(0), -p(1)) \in N_S(\bar{x}(0), \bar{x}(1)) + \eta\partial l(\bar{x}(0), \bar{x}(1))$.

In the presence of the mixed constraints, the weak maximum principle was shown to be valid under a uniform full rank assumption. We will reproduce here such a condition and the weak maximum principle for (P) without inequality constraints, since this version will be used later in the paper.

(FRC) The uniform full rank condition is said to be satisfied at a feasible process $(\bar{x}, \bar{u}, \bar{v})$ if there exists $K > 0$ such that

$$\det(\nabla_u h(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))\nabla_u h(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))^T) \geqslant K \text{ a.e. in } [0, 1].$$

Now,

$$H(t, x, p, \mu, u, v) := p \cdot f(t, x, u, v) + \mu \cdot h(t, x, u, v).$$

THEOREM 2.2 (de Pinho & Ilchmann, 2002). Let $(\bar{x}, \bar{v})$ be a weak local optimal process of (P) with $m_g = 0$. Assume that the basic hypotheses (H1)–(H5) are valid at $(\bar{x}, \bar{v})$ and FRC is satisfied. Then there exist $\eta \geqslant 0$, $p \in W^{1,1}([0, 1]; \mathbb{R}^n)$, $\mu \in L^1([0, 1]; \mathbb{R}^r)$ and $\zeta \in L^1([0, 1]; \mathbb{R}^{k_v})$ such that

(i) $\|p\|_\infty + \eta \neq 0$;

(ii) $(-\dot{p}(t), 0, \zeta(t)) \in \operatorname{co} \partial_{x,u,v} H(t, \bar{x}(t), p(t), \mu(t), \bar{u}(t), \bar{v}(t))$ a.e. in $[0, 1]$;

(iii) $\zeta(t) \in \operatorname{co} N_{V(t)}(\bar{v}(t))$ a.e. in $[0, 1]$;

(iv) $(p(0), -p(1)) \in N_S(\bar{x}(0), \bar{x}(1)) + \eta\partial l(\bar{x}(0), \bar{x}(1))$.

Furthermore, there exists $M > 0$ such that $|\mu(t)| \leqslant k_f(t)Mk_{h,g}|p(t)|$ a.e. in $[0, 1]$.

Below the so-called uniform inverse mapping theorem is presented.

THEOREM 2.3 (de Pinho & Vinter, 1997). Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, n-vectors x_0 and y_0 and a family of functions $\{F_a : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in A}$ satisfying $y_0 = F_a(x_0)$ for all $a \in A$. It is assumed that

(i) F_a is continuously differentiable on $x_0 + \alpha B$ for all $a \in A$;

(ii) there exists a monotone increasing function $\theta : (0, \infty) \to (0, \infty)$, with $\theta(s) \downarrow 0$ as $s \downarrow 0$, such that

$$|\nabla F_a(x) - \nabla F_a(\tilde{x})| \leqslant \theta(|x - \tilde{x}|) \ \forall x, \tilde{x} \in x_0 + \alpha B, a \in A;$$

(iii) $\nabla F_a(x_0)$ is non-singular for each $a \in A$ and there exists $c > 0$ such that

$$|[\nabla F_a(x_0)]^{-1}| \leqslant c \ \forall a \in A.$$

Then there exist numbers $\varepsilon \in (0, \alpha)$ and $\delta > 0$, and a family of continuously differentiable functions $\{G_a : y_0 + \delta B \to x_0 + \alpha B\}_{a\in A}$, which are Lipschitz continuous with a common Lipschitz constant K such that

$$F_a(G_a(y)) = y \ \forall y \in y_0 + \delta B, a \in A,$$

$$G_a(F_a(x)) = x \ \forall x \in x_0 + \varepsilon B, a \in A.$$

The numbers ε and δ depend only on α, $\theta(\cdot)$ and c. Furthermore, if A is a Borel set and $a \mapsto F_a(x)$ is Borel measurable for each $x \in x_0 + \alpha B$, then $a \mapsto G_a(y)$ is Borel measurable for each $y \in y_0 + \delta B$.

REMARK 2.1 As can be seen in Step 1 of the proof of the uniform inverse mapping theorem in de Pinho & Vinter (1997), $\nabla F_a(x)$ is non-singular in $x_0 + \varepsilon B$ for each $a \in A$.

Now, a corollary of Weyl's theorem. Specifically, Corollary 4.3.12 on page 242 of Horn & Johnson (2013).

THEOREM 2.4 (Horn & Johnson, 2013). Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices and let the eigenvalues of A and $A + B$ be $\{\lambda_i(A)\}_{i=1}^n$ and $\{\lambda_i(A + B)\}_{i=1}^n$, each algebraically ordered in non-decreasing order. Assume that B is positive semi-definite. Then $\lambda_i(A) \leqslant \lambda_i(A + B)$, $i = 1, \dots, n$.

3. Auxiliary technical results

In this section, three technical lemmas are presented, the last one being of fundamental importance in the paper. These results are generalizations of two lemmas given in Andreani *et al.* (2010) in the context of non-linear programming.

LEMMA 3.1 Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, n-vectors x_0 and y_0 and a family of functions $\{F_a = (f_1^a, \dots, f_n^a) : \mathbb{R}^n \to \mathbb{R}^n\}_{a\in A}$ satisfying $y_0 = F_a(x_0)$ for all $a \in A$ and all the assumptions of Theorem 2.3. Let $\{f^a : \mathbb{R}^n \to \mathbb{R}\}_{a\in A}$ be a second family of functions, which are continuously differentiable on $x_0 + \alpha B$ for all $a \in A$. Assume that, for each $a \in A$, $\nabla f^a(x)$ is a linear combination of $\nabla f_1^a(x), \dots, \nabla f_q^a(x)$ for all $x \in x_0 + \alpha B$, for some integer $0 < q \leqslant n$. Then there exists $\delta > 0$ such that $\varphi^a : y_0 + \delta B \to \mathbb{R}$, given by

$$\varphi^a(u) = f^a(F_a^{-1}(u)), a \in A, \tag{3.1}$$

satisfies

$$\frac{\partial \varphi^a}{\partial u_j}(u) = 0 \ \forall u \in y_0 + \delta B, j = q + 1, \dots, n, a \in A. \tag{3.2}$$

Proof. By Theorem 2.3, there exist $0 < \varepsilon < \alpha$ and $\delta > 0$ and a family of continuously differentiable functions $\{G_a = F_a^{-1} : y_0 + \delta B \to x_0 + \varepsilon B\}_{a\in A}$ such that $G_a(F_a(x)) = x$ for all $x \in x_0 + \varepsilon B, a \in A$. Then, by the chain rule and Remark 2.1,

$$\nabla G_a(F_a(x))\nabla F_a(x) = I \ \forall x \in x_0 + \varepsilon B \Rightarrow \nabla G_a(y) = \nabla F_a(G_a(y))^{-1} \ \forall y \in y_0 + \delta B, a \in A.$$

Applying the chain rule in (3.1) one has, for $u \in y_0 + \delta B$,

$$\nabla \varphi^a(u) = \nabla G_a(u)^T \nabla f_a(G_a(u)) = \nabla F_a(G_a(u))^{-T} \nabla f_a(G_a(u)),$$

so that

$$\nabla F_a(G_a(u))^T \nabla \varphi^a(u) = \nabla f_a(G_a(u)), a \in A.$$

Thence,

$$\frac{\partial \varphi^a}{\partial u_1}(u)\nabla f_1^a(G_a(u)) + \dots + \frac{\partial \varphi^a}{\partial u_n}(u)\nabla f_n^a(G_a(u)) = \nabla f^a(G^a(u)) \ \forall u \in y_0 + \delta B, a \in A.$$

Notice that $\nabla f_1^a(G^a(u)),\dots,\nabla f_n^a(G^a(u))$ are linearly independent for all $u \in y_0+\delta B$, for each $a \in A$, since $\nabla F^a(G^a(u))$ is non-singular (see Remark 2.1). On the other hand, if $u \in y_0+\delta B$, then $G^a(u) \in x_0+\epsilon B \subset x_0+\alpha B$, for all $a \in A$, so that, by hypothesis, $\nabla f^a(G^a(u))$ can only be written as a linear combination of $\nabla f_1^a(G^a(u)),\dots,\nabla f_q^a(G^a(u))$. Therefore, one sees that (3.2) does hold. ☐

LEMMA 3.2 Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, n-vectors x_0 and y_0 and a family of functions $\{\tilde{F}_a = (f_1^a,\dots,f_q^a) : \mathbb{R}^n \to \mathbb{R}^q\}_{a\in A}$, $0 < q \leqslant n$, satisfying $\tilde{F}_a(x_0) = z_0$ for all $a \in A$, where $y_0 = (z_0, w_0) \in \mathbb{R}^q \times \mathbb{R}^{n-q}$. Suppose that the following conditions hold:

(a) $\tilde{F}_a$ is continuously differentiable on $x_0+\alpha B$ for all $a \in A$;

(b) There exists $\theta : (0,\infty) \to (0,\infty)$, $\theta(s) \downarrow 0$ when $s \downarrow 0$, such that

$$|\nabla \tilde{F}_a(x) - \nabla \tilde{F}_a(\bar{x})| \leqslant \theta(|x-\bar{x}|),$$

for all $x, \bar{x} \in x_0+\alpha B,\ a \in A$; there exists $\tilde{K} > 0$ such that

$$|\nabla \tilde{F}_a(x_0)| \leqslant \tilde{K}, a \in A;$$

(c) There exists $K > 0$ such that

$$\det\{[\nabla \tilde{F}_a(x_0)][\nabla \tilde{F}_a(x_0)]^T\} \geqslant K, a \in A.$$

Then there exists a family of continuously differentiable functions $\{\hat{F} : \mathbb{R}^n \to \mathbb{R}^{n-q}\}_{a\in A}$ such that $F_a = (\tilde{F}_a, \hat{F}_a)$ satisfies $F_a(x_0) = y_0$ for all $a \in A$ and all assumptions of Theorem 2.3.

