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AN EXTREMAL NONNEGATIVE SINE POLYNOMIAL

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ABSTRACT. For any positive integer n, the sine polynomials that are nonnegative in $[0,\pi]$ and which have the maximal derivative at the origin are determined in an explicit form. Associated cosine polynomials $K_n(\theta)$ are constructed in such a way that $\{K_n(\theta)\}$ is a summability kernel. Thus, for each p, $1 \le p \le \infty$ and for any 2π-periodic function $f \in L_p[-\pi,\pi]$, the sequence of convolutions $K_n * f$ is proved to converge to f in $L_p[-\pi,\pi]$. The pointwise and almost everywhere