Proof. For each $a \in A$, let M_a be a matrix whose columns form an orthonormal basis for the orthogonal complement to the subspace generated by the rows of $\nabla \tilde{F}_a(x_0)$.

For each $a \in A$, define $\hat{F}_a : \mathbb{R}^n \to \mathbb{R}^{n-q}$ and $F_a : \mathbb{R}^n \to \mathbb{R}^n$ respectively as

$$\hat{F}_a(x) = {M_a}^T(x-x_0) + w_0 \quad \text{and} \quad F_a(x) = (\tilde{F}_a(x), \hat{F}_a(x)).$$

Then,

$$F_a(x_0) = (\tilde{F}_a(x_0), \hat{F}_a(x_0)) = (z_0, w_0) = y_0, a \in A,$$

and

$$\nabla F_a(x)^T = [\nabla \tilde{F}_a(x)^T \nabla\, \hat{F}_a(x)^T] = [\nabla \tilde{F}_a(x)^T M_a]\ \forall\, x, a \in A.$$

Moreover,

(i) from assumption (a) and the definition of $\hat{F}_a$, it follows that F_a is continuously differentiable on $x_0+\alpha B$ for all $a \in A$;

(ii) from (b), for $x, \bar{x} \in x_0+\alpha B$ one has

$$|\nabla F_a(x) - \nabla F_a(\bar{x})| = |\nabla \tilde{F}_a(x) - \tilde{F}_a(\bar{x})| \leqslant \theta(|x-\bar{x}|),$$

where $\theta : (0,\infty) \to (0,\infty),\ \theta(s) \downarrow 0$ when $s \downarrow 0$;

(iii) by construction, $|\nabla\hat{F}_a(x_0)| = |M_a{}^T| = 1$. Thence,

$$|\nabla F_a(x_0)| = \left|\begin{bmatrix} \nabla\tilde{F}_a(x_0) \\ M_a{}^T \end{bmatrix}\right| \leqslant \sqrt{|\nabla\tilde{F}_a(x_0)|^2 + |M_a{}^T|^2} \leqslant \sqrt{\tilde{K}^2 + 1}.$$

Also by construction,

$$\det([\nabla F_a(x_0)][\nabla F_a(x_0)]^T) = \det([\nabla\tilde{F}_a(x_0)][\nabla\tilde{F}_a(x_0)]^T) \geqslant K, a \in A,$$

implying that $\det(\nabla F_a(x_0)) \geqslant \sqrt{K} > 0,\ a \in A$. Therefore, $\nabla F_a(x_0)$ is non-singular. It follows that there exists $M > 0$ such that

$$|[\nabla F_a(x_0)]^{-1}| \leqslant M, a \in A.$$

Thus, the family $\{F_a\}_{a\in A}$ satisfies all the assumptions of Theorem 2.3. □

LEMMA 3.3 Consider a set $A \subset \mathbb{R}^k$, a number $\alpha > 0$, n-vectors x_0 and $y_0 \in \mathbb{R}^n$ and families of functions $\{f^a : \mathbb{R}^n \to \mathbb{R}\}_{a\in A}$ and $\{\tilde{F}_a = (f_1^a, \dots, f_q^a) : \mathbb{R}^n \to \mathbb{R}^q\}_{a\in A}$, $0 < q \leqslant n$, satisfying $\tilde{F}_a(x_0) = z_0$ for all $a \in A$, where $y_0 = (z_0, w_0) \in \mathbb{R}^q \times \mathbb{R}^{n-q}$. Assume that f^a is continuously differentiable on $x_0 + \alpha B$ for all $a \in A$, and that $\{\tilde{F}_a\}_{a\in A}$ satisfies assumptions (a)–(c) of Lemma 3.2. In addition, assume that $\nabla f^a(x)$ is a linear combination of $\nabla f_1^a(x), \dots, \nabla f_q^a(x)$ for all $a \in A$ and $x \in x_0 + \alpha B$. Specifically, for $x = x_0$,

$$\nabla f^a(x_0) = \sum_{i=1}^{q} \lambda_i^a \nabla f_i^a(x_0), a \in A.$$

Then there exist $\sigma \in (0, \alpha)$, $\rho > 0$ and a family of continuously differentiable functions $\{\chi^a : z_0 + \rho B \to \mathbb{R}\}_{a\in A}$ such that for all $x \in x_0 + \sigma B$ and each $a \in A$,

$$(f_1^a(x), \dots, f_q^a(x)) = \tilde{F}_a(x) \in z_0 + \rho B$$

and

$$f^a(x) = \chi^a(\tilde{F}_a(x)) = \chi^a(f_1^a(x), \dots, f_q^a(x)).$$

The numbers σ and ρ depend only on α, θ, $\tilde{K}$ and K. Furthermore,

$$\frac{\partial\chi^a}{\partial u_i}(\tilde{F}_a(x_0)) = \lambda_i^a, i = 1, \dots, q, a \in A.$$

Proof. By Lemma 3.2, for each $a \in A$, one can define $n - q$ functions $f_{q+1}^a, \dots, f_n^a$ in such a way that $F_a = (\tilde{F}_a, \hat{F}_a) = (f_1^a, \dots, f_q^a, f_{q+1}^a, \dots, f_n^a)$ satisfies the hypothesis of Lemma 3.1. Then, by (3.2), the function φ^a as defined in (3.1) does not depend on the variables $u_{q+1}, \dots, u_n$, for all $a \in A$. Provided $f_1^a, \dots, f_n^a$ are continuous for all $a \in A$, there exist open balls $x_0 + \sigma B \subset x_0 + \alpha B$ and $(z_0 + \rho_1 B) \times$

$(w_0 + \rho_2 B) \subset (z_0, w_0) + \delta B = y_0 + \delta B$ such that

$$(f_1^a(x), \ldots, f_q^a(x)) = \tilde{F}_a(x) \in z_0 + \rho_1 B \forall x \in x_0 + \sigma B. \tag{3.3}$$

For each $a \in A$, define $\chi^a : z_0 + \rho_1 B \to \mathbb{R}$ as

$$\chi^a(u) = \varphi^a(u, \hat{F}_a(x_0)) = \varphi^a(u_1, \ldots, u_q, f_{q+1}^a(x_0), \ldots, f_n^a(x_0)).$$

Clearly, χ^a is continuously differentiable for all $a \in A$. Putting $\rho = \rho_1$, one has, by (3.3), for all $x \in x_0 + \sigma B$ and each $a \in A$, that

$$(f_1^a(x), \ldots, f_q^a(x)) = \tilde{F}_a(x) \in z_0 + \rho B$$

and, by the fact that φ^a does not depend on the last $n - q$ variables,

$$\begin{aligned} \chi^a(\tilde{F}_a(x)) &= \chi^a(f_1^a(x), \ldots, f_q^a(x)) \\ &= \varphi^a(f_1^a(x), \ldots, f_q^a(x), f_{q+1}^a(x_0), \ldots, f_n^a(x_0)) \\ &= \varphi^a(f_1^a(x), \ldots, f_q^a(x), f_{q+1}^a(x), \ldots, f_n^a(x)) \\ &= \varphi^a(F_a(x)) = f^a(x), \end{aligned}$$

where (3.1) was used in the last inequality. By applying the chain rule in $f^a(x) = \chi^a(\tilde{F}_a(x)) = \chi^a(f_1^a(x), \ldots, f_q^a(x))$ one gets

$$\nabla f^a(x) = \sum_{i=1}^{q} \frac{\partial \chi^a}{\partial u_i}(\tilde{F}_a(x)) \nabla f_i^a(x) \forall x \in x_0 + \sigma B, a \in A.$$

On the other hand, by assumption,

$$\nabla f^a(x_0) = \sum_{i=1}^{q} \lambda_i^a \nabla f_i^a(x_0), a \in A,$$

and $\nabla f_1^a(x_0), \ldots, \nabla f_q^a(x_0)$ are linearly independent for all $a \in A$. It follows that

$$\frac{\partial \chi_a}{\partial u_j}(\tilde{F}_a(x_0)) = \lambda_j^a, j = 1, \ldots, q, a \in A.$$

□

4. Problems with mixed equality constraints

The weak maximum principle is obtained in this section for problems with mixed equality constraints only. We start, in Section 4.1, by treating problems with linear equality constraints. We then turn, in Section 4.2, to the general non-linear case, establishing the maximum principle under a constant rank condition.

4.1. *The linear case*

It is well known in non-linear programming that linearity is itself a constraint qualification. This fact can be checked (i) directly; (ii) as a consequence of the linear independence constraint qualification (by discarding redundant constraints and obtaining a full rank coefficient matrix); or (iii) as a consequence of the constant rank constraint qualification (since the rank of the Jacobian of the constraints does not change in any neighbourhood of the optimal point). In optimal control, this is not the case. We will see in this subsection that the maximum principle may not be valid for problems with linear equality constraints, and we give conditions under which it is valid.

The following optimal control problem with mixed linear equality constraints is treated here:

$$\begin{aligned} \text{minimize} \quad & l(x(0), x(1)) \\ \text{subject to} \quad & \dot{x}(t) = f(t, x(t), u(t), v(t)) \text{ a.e. in } [0,1], \\ & h(t, x(t), u(t), v(t)) = 0 \text{ a.e. in } [0,1], \\ & v(t) \in V(t) \text{ a.e. in } [0,1], \\ & (x(0), x(1)) \in S, \end{aligned} \tag{LP}$$

with

$$h(t,x,u,v) = A(t)x + B(t)u + C(t)v - b(t),$$

where $(A,B,C) : [0,1] \to \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times k_u} \times \mathbb{R}^{m\times k_v}$ and $b : [0,1] \to \mathbb{R}^m$ are $\mathscr{L}$ measurable functions.

We now present a simple example of an optimal control problem with mixed linear equality constraints in which the maximum principle is not valid.

EXAMPLE 4.1 Let us examine the following problem:

$$\begin{aligned} \text{minimize} \quad & l(x(0), x(1)) = x(1) \\ \text{subject to} \quad & \dot{x}(t) = u_1(t) \text{ a.e. in } [0,1], \\ & x(t) + u_1(t) + 2u_2(t) = 0 \text{ a.e. in } [0,1], \\ & u_1(t) + 2u_2(t) = 0 \text{ a.e. in } [0,1]. \end{aligned}$$

The only feasible process is $(x, u_1, u_2) = (0,0,0)$, so that it is optimal. The conditions from the weak maximum principle (see Theorem 2.2) are written as

$$\begin{aligned} & \|p\|_\infty + \eta \neq 0; \\ & -\dot{p}(t) = \mu_1(t) \text{ a.e. in } [0,1], \\ & p(t) + \mu_1(t) + \mu_2(t) = 0 \text{ a.e. in } [0,1], \\ & 2\mu_1(t) + 2\mu_2(t) = 0 \text{ a.e. in } [0,1], \\ & (p(0), -p(1)) = \eta(0,1). \end{aligned}$$

The system above does not have any solution.

The example above shows us that regularity conditions on the mixed constraints are necessary to the validity of the weak maximum principle, even in the linear case, where the rank of the Jacobian of the constraints is constant (in function of points in $T_\varepsilon(t)$ a.e. in $[0,1]$).

DEFINITION 4.1 The regularity condition (RC) is said to be satisfied if

(a) $\mathrm{range}([\,A(t)\ C(t)\,]) \subset \mathrm{range}(B(t))$ a.e. in $[0,1]$;

(b) there exists $k_B > 0$ such that $\sigma_{r(t)}(B(t)) \geqslant k_B$ a.e. in $[0,1]$,

where $r(t) = \mathrm{rank}(B(t))$ a.e. in $[0,1]$.

REMARK 4.1

(i) It is easy to see that RC is valid when $\det(B(t)B(t)^T) \geqslant K$ a.e. in $[0,1]$ for some $K > 0$, that is, the full rank condition (FRC) implies the regularity condition (RC).

(ii) Condition RC-(a) is equivalent to $\mathrm{range}(A(t)) \subset \mathrm{range}(B(t))$ and $\mathrm{range}(C(t)) \subset \mathrm{range}(B(t))$ a.e. in $[0,1]$.

Next, we have simple cases in which RC holds true while FRC does not.

EXAMPLE 4.2 Let us consider

$$A(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ a.e. in } [0,1].$$

It is clear that FRC is not valid and RC holds (with $k_B = \sqrt{10}$).

EXAMPLE 4.3 Let

$$\begin{aligned} A(t) &= \begin{bmatrix} 1 \\ t \end{bmatrix} \text{ a.e. in } [0,1], \\ B(t) &= \begin{cases} \begin{bmatrix} 1 & 1 \\ 1 & t \end{bmatrix} & \text{a.e. in } [0,1/2], \\ \begin{bmatrix} 1 & 1 \\ t & t \end{bmatrix} & \text{a.e. in } (1/2,1], \end{cases} \\ C(t) &= \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{a.e. in } [0,1/2], \\ \begin{bmatrix} 2 & 0 \\ 2t & 0 \end{bmatrix} & \text{a.e. in } (1/2,1]. \end{cases} \end{aligned}$$

It is easy to see that RC-(a) holds true and FRC does not. Condition RC-(b) is valid with $k_B = 1/4$.

In this subsection,

$$H(t,x,p,\mu,u,v) := p \cdot f(t,x,u,v) + \mu \cdot h(t,x,u,v).$$

THEOREM 4.1 Let $(\bar{x},\bar{u},\bar{v})$ be a weak local optimal process of (LP). Assume that the basic hypotheses (H1)–(H5) are valid at $(\bar{x},\bar{u},\bar{v})$ and the regularity condition (RC) is satisfied. Then there exist $\eta \geqslant 0$, $p \in W^{1,1}([0,1];\mathbb{R}^n)$, $\mu \in L^1([0,1];\mathbb{R}^m)$ and $\zeta \in L^1([0,1];\mathbb{R}^{k_v})$ such that, for almost all $t \in [0,1]$,

(i) $\eta + \|p\|_\infty \neq 0$;

(ii) $(-\dot{p}(t), 0, \zeta(t)) \in \mathrm{co}\, \partial_{x,u,v} H(t,\bar{x}(t),p(t),\mu(t),\bar{u}(t),\bar{v}(t))$;

(iii) $\zeta(t) \in \operatorname{co} N_{V(t)}(\bar{v}(t))$;

(iv) $(p(0), -p(1)) \in N_S(\bar{x}(0), \bar{x}(1)) + \eta\partial l(\bar{x}(0), \bar{x}(1))$.

Moreover,

$$|\mu(t)| \leqslant k_B^{-1} k_f(t)|p(t)| \text{ a.e. in } [0,1].$$

Proof. If $(\bar{x}, \bar{u}, \bar{v})$ is a weak local optimal process of (LP), there exists $\varepsilon > 0$ such that $l(\bar{x}(0), \bar{x}(1)) \leqslant l(x(0), x(1))$ for all feasible processes (x, u, v), which satisfy $(x(t), u(t), v(t)) \in T_\varepsilon(t)$ a.e. in $[0, 1]$. Let us consider the auxiliary problem below:

$$\begin{aligned}
\text{minimize} \quad & l(x(0), x(1)) \\
\text{subject to} \quad & \dot{x}(t) = \phi(t, x(t), u(t), v(t)) \text{ a.e. in } [0,1], \\
& (u(t), v(t)) \in U_\varepsilon(t) \times V_\varepsilon(t) \text{ a.e. in } [0,1], \\
& (x(0), x(1)) \in S,
\end{aligned} \tag{4.1}$$

where

$$\phi(t, x, u, v) = f(t, x, u + \psi(t, x, u, v), v),$$

$$\psi(t, x, u, v) = B(t)^+[A(t)(\bar{x}(t) - x) + B(t)(\bar{u}(t) - u) + C(t)(\bar{v}(t) - v)],$$

and

$$U_\varepsilon(t) \times V_\varepsilon(t) = (\bar{u}(t) + \varepsilon\bar{B}) \times ((\bar{v}(t) + \varepsilon\bar{B}) \cap V(t)) \text{ a.e. in } [0,1].$$

Then $(\bar{x}, \bar{u}, \bar{v})$ is a weak local optimal process of (4.1). Indeed, it is clear that $(\bar{x}, \bar{u}, \bar{v})$ is feasible for (4.1). Let one assume the contrary, that given $0 < \delta < \min\{\varepsilon/4, k_B\varepsilon/(4k_{g,h})\}$, there exists a feasible process $(\tilde{x}, \tilde{u}, \tilde{v})$ of (4.1) with $(\tilde{x}(t), \tilde{u}(t), \tilde{v}(t)) \in T_\delta(t)$ a.e. in $[0, 1]$ and $l(\tilde{x}(0), \tilde{x}(1)) < l(\bar{x}(0), \bar{x}(1))$. Take

$$\hat{u}(t) = \tilde{u}(t) + \psi(t, \tilde{x}(t), \tilde{u}(t), \tilde{v}(t)) \text{ a.e. in } [0,1].$$

One has that

$$\begin{aligned}
|\hat{u}(t) - \bar{u}(t)| &\leqslant |\tilde{u}(t) - \bar{u}(t)| + |B(t)^+A(t)| \cdot |\bar{x}(t) - \tilde{x}(t)| \\
&\quad + |B(t)^+B(t)| \cdot |\bar{u}(t) - \tilde{u}(t)| + |B(t)^+C(t)| \cdot |\bar{v}(t) - \tilde{v}(t)| \\
&\leqslant \delta + [\sigma_{r(t)}(B(t))]^{-1}|A(t)|\delta + \delta + [\sigma_{r(t)}(B(t))]^{-1}|C(t)|\delta \\
&\leqslant \delta + \frac{k_{g,h}}{k_B}\delta + \delta + \frac{k_{g,h}}{k_B}\delta < \varepsilon.
\end{aligned}$$

Thence, $(\tilde{x}(t), \hat{u}(t), \tilde{v}(t)) \in T_\varepsilon(t)$ a.e. in $[0, 1]$. It is clear that

$$\dot{\tilde{x}}(t) = f(t, \tilde{x}(t), \hat{u}(t), \tilde{v}(t)) \text{ a.e. in } [0,1].$$

Furthermore,

$$\begin{aligned} A(t)\tilde{x}(t) + B(t)\hat{u}(t) + C(t)\tilde{v}(t) &= A(t)\tilde{x}(t) + B(t)\tilde{u}(t) + B(t)B(t)^{+}A(t)(\bar{x}(t) - \tilde{x}(t)) \\ &\quad + B(t)B(t)^{+}B(t)(\bar{u}(t) - \tilde{u}(t)) + B(t)B(t)^{+}C(t)(\bar{v}(t) - \tilde{v}(t)) \\ &\quad + C(t)\tilde{v}(t) \\ &= A(t)\tilde{x}(t) + B(t)\tilde{u}(t) + A(t)(\bar{x}(t) - \tilde{x}(t)) \\ &\quad + B(t)(\bar{u}(t) - \tilde{u}(t)) + C(t)(\bar{v}(t) - \tilde{v}(t)) + C(t)\tilde{v}(t) \\ &= A(t)\bar{x}(t) + B(t)\bar{u}(t) + C(t)\bar{v}(t) = b(t), \end{aligned}$$

where RC-(a) and the fact that $B(t)B(t)^{+}$ is a orthogonal projector into the range$(B(t))$ a.e. in $[0,1]$ were used in the penultimate equality above. Therefore, $(\tilde{x}, \hat{u}, \tilde{v})$ is a feasible process of (LP) with $(\tilde{x}(t), \hat{u}(t), \tilde{v}(t)) \in T_{\varepsilon}(t)$ a.e. in $[0,1]$ and $l(\tilde{x}(0), \tilde{x}(1)) < l(\bar{x}(0), \bar{x}(1))$. This contradicts the optimality of $(\bar{x}, \bar{u}, \bar{v})$ in (LP).

The standard maximum principle (Theorem 2.1) will be applied. It follows that there exist $\eta \geqslant 0$, $p \in W^{1,1}([0,1]; \mathbb{R}^n)$ and $\zeta \in L^1([0,1]; \mathbb{R}^{k_v})$ such that, for almost every $t \in [0,1]$,

$$\begin{aligned} &\eta + \|p\|_{\infty} \neq 0; \\ &(-\dot{p}(t), \zeta_1(t), \zeta_2(t)) \in \operatorname{co} \partial_{x,u,v}\{p(t) \cdot \phi(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))\}; \\ &(\zeta_1(t), \zeta_2(t)) \in \operatorname{co} N_{U_{\varepsilon}(t)}(\bar{u}(t)) \times \operatorname{co} N_{V_{\varepsilon}(t)}(\bar{v}(t)); \\ &(p(0), -p(1)) \in N_S(\bar{x}(0), \bar{x}(1)) + \eta \partial l(\bar{x}(0), \bar{x}(1)). \end{aligned}$$

By making use of the non-smooth chain rule (see Theorem 2.3.9 in Clarke, 1983), one has

$$\begin{aligned} &\operatorname{co} \partial_{x,u,v}\{p \cdot \phi(t, x, u, v)\} = \operatorname{co} \partial_{x,u,v}\{p \cdot f(t, x, u + \psi(t, x, u, v), v)\} \\ &\subset \Big\{ \Big(\nu - (B(t)^{+}A(t))^T \rho, \rho - (B(t)^{+}B(t))^T \rho, \pi - (B(t)^{+}C(t))^T \rho \Big) : \\ &\qquad (\nu, \rho, \pi) \in \operatorname{co} \partial_{x,u,v}\{p \cdot f(t, x, u, v)\} \Big\}. \end{aligned}$$

Therefore, through a suitable selection theorem, one sees that there exist measurable functions ν, ρ and π such that

$$(-\dot{p}(t), \zeta_1(t), \zeta_2(t)) = \Big(\nu(t) + A(t)^T \mu(t), \rho(t) + B(t)^T \mu(t), \pi(t) + C(t)^T \mu(t) \Big) \text{ a.e. in } [0,1]$$

with

$$(\nu(t), \rho(t), \pi(t)) \in \operatorname{co} \partial_{x,u,v}\{p(t) \cdot f(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))\} \text{ a.e. in } [0,1],$$

where $\mu(t) = -(B(t)^{+})^T \rho(t)$ a.e. in $[0,1]$. Moreover, $N_{U_{\varepsilon}(t)}(\bar{u}(t)) = \{0\}$ and $N_{V_{\varepsilon}(t)}(\bar{v}(t)) = N_{V(t)}(\bar{v}(t))$. Thus, $\zeta_1(t) = 0$ a.e. in $[0,1]$, and defining $\zeta(t) = \zeta_2(t)$ a.e. in $[0,1]$, one obtains

$$(-\dot{p}(t), 0, \zeta(t)) \in \operatorname{co} \partial_{x,u,v} H(t, \bar{x}(t), p(t), \mu(t), \bar{u}(t), \bar{v}(t)) \text{ a.e. in } [0,1]$$

and

$$\zeta(t) \in \operatorname{co} N_{V(t)}(\bar{v}(t)) \text{ a.e. in } [0,1].$$

From (H2) and RC-(b) it follows that

$$|\mu(t)| \leqslant k_B^{-1} k_f(t)|p(t)| \text{ a.e. in } [0,1].$$

□

Below we have some illustrative examples. In both of them, the full rank condition is not satisfied, and the weak basic constraint qualification is not applied since the problems are non-autonomous.

EXAMPLE 4.5 Let us consider the following optimal control problem:

$$\begin{array}{ll} \text{minimize} & l(x(0),x(1)) = x(1) \\ \text{subject to} & \dot{x}(t) = tu_1(t)^2 + u_2(t)^2 \text{ a.e. in } [0,1], \\ & x(t) + u_1(t) + u_2(t) = 0 \text{ a.e. in } [0,1], \\ & 2x(t) + 2u_1(t) + 2u_2(t) = 0 \text{ a.e. in } [0,1], \\ & (x(0),x(1)) \in \{0\} \times \mathbb{R}. \end{array}$$

Since $\dot{x}(t) \geqslant 0$ a.e. in $[0,1]$ and $x(0) = 0$, we have $x(1) \geqslant 0$. Then $(\bar{x},\bar{u}_1,\bar{u}_2) = (0,0,0)$ is the only optimal process. As we saw in Example 4.2, RC is valid. The maximum principle is verified with $\eta = 1,\ p(t) = -1$ and $\mu_1(t) = \mu_2(t) = 0$ a.e. in $[0,1]$.

EXAMPLE 4.6 Let us consider the optimal control problem below:

$$\begin{array}{ll} \text{minimize} & l(x(0),x(1)) = x(1) \\ \text{subject to} & \dot{x}(t) = x(t) + 2u_1(t) + 2u_2(t) \text{ a.e. in } [0,1], \\ & A(t)x(t) + B(t)u(t) = b(t) \text{ a.e. in } [0,1], \\ & (x(0),x(1)) \in \{1\} \times \mathbb{R}_+, \end{array}$$

where matrices $A(t)$ and $B(t)$ are given in Example 4.3 and

$$b(t) = \begin{bmatrix} (1-2t)\exp(t)/2 \\ (t-2t^2)\exp(t)/2 \end{bmatrix} \text{ a.e. in } [0,1].$$

Let $(\bar{x}(t),\bar{u}_1(t),\bar{u}_2(t)) = ((1-t)\exp(t),0,-\exp(t)/2)$ a.e. in $[0,1]$. It is clear that this is a feasible process. Since $\bar{x}(1) = 0$, we can see that it is optimal. As we saw in Example 4.3, RC holds true. The conditions of the maximum principle are fulfilled, for example, with $\eta = 1,\ p(t) = \exp(t-1),\ \mu_1(t) = -2\exp(t-1)$ and $\mu_2(t) = 0$ a.e. in $[0,1]$.

4.2. *The non-linear case*

Different from what might be expected, we will not use the result from the last section as a basis for the study of the non-linear case. The approach used here is independent.

We consider the optimal control problem with non-linear equality constraints:

$$\begin{array}{ll} \text{minimize} & l(x(0),x(1)) \\ \text{subject to} & \dot{x}(t)=f(t,x(t),u(t),v(t)) \text{ a.e. in } [0,1], \\ & h(t,x(t),u(t),v(t))=0 \text{ a.e. in } [0,1], \\ & v(t)\in V(t) \text{ a.e. in } [0,1], \\ & (x(0),x(1))\in S. \end{array} \tag{$\mathrm{P}_=$}$$

DEFINITION 4.2 The constant rank condition (CRC) is said to be satisfied at a feasible process $(\bar{x},\bar{u},\bar{v})$ if

$$\operatorname{rank}(\nabla_u h(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))) = r(t) = r \text{ a.e. in } [0,1]$$

and there exist $K>0$ and a sub-matrix containing r rows of $\nabla_u h(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))$, say

$$\Gamma(t)^T = \left[\ \nabla_u h_{i_1}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) \ \cdots \ \nabla_u h_{i_r}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\ \right],$$

such that

(i) $\det(\Gamma(t)\Gamma(t)^T)\geqslant K$ a.e. in $[0,1]$;

(ii) $\{\nabla_{x,u,v}h_i(t,x,u,v)\}\cup\{\nabla_{x,u,v}h_{i_j}(t,x,u,v)\}_{j=1}^r$ has constant rank equal to r in $T_\epsilon(t)$ a.e. in $[0,1]$, for each $i\in\{1,\ldots,m_h\}\setminus\{i_1,\ldots,i_r\}$.

As we have already mentioned, in mathematical programming, the constant rank constraint qualification is automatically verified in the case of linear constraints. If the $r(t)$ above is not assumed to be constant, it is an easy task to verify that the regularity condition RC is satisfied when CRC is valid. For technical reasons, such an assumption is necessary, and we cannot compare these two regularity conditions.

As in the last subsection, the Hamiltonian function is

$$H(t,x,p,\mu,u,v) := p\cdot f(t,x,u,v) + \mu\cdot h(t,x,u,v).$$

THEOREM 4.7 Let $(\bar{x},\bar{u},\bar{v})$ be a weak local optimal process of $(\mathrm{P}_=)$. Assume that the basic hypotheses (H1)–(H5) are valid and the constant rank condition (CRC) is satisfied at $(\bar{x},\bar{u},\bar{v})$. Then there exist $\eta\geqslant 0$, $p\in W^{1,1}([0,1];\mathbb{R}^n)$, $\mu\in L^1([0,1];\mathbb{R}^m)$ and $\zeta\in L^1([0,1];\mathbb{R}^{k_v})$ such that, for almost all $t\in[0,1]$,

(i) $\eta+\|p\|_\infty\neq 0$;

(ii) $(-\dot{p}(t),0,\zeta(t))\in \operatorname{co}\partial_{x,u,v}H(t,\bar{x}(t),p(t),\mu(t),\bar{u}(t),\bar{v}(t))$;

(iii) $\zeta(t)\in\operatorname{co}N_{V(t)}(\bar{v}(t))$;

(iv) $(p(0),-p(1))\in N_S(\bar{x}(0),\bar{x}(1))+\eta\partial l(\bar{x}(0),\bar{x}(1))$.

Moreover, there exists $M>0$ such that

$$|\mu(t)|\leqslant k_f(t)Mk_{g,h}|p(t)| \text{ a.e. in } [0,1].$$

Proof. Let T_0 be the largest subset of $[0,1]$ in which conditions (H1)–(H5) and CRC do not hold for every $t \in T_0$. Provided T_0 has Lebesgue measure zero, it follows that there exists a Borel set T_1 (being the intersection of a countable family of open sets) with $T_0 \subset T_1$ such that $T_1 \setminus T_0$ is of measure zero (see Rudin, 1964). Set $T = [0,1] \setminus T_1$.

If $(\bar{x}, \bar{u}, \bar{v})$ is a weak local optimal process of $(P_=)$, there exists $\varepsilon > 0$ such that $l(\bar{x}(0), \bar{x}(1)) \leqslant l(x(0), x(1))$ for all feasible processes (x, u, v) which satisfy $(x(t), u(t), v(t)) \in T_\varepsilon(t)$ a.e. in $[0,1]$.

Consider the auxiliary problem below:

$$\begin{array}{ll} \text{minimize} & l(x(0), x(1)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t), v(t)) \text{ a.e. in } [0,1], \\ & \hat{h}(t, x(t), u(t), v(t)) = 0 \text{ a.e. in } [0,1], \\ & v(t) \in V(t) \text{ a.e. in } [0,1], \\ & (x(0), x(1)) \in S, \end{array} \tag{4.2}$$

where $\hat{h} : [0,1] \times \mathbb{R}^n \times \mathbb{R}^{k_u} \times \mathbb{R}^{k_v} \to \mathbb{R}^r$ is defined as

$$\hat{h}_j(t, x, u, v) = h_{i_j}(t, x, u, v), j = 1, \ldots, r.$$

Process $(\bar{x}, \bar{v}, \bar{u})$ is feasible for (4.2), since it is for $(P_=)$. It will be shown that $(\bar{x}, \bar{v}, \bar{u})$ is a weak local optimal process of (4.2). On the contrary, assume that given $0 < \delta < \varepsilon$, there exists a feasible process $(\tilde{x}, \tilde{u}, \tilde{v})$ of (4.2) with $(\tilde{x}(t), \tilde{u}(t), \tilde{v}(t)) \in T_\delta(t)$ a.e. in $[0,1]$ and $l(\tilde{x}(0), \tilde{x}(1)) < l(\bar{x}(0), \bar{x}(1))$. If one has that $(\tilde{x}, \tilde{u}, \tilde{v})$ is a feasible process of $(P_=)$, a contradiction to the optimality of it will follow. It is enough to show that $h_i(t, \tilde{x}(t), \tilde{u}(t), \tilde{v}(t)) = 0$ a.e. in $[0,1]$, $i \in \{1, \ldots, m_h\} \setminus \{i_1, \ldots, i_r\}$.

Let $i_0 \in \{1, \ldots, m_h\} \setminus \{i_1, \ldots, i_r\}$. In Lemma 3.3, let us identify t with a, T with A, $(0,0,0)$ with x_0, εB with $x_0 + \alpha B$, (x, u, v) with x, and

$$f_0^t(x, u, v) = h_{i_0}(t, \bar{x}(t) + x, \bar{u}(t) + u, \bar{v}(t) + v),$$

$$f_j^t(x, u, v) = h_{i_j}(t, \bar{x}(t) + x, \bar{u}(t) + u, \bar{v}(t) + v), j = 1, \ldots, r.$$

In order to apply Lemma 3.3, assumptions (a)–(c) of Lemma 3.2 should be satisfied. Assumptions (a) and (b) follow directly from (H3). Assumption (c) follows from CRC-(i) and an application of Weyl's Theorem 2.4. Furthermore, from the definition of $\Gamma(t)$ and CRC-(i) one has that $\{\nabla_u f_j^t(0,0,0) = \nabla_u h_{i_j}(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))\}_{j=1}^r$, $t \in T$, is linearly independent. Then, the set $\{\nabla_{x,u,v} f_j^t(0,0,0)\}_{j=1}^r$, $t \in T$, is also linearly independent. So, from CRC-(ii) it comes that the set $\{\nabla f_{i_0}^t(x, u, v)\} \cup \{\nabla f_j^t(x, u, v)\}_{j=1}^r$ has constant rank equal to r in εB, $t \in T$, that is, $\nabla f_0^t(x, u, v)$ is a linear combination of $\nabla f_1^t(x, u, v), \ldots, \nabla f_r^t(x, u, v)$, $t \in T$, for all $(x, u, v) \in \varepsilon B$. It follows from Lemma 3.3 that there exist $\sigma \in (0, \varepsilon)$, $\rho > 0$ and a family of continuously differentiable functions $\chi^t : \rho B \to \mathbb{R}$ such that, for all $(x, u, v) \in \sigma B$, one has that

$$(f_1^t(x, u, v), \ldots, f_r^t(x, u, v)) \in \rho B, t \in T,$$

and

$$f_0^t(x, u, v) = \chi^t(f_1^t(x, u, v), \ldots, f_r^t(x, u, v)), t \in T.$$

Therefore, diminishing δ if necessary,

$$\begin{aligned}
&h_{i_0}(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))\\
&\quad = f_0^t(\tilde{x}(t)-\bar{x}(t),\tilde{u}(t)-\bar{u}(t),\tilde{v}(t)-\bar{v}(t))\\
&\quad = \chi^t(f_1^t(\tilde{x}(t)-\bar{x}(t),\tilde{u}(t)-\bar{u}(t),\tilde{v}(t)-\bar{v}(t)),\ldots,f_r^t(\tilde{x}(t)-\bar{x}(t),\tilde{u}(t)-\bar{u}(t),\tilde{v}(t)-\bar{v}(t)))\\
&\quad = \chi^t(h_{i_1}(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t)),\ldots,h_{i_r}(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t)))\\
&\quad = \chi^t(0,\ldots,0)\\
&\quad = \chi^t(h_{i_1}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)),\ldots,h_{i_r}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)))\\
&\quad = \chi^t(f_1^t(0,0,0),\ldots,f_r^t(0,0,0))\\
&\quad = f_0^t(0,0,0) = h_{i_0}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) = 0 \text{ a.e. in } [0,1].
\end{aligned}$$

It follows from Theorem 2.2 that there exist $\eta \geqslant 0$, $p \in W^{1,1}([0,1];\mathbb{R}^n)$, $\mu \in L^1([0,1];\mathbb{R}^r)$ and $\zeta \in L^1([0,1];\mathbb{R}^{k_v})$ such that

(i) $\|p\|_\infty + \eta \neq 0$;

(ii) $(-\dot{p}(t),0,\zeta(t)) \in \operatorname{co}\partial_{x,u,v}H(t,\bar{x}(t),p(t),\mu(t),\bar{u}(t),\bar{v}(t))$ a.e. in $[0,1]$;

(iii) $\zeta(t) \in \operatorname{co}N_{V(t)}(\bar{v}(t))$ a.e. in $[0,1]$;

(iv) $(p(0),-p(1)) \in N_S(\bar{x}(0),\bar{x}(1)) + \eta\partial l(\bar{x}(0),\bar{x}(1))$.

Furthermore, there exists $M > 0$ such that $|\mu(t)| \leqslant k_f(t)Mk_{h,g}|p(t)|$ a.e. in $[0,1]$. The result follows defining $\tilde{\mu} : [0,1] \to \mathbb{R}^{m_h}$ almost everywhere in $[0,1]$ as

$$\tilde{\mu}_i(t) = \begin{cases} \mu_i(t), & i \in \{1,\ldots,r\},\\ 0, & i \in \{1,\ldots,m_h\} \setminus \{1,\ldots,r\}. \end{cases}$$

□

Below we have an instance of a non-linear problem in which CRC is satisfied while the full rank condition is not valid. The weak basic constraint qualification is not applied in as much as the problem is not autonomous.

EXAMPLE 4.8 We analyse the problem presented next.

$$\begin{array}{ll}
\text{Minimize} & l(x(0),x(1)) = -x(1)\\
\text{subject to} & \dot{x}(t) = x(t) + u_1(t)u_2(t) \text{ a.e. in } [0,1],\\
& (t+1)u_1(t) - t^2 - t = 0 \text{ a.e. in } [0,1],\\
& u_1(t)^2 - t^2 = 0 \text{ a.e. in } [0,1],\\
& (x(0),x(1)) \in \{0\} \times [-1,1].
\end{array}$$

Let $(\bar{x}(t),\bar{u}_1(t),\bar{u}_2(t)) = (t^2\exp(t-1), t, 2\exp(t-1))$ a.e. in $[0,1]$. We have $\bar{x}(1) = 1$, so that this feasible process is optimal. CRC is satisfied at $(\bar{x},\bar{u}_1,\bar{u}_2)$. Indeed,

$$\nabla_u h(t,\bar{x}(t),\bar{u}_1(t),\bar{u}_2(t)) = \begin{bmatrix} t+1 & 0\\ 2t & 0 \end{bmatrix}$$

and $\operatorname{rank}(\nabla_u h(t, \bar{x}(t), \bar{u}_1(t), \bar{u}_2(t))) = 1$ a.e. in $[0, 1]$. Taking $\Gamma(t) = [t + 10]$ a.e. in $[0, 1]$, we have $\det(\Gamma(t)\Gamma(t)^T) = (t + 1)^2 \geqslant 1$ a.e. in $[0, 1]$. It is clear that

$$\{\nabla_{x,u_1,u_2} h_2(t, x, u_1, u_2)\} \cup \{\nabla_{x,u_1,u_2} h_1(t, x, u_1, u_2)\} = \{(0, 2u_1, 0), (0, t + 1, 0)\}$$

has constant rank equal to 1 in $T_\epsilon(t)$ a.e. in $[0, 1]$. By Theorem 4.7, the maximum principle is valid. In fact, conditions (i)–(iv) are verified with $\eta = 1$, $p(t) = \mu_1(t) = \mu_2(t) = 0$ a.e. in $[0, 1]$.

5. Problems with mixed equality and inequality constraints

In this last section, the weak maximum principle is established for some instances of the general problem, which we restate here for the readers' convenience:

$$\begin{array}{ll} \text{minimize} & l(x(0), x(1)) \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t), v(t)) \text{ a.e. in } [0, 1], \\ & g(t, x(t), u(t), v(t)) \leqslant 0 \text{ a.e. in } [0, 1], \\ & h(t, x(t), u(t), v(t)) = 0 \text{ a.e. in } [0, 1], \\ & v(t) \in V(t) \text{ a.e. in } [0, 1], \\ & (x(0), x(1)) \in S. \end{array} \tag{P}$$

The proof technique employed here to demonstrate the weak maximum principle for (P) consists in transforming the inequality constraints into equality ones, by introducing slackness variables, and adapting CRC to the obtained equality constrained problem. We, then, apply Theorem 4.7.

We assume throughout this section that $m_h \geqslant 1$, that is, at least one equality constraint is present.

We start with some definitions related to the inequality constraints. Let $(\bar{x}, \bar{u}, \bar{v})$ be a feasible process. We will denote $I = \{1, \ldots, m_g\}$ and $J = \{1, \ldots, m_h\}$. The index set of the active constraints is defined as

$$I_a(t) = \{i \in I : g_i(t, \bar{x}(t), \bar{u}(t), \bar{v}(t)) = 0\} \text{ a.e. in } [0, 1].$$

Its complement will be denoted by $I_c(t)$, that is, $I_c(t) = I \setminus I_a(t)$ a.e. in $[0, 1]$.

Given any set of indices $\mathscr{I} \subset \mathbb{N}$, its cardinality will be denoted by $|\mathscr{I}|$. Given any matrix $A \in \mathbb{R}^{m \times k}$ and a subset of indices $\mathscr{I} \subset \{1, \ldots, m\}$, the matrix obtained from A after removing the rows with indices not belonging to $\mathscr{I}$ will be denoted as $A^{\mathscr{I}}$.

The following notation will be used in the sequel:

$$G(t) = \operatorname{diag}\{-g_j(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))\}_{j=1}^{m_g} \text{ a.e. in } [0, 1].$$

DEFINITION 5.1 The constant rank condition (CRC) is said to be satisfied at a feasible process $(\bar{x}, \bar{u}, \bar{v})$ if

$$\operatorname{rank}\left(\begin{bmatrix} \nabla_u g^{I_a(t)}(t, \bar{x}(t), \bar{u}(t), \bar{v}(t)) \\ \nabla_u h(t, \bar{x}(t), \bar{u}(t), \bar{v}(t)) \end{bmatrix}\right) = r(t) \text{ a.e. in } [0, 1]$$

and there exist $K > 0$ and index subsets $J_1(t) = \{i_1, \ldots, i_{r_1(t)}\} \subset I_a(t)$ and $J_2(t) = \{j_1, \ldots, j_{r_2(t)}\} \subset J$, $J_2(t) \neq \emptyset$, with $|J_1(t)| + |J_2(t)| = r(t)$ and $|I_c(t)| + r(t) = \rho(t) = \rho$ a.e. in $[0, 1]$, such that

(i) $\det(\Gamma(t)\Gamma(t)^T) \geqslant K$ a.e. in $[0,1]$, where

$$\Gamma(t) = \begin{bmatrix} \nabla_u g^{I_c(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & G^{I_c(t)}(t) \\ \nabla_u g^{J_1(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & 0 \\ \nabla_u h^{J_2(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & 0 \end{bmatrix} \text{ a.e. in } [0,1];$$

(ii) for each $i \in I_a(t) \setminus J_1(t)$, $\{\nabla_{x,u,v} g_i(t,x,u,v)\} \cup \{\nabla_{x,u,v} h_j(t,x,u,v)\}_{j\in J_2(t)}$ and for each $i \in J \setminus J_2(t)$, $\{\nabla_{x,u,v} h_i(t,x,u,v)\} \cup \{\nabla_{x,u,v} h_j(t,x,u,v)\}_{j\in J_2(t)}$ have constant rank equal to $|J_2(t)|$ in $T_\varepsilon(t)$ a.e. in $[0,1]$.

In the presence of equality and inequality constraints, the Hamiltonian is

$$H(t,x,p,\lambda,\mu,u,v) := p\cdot f(t,x,u,v) + \lambda\cdot g(t,x,u,v) + \mu\cdot h(t,x,u,v).$$

THEOREM 5.1 Let $(\bar{x},\bar{u},\bar{v})$ be a weak local optimal process of (P). Assume that the basic hypotheses (H1)–(H5) are valid at $(\bar{x},\bar{u},\bar{v})$ and the constant rank condition CRC is satisfied. Then there exist $\eta \geqslant 0$, $p \in W^{1,1}([0,1];\mathbb{R}^n)$, $\lambda \in L^1([0,1];\mathbb{R}^{m_g})$, $\mu \in L^1([0,1];\mathbb{R}^{m_h})$ and $\zeta \in L^1([0,1];\mathbb{R}^{k_v})$ such that, for almost all $t\in[0,1]$,

(i) $\eta + \|p\|_\infty \neq 0$;

(ii) $(-\dot{p}(t), 0, \zeta(t)) \in \operatorname{co}\partial_{x,u,v} H(t,\bar{x}(t),p(t),\lambda(t),\mu(t),\bar{u}(t),\bar{v}(t))$;

(iii) $\zeta(t) \in \operatorname{co} N_{V(t)}(\bar{v}(t))$;

(iv) $\lambda_i(t) \leqslant 0$ and $\lambda_i(t) g_i(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) = 0$ a.e. in $[0,1]$, $i=1,\dots,m_g$;

(v) $(p(0),-p(1)) \in N_S(\bar{x}(0),\bar{x}(1)) + \eta\partial l(\bar{x}(0),\bar{x}(1))$.

Moreover, there exists $M>0$ such that

$$|\,(\lambda(t),\mu(t))\,| \leqslant k_f(t) M k_{g,h} |p(t)| \text{ a.e. in } [0,1].$$

Proof. For each $j\in\{1,\dots,m_g\}$ we define the auxiliary function $\delta_j : [0,1]\to\mathbb{R}$ as

$$\delta_j(t) = \begin{cases} 1 & \text{if } j \in J_1(t), \\ 0 & \text{otherwise,} \end{cases}$$

and $\Delta : [0,1] \to \mathbb{R}^{m_g\times m_g}$ as

$$\Delta(t) = \operatorname{diag}\{\delta_j(t)\}_{j=1}^{m_g}.$$

Consider the auxiliary optimal control problem posed as

$$\begin{array}{ll} \text{minimize} & l(x(0),x(1)) \\ \text{subject to} & \dot{x}(t) = f(t,x(t),u(t),v(t)) \text{ a.e. in } [0,1], \\ & g(t,x(t),u(t),v(t)) + G(t)w(t) + \Delta(t)z(t) = 0 \text{ a.e. in } [0,1], \\ & h(t,x(t),u(t),v(t)) = 0 \text{ a.e. in } [0,1], \\ & (u(t),v(t),w(t),z(t)) \in \mathbb{R}^{k_u}\times V(t)\times\mathbb{R}^{m_g}\times\mathbb{R}_+^{m_g} \text{ a.e. in } [0,1], \\ & (x(0),x(1)) \in S, \end{array} \tag{5.1}$$

where $w, z : [0,1] \to \mathbb{R}^{m_g}$ are auxiliary control functions, which are supposed to be measurable.

If $(\bar{x},\bar{u},\bar{v})$ is a weak local optimal process of (P), there exists $0<\varepsilon<1$ such that $l(\bar{x}(0),\bar{x}(1))\leqslant l(x(0),x(1))$ for all feasible processes (x,u,v) that satisfy $(x(t),u(t),v(t))\in T_\varepsilon(t)$ a.e. in $[0,1]$.

For each $j\in\{1,\dots,m_g\}$, defining $\bar{w}_j(t)=1$ and $\bar{z}_j(t)=-g_j(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))$ a.e. in $[0,1]$, the process $(\bar{x},\bar{u},\bar{v},\bar{w},\bar{z})$ is a weak local optimal process of (5.1). Indeed, it is clear that it is feasible in (5.1). Let $(\tilde{x},\tilde{u},\tilde{v},\tilde{w},\tilde{z})$ be an arbitrary feasible process of (5.1) with $(\tilde{x}(t),\tilde{u}(t),\tilde{v}(t),\tilde{w}(t),\tilde{z}(t))\in T_\varepsilon(t)\times(\bar{w}(t)+\varepsilon\bar{B})\times\left((\bar{z}(t)+\varepsilon\bar{B})\cap\mathbb{R}^{m_g}_+\right)$ a.e. in $[0,1]$. Let us show that $(\tilde{x},\tilde{u},\tilde{v})$ is feasible in (P). It is enough to show that the inequality constraints are satisfied, since all other constraints clearly are. Fix an arbitrary $t\in[0,1]$. For $j\in J_1(t)$, one has

$$g_j(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))-g_j(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\tilde{w}_j(t)+\delta_j(t)\tilde{z}_j(t)=0\Leftrightarrow g_j(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))=-\tilde{z}_j(t)\leqslant 0.$$

For $j\in I_a(t)\setminus J_1(t)$, one has

$$g_j(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))-g_j(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\tilde{w}_j(t)+\delta_j(t)\tilde{z}_j(t)=0\Leftrightarrow g_j(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))=0.$$

For $j\in I_c(t)$, one has

$$\begin{aligned}&g_j(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))-g_j(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\tilde{w}_j(t)+\delta_j(t)\tilde{z}_j(t)=0\\ &\Leftrightarrow g_j(t,\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))=g_j(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\tilde{w}_j(t)<0,\end{aligned}$$

since $\tilde{w}(t)\in\bar{w}(t)+\varepsilon\bar{B}$ together with $\bar{w}_j(t)=1$, $j\in\{1,\dots,m_g\}$, and $0<\varepsilon<1$ imply that $\tilde{w}_j(t)>0$, $j\in\{1,\dots,m_g\}$. Thence, $(\tilde{x},\tilde{u},\tilde{v})$ is feasible in (P) with $(\tilde{x}(t),\tilde{u}(t),\tilde{v}(t))\in T_\varepsilon(t)$ a.e. in $[0,1]$. By the local optimality of $(\bar{x},\bar{u},\bar{v})$, one has $l(\bar{x}(0),\bar{x}(1))\leqslant l(\tilde{x}(0),\tilde{x}(1))$.

We will apply Theorem 4.7. Hypotheses (H1)–(H5) and CRC (Definition 4.2) should be satisfied at $(\bar{x},\bar{u},\bar{v},\bar{w},\bar{z})$ for (5.1). Define $\psi:[0,1]\times\mathbb{R}^n\times\mathbb{R}^{k_u}\times\mathbb{R}^{k_v}\times\mathbb{R}^{m_g}\times\mathbb{R}^{m_g}\to\mathbb{R}^{m_h+m_g}$ as

$$\psi(t,x,u,v,w,z)=\begin{pmatrix}g(t,x,u,v)+G(t)w+\varDelta(t)z\\ h(t,x,u,v)\end{pmatrix}.$$

Hypothesis (H1)–(H5) are immediate. Let us check CRC. We have that

$$\nabla_{u,w}\psi(t,\bar{x}(t),\bar{u}(t),\bar{v}(t),\bar{w}(t),\bar{z}(t))=\begin{bmatrix}\nabla_u g(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & G(t)\\ \nabla_u h(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & 0\end{bmatrix}$$

and

$$\nabla_{x,u,v,w,z}\psi(t,x,u,v,w,z)=\begin{bmatrix}\nabla_{x,u,v}g(t,x,u,v) & G(t) & \varDelta(t)\\ \nabla_{x,u,v}h(t,x,u,v) & 0 & 0\end{bmatrix}.$$

From Definition 5.1, we have that

$$\operatorname{rank}(\nabla_{u,w}\psi(t,\bar{x}(t),\bar{u}(t),\bar{v}(t),\bar{w}(t),\bar{z}(t)))=|I_c(t)|+r(t)=\rho(t)=\rho\text{ a.e. in }[0,1].$$

Note that matrix $\varGamma(t)$ in Definition 5.1 is a sub-matrix containing ρ rows of $\nabla_{u,w}\psi(t,\bar{x}(t),\bar{u}(t),\bar{v}(t),\bar{w}(t),\bar{z}(t))$ a.e. in $[0,1]$. Then condition (i) in Definition 4.2 for problem (5.1) follows from condition (i) in Definition 5.1. Let us denote by $J(t)$ the set of indices of rows of $\nabla_{x,u,v,w,z}\psi(t,x,u,v,w,z)$ corresponding to the rows in $\Gamma(t)$, that is, corresponding to $I_c(t)\cup J_1(t)\cup J_2(t)$. If $\{\nabla_{x,u,v,w,z}\psi_i(t,x,u,v,w,z)\}\cup\{\nabla_{x,u,v,w,z}\psi_j(t,x,u,v,w,z)\}_{j\in J(t)}$ has rank equal to $\rho+1$ for some $i\in\{1,\dots,m_g+m_h\}\setminus J(t)$ and some $(x,u,v,w,z)\in T_\varepsilon(t)\times(\bar{w}(t)+\varepsilon\bar{B})\times\left((\bar{z}(t)+\varepsilon\bar{B})\cap\mathbb{R}_+^{m_g}\right)$ a.e. in $[0,1]$, then this contradicts condition (ii) in Definition 5.1. Thence, condition (ii) in Definition 4.2 is satisfied.

It follows from Theorem 4.7 that there exist $\eta\geqslant 0$, $p\in W^{1,1}([0,1];\mathbb{R}^n)$, $\lambda\in L^1([0,1];\mathbb{R}^{m_g})$, $\mu\in L^1([0,1];\mathbb{R}^m)$ and $(\zeta_1,\zeta_2)\in L^1([0,1];\mathbb{R}^{k_v}\times\mathbb{R}^{m_g})$ such that, for almost all $t\in[0,1]$,

$$\eta+\|p\|_\infty\neq 0; \tag{5.2}$$

$$\begin{aligned}(-\dot{p}(t),0,\zeta_1(t),0,\zeta_2(t))\in\operatorname{co}\partial_{x,u,v,w,z}\{&p(t)\cdot f(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))+\lambda(t)\cdot[g(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\\&+G(t)\bar{w}(t)+\Delta(t)\bar{z}(t)]+\mu(t)\cdot h(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))\};\end{aligned} \tag{5.3}$$

$$(\zeta_1(t),\zeta_2(t))\in\operatorname{co}N_{V(t)}(\bar{v}(t))\times\operatorname{co}N_{\mathbb{R}_+^{m_g}}(\bar{z}(t)); \tag{5.4}$$

$$(p(0),-p(1))\in N_S(\bar{x}(0),\bar{x}(1))+\eta\partial l(\bar{x}(0),\bar{x}(1)). \tag{5.5}$$

Moreover, there exists $M>0$ such that

$$|(\lambda(t),\mu(t))|\leqslant k_f(t)Mk_{g,h}|p(t)|\text{ a.e. in }[0,1]. \tag{5.6}$$

Deriving with respect to w and z, one sees, from (5.3), that $G(t)\lambda(t)=0$, so that $\lambda_i(t)g_i(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))=0$, $i\in I$, and $\Delta(t)\lambda(t)=\zeta_2(t)$ a.e. in $[0,1]$. From (5.4), one sees that $\zeta_2(t)\leqslant 0$ a.e. in $[0,1]$. Then $\lambda_i(t)\leqslant 0$ for $i\in J_1(t)$ a.e. in $[0,1]$. Furthermore, one knows from the proof of Theorem 4.7 that $\lambda_i(t)=0$ for $I\setminus J_1(t)$ and $\mu_j(t)=0$ for $j\in J\setminus J_2(t)$ a.e. in $[0,1]$. Therefore, conditions (i) to (v) follow from (5.2)–(5.6). □

Remark 5.1 If $m_h=0$, this proof technique works only if $\nabla_u g^{I_a(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t))$ has full rank a.e. in $[0,1]$.

The constant rank condition CRC is satisfied at the optimal solution of the optimal control problem in the next example, while the full rank (de Pinho, 2003) as well as the Mangasarian–Fromovitz (de Pinho & Rosenblueth, 2008) conditions are not.

Example 5.2 We will verify the constant rank condition at the optimal solution of the problem below:

$$\begin{array}{ll}\text{minimize} & l(x(0),x(1))=\exp(x(1))\\ \text{subject to} & \dot{x}(t)=x(t)-v(t)\text{ a.e. in}[0,1],\\ & 2u_1(t)-v(t)\leqslant 0\text{ a.e. in}[0,1],\\ & u_1(t)+u_2(t)+v(t)=0\text{ a.e. in}[0,1],\\ & -u_1(t)+u_2(t)+2v(t)=0\text{ a.e. in}[0,1],\\ & v(t)\in\mathbb{R}_+\text{ a.e. in}[0,1],\\ & (x(0),x(1))\in\{1\}\times\mathbb{R}_+.\end{array}$$

Let $(\bar{x}(t), \bar{u}_1(t), \bar{u}_2(t), \bar{v}(t)) = ((1-t)\exp(t), \exp(t)/2, -3\exp(t)/2, \exp(t))$ a.e. in $[0,1]$. We have $\bar{x}(1) = 0$, so then this feasible process is optimal. In this case, $I_a(t) = \{1\}$ a.e. in $[0,1]$. We have that

$$\operatorname{rank}\left(\begin{bmatrix} \nabla_u g^{I_a(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) \\ \nabla_u h(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\right) = 2 \text{ a.e. in } [0,1].$$

Let us take $J_1(t) = \emptyset$ and $J_2 = \{1,2\}$ a.e. in $[0,1]$. Then,

$$\Gamma(t) = \begin{bmatrix} \nabla_u g^{I_c(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & G^{I_c(t)}(t) \\ \nabla_u g^{J_1(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & 0 \\ \nabla_u h^{J_2(t)}(t,\bar{x}(t),\bar{u}(t),\bar{v}(t)) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \text{ a.e. in } [0,1]$$

and $\det(\Gamma(t)\Gamma(t)^T) = 4$ a.e. in $[0,1]$, so that condition (i) in CRC is satisfied. Let us check condition (ii). Let $i \in I_a(t) \setminus J_1(t)$, that is, let $i = 1$. Then $\{\nabla_{x,u,v} g_1(t,x,u,v)\} \cup \{\nabla_{x,u,v} h_j(t,x,u,v)\}_{j\in J} = \{(0,2,0,-1),(0,1,1,1),(0,-1,1,2)\}$ has constant rank equal to $|J_2(t)| = 2$ in $T_\varepsilon(t)$ a.e. in $[0,1]$. Therefore, CRC is satisfied. The weak maximum principle is verified with $\eta = 1$, $p(t) = \lambda(t) = \mu_1(t) = \mu_2(t) = 0$ a.e. in $[0,1]$.

6. Concluding remarks

The paper is dedicated to obtaining the weak maximum principle for optimal control problems involving mixed state-control equality constraints under a new regularity condition of constant rank type (CRC). In fact, the constant rank constraint qualification is not new. It was introduced by Janin (1984) for mathematical programming problems. To the best of our knowledge, it is new however in the optimal control context. As in the mathematical programming setting, the constant rank condition is weaker than the full rank one and this is an alternative regularity condition to the Mangasarian–Fromovitz constraint qualification.

Optimal control problems in which mixed inequality constraints are present together with at least one equality constraint were also considered. The general case, including problems with inequality constraints only, was not covered, due to the proof technique. Analysing the proof of the last theorem, we see that CRC is never satisfied at the optimal solution of (5.1) when $J_2(t) = \emptyset$ a.e. in $[0,1]$. When $m_h = 0$, CRC to (5.1) is satisfied only if the full rank condition holds, as we already mentioned in Remark 5.1.

The example below shows that the weak maximum principle may be valid when CRC is not satisfied, from where we conjecture that it is possible to formulate another constraint qualification, possibly of constant rank type, which is weaker than CRC. That is going to be a topic for future research.

EXAMPLE 6.1 Let us consider the optimal control problem given in what follows:

$$\begin{array}{ll} \text{minimize} & l(x(0),x(1)) = x(1) \\ \text{subject to} & \dot{x}(t) = (v(t)+t)^2 \text{ a.e. in } [0,1], \\ & -u_1(t) \leqslant 0 \text{ a.e. in } [0,1], \\ & u_2(t) + v(t) \leqslant t \text{ a.e. in } [0,1], \\ & -u_1(t)^2 + u_2(t) + v(t) = t \text{ a.e. in } [0,1], \\ & v(t) \in \mathbb{R}_- \text{ a.e. in } [0,1], \\ & (x(0),x(1)) \in \{0\} \times \mathbb{R}. \end{array}$$

The feasible process $(\bar{x}(t), \bar{u}_1(t), \bar{u}_2(t), \bar{v}(t)) = (0, 0, 2t, -t)$ a.e. in $[0, 1]$ is clearly optimal. In this case, $I_a(t) = \{1, 2\}$ a.e. in $[0, 1]$. We have that

$$\operatorname{rank}\left(\begin{bmatrix} \nabla_u g^{I_a(t)}(t, \bar{x}(t), \bar{u}(t), \bar{v}(t)) \\ \nabla_u h(t, \bar{x}(t), \bar{u}(t), \bar{v}(t)) \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = 2 \text{ a.e. in } [0, 1].$$

The possible choices for the sets J_1 and J_2 are $J_1(t) = \{1\}$ and $J_2(t) = \{1\}$, $J_1(t) = \{1, 2\}$ and $J_2(t) = \emptyset$ or $J_1(t) = \{2\}$ and $J_2(t) = \{1\}$ a.e. in $[0, 1]$. In the first option, condition (ii) is not satisfied. The second one is not allowed, since $J_2 = \emptyset$. In the third one, condition (i) fails. Note that neither the full rank nor the Mangasarian–Fromovitz conditions are satisfied. The weak basic constraint qualification is not applied due to the fact that the problem is not autonomous. Nevertheless, the weak maximum principle holds, for example, with $\eta = 1$, $p(t) = -1$, $\lambda_1(t) = 0$, $\lambda_2(t) = -t$ and $\mu(t) = t$ a.e. in $[0, 1]$.

Another topic for future work is related to the implications of constant rank type conditions in the convergence study of numerical methods for mixed constrained optimal control problems. We have successfully applied the ICLOCS (Imperial College London Optimal Control Software) package (see Falugi *et al.*, 2010) to solve all the examples presented in the paper. The fact that full rank is not valid in any of them led us to think that convergence of computational algorithms to solve optimal control problems with mixed constraints is likely to occur under constant rank assumptions.

A last comment concerns the assumption that $\operatorname{rank}(\nabla_u h(t, \bar{x}(t), \bar{u}(t), \bar{v}(t))) = r(t) = r$ a.e. in $[0, 1]$ in CRC (Definition 4.2), that is, the rank of the Jacobian is assumed to be constant with respect to t. In the linear case, in Definition 4.1, this kind of assumption was unnecessary, so we think it may be possible to remove it in the non-linear case. This is going to be another topic for future research as well.

Funding

São Paulo Research Foundation (FAPESP) [2015/25610-1 to J.T.P., 2013/07375-0 to V.A.O, R.A. and G.N.S., 2016/03540-4 to V.A.O.]; National Council for Scientific and Technological Development (CNPq) [457785/2014-4 to V.A.O., 310955/2015-7 to V.A.O.].

Acknowledgements

The authors are grateful to anonymous referees for their valuable comments, which improved the final version of this work.

References

Andreani, R., Birgin, E. G., Martínez, J. M. & Schuverdt, M. L. (2007) On augmented Lagrangian methods with general lower-level constraints. *SIAM J. Optim.*, **18**, 1286–1309.

Andreani, R., Echagüe, C. E. & Schuverdt, M. L. (2010) Constant-rank condition and second-order constraint qualification. *J. Optim. Theory Appl.*, **146**, 255–266.

Arutyunov, A. V. (2000) *Optimality Conditions: Abnormal and Degenerate Problems*. Dordrecht: Kluwer Academic Publishers.

Arutyunov, A. V., Karamzin, D. Y., Pereira, F. L. & Silva, G. N. (2016) Investigation of regularity conditions in optimal control problems with geometric mixed constraints. *Optimization*, **65**, 185–206.

Biswas, M. H. A. & de Pinho, M. R. (2015) A maximum principle for optimal control problems with state and mixed constraints. *ESAIM Control Optim. Calc. Var.*, **21**, 939–957.

Boccia, A., de Pinho, M. D. R. & Vinter, R. B. (2016) Optimal control problems with mixed and pure state constraints. *SIAM J. Control Optim.*, **54**, 3061–3083.

Clarke, F. H. (1983) *Optimization and Nonsmooth Analysis*. New York: Wiley-Interscience.

Clarke, F. H. (2005) The maximum principle in optimal control then and now. *J. Control Cybern.*, **34**, 709–722.

Clarke, F. H. & de Pinho, M. R. (2010) Optimal control problems with mixed constraints. *SIAM J. Control Optim.*, **48**, 4500–4524.

Clarke, F., Ledyaev, Y. & de Pinho, M. R. (2011) An extension of the Schwarzkopf multiplier rule in optimal control. *SIAM J. Control Optim.*, **49**, 599–610.

Devdariani, E. N. & Ledyaev, Y. S. (1999) Maximum principle for implicit control systems. *Appl. Math. Optim.*, **40**, 79–103.

Dmitruk, A. V. (1993) Maximum principle for the general optimal control problem with phase and regular mixed constraints. *Comput. Math. Model.*, **4**, 364–377.

Dmitruk, A. V. & Osmolovskii, N. P. (2014) Necessary conditions for a weak minimum in optimal control problems with integral equations subject to state and mixed constraints. *SIAM J. Control Optim.*, **52**, 3437–3462.

Dontchev, A. L., Hager, W. W. & Malanowski, K. (2000) Error bounds for Euler approximation of a state and control constrained optimal control problem. *Numer. Funct. Anal. Optim.*, **21**, 653–682.

Falugi, P., Kerrigan, E. & van Wyk, E. (2010) *Imperial College London Optimal Control Software User Guide (ICLOCS)*. London, England, UK: Imperial College London.

Gayte, I., Guillén-González, F. & Rojas-Medar, M. (2010) Dubovitskii–Milyutin formalism applied to optimal control problems with constraints given by the heat equation with final data. *IMA J. Math. Control Inform.*, **27**, 57–76.

Girsanov, I. V. (1972) *Lectures on Mathematical Theory of Extremum Problems* (B. T. Poljak ed.). Lecture Notes in Economics and Mathematical Systems, Vol. 67. Berlin-New York: Springer. Translated from Russian by D. Louvish.

Hestenes, M. R. (1966) *Calculus of Variations and Optimal Control Theory*. New York: John Wiley.

von Heusinger, A., Kanzow, C. & Fukushima, M. (2012) Newton's method for computing a normalized equilibrium in the generalized Nash game through fixed point formulation. *Math. Program.*, **132**, 99–123.

Horn, R. A. & Johnson, C. R. (2013) *Matrix Analysis*, 2nd edn. Cambridge: Cambridge University Press.

Janin, R. (1984) Directional derivative of the marginal function in nonlinear programming. *Math. Program. Stud.*, **21**, 110–126.

Ledzewicz, U. (1993) On abnormal optimal control problems with mixed equality and inequality constraints. *J. Math. Anal. Appl.*, **173**, 18–42.

Li, A. & Ye, J. J. (2016) Necessary optimality conditions for optimal control problems with nonsmooth mixed state and control constraints. *Set-Valued Var. Anal.*, **24**, 449–470.

Li, A. & Ye, J. J. (2018) Necessary optimality conditions for implicit control systems with applications to control of differential algebraic equations. *Set-Valued Var. Anal.*, **26**, 179–203.

Milyutin, A. A. & Osmolovskii, N. P. (1998) *Calculus of Variations and Optimal Control*. Providence: Amer. Math. Soc.

Mordukhovich, B. S. (2006) *Variational Analysis and Generalized Differentiation I. Basic Theory*. Grundlehren der Mathematischen Wissenschaften, vol. 330 [Fundamental Principles of Mathematical Sciences]. Berlin: Springer.

Noble, B. & Daniel, J. W. (1977) *Applied Linear Algebra*, 2nd edn. Englewood Cliffs, N.J.: Prentice-Hall, Inc.

Páles, Z. & Zeidan, V. (1994) First and second order necessary conditions for optimal control problems with constraints. *Trans. Amer. Math. Soc.*, **346**, 421–453.

de Pinho, M. D. R. & Vinter, R. B. (1995) An Euler–Lagrange inclusion for optimal control problems. *IEEE Trans. Automat. Control*, **40**, 1191–1198.

de Pinho, M. R. (2003) Mixed constrained control problems. *J. Math. Anal. Appl.*, **278**, 293–307.

de Pinho, M. R. (2016) On necessary conditions for implicit control systems. *Pure Appl. Funct. Anal.*, **1**, 185–196.

de Pinho, M. R. & Ilchmann, A. (2002) Weak maximum principle for optimal control problems with mixed constraints. *Nonlinear Anal.*, **48**, 1179–1196.

de Pinho, M. R. & Rosenblueth, J. F. (2008) Necessary conditions for constrained problems under Mangasarian–Fromowitz conditions. *SIAM J. Control Optim.*, **47**, 535–552.

de Pinho, M. R. & Vinter, R. B. (1997) Necessary conditions for optimal control problems involving nonlinear differential algebraic equations. *J. Math. Anal. Appl.*, **212**, 493–516.

de Pinho, M. R., Vinter, R. B. & Zheng, H. (2001) A maximum principle for optimal control problems with mixed constraints. *IMA J. Math. Control Inform.*, **18**, 189–205.

Rudin, W. (1964) *Principles of Mathematical Analysis*. New York: McGraw-Hill.

Sun, B. (2017) Maximum principle for optimal control of vibrations of a dynamic Gao beam in contact with a rigid foundation. *Int. J. Syst. Sci.*, **48**, 3522–3529.

Vinter, R. (2000) *Optimal Control*. Boston: Birkhäuser.

Xu, D. C., Han, J. Y. & Chen, Z. W. (2004) Nonmonotone trust-region method for nonlinear programming with general constraints and simple bounds. *J. Optim. Theory Appl.*, **122**, 185–206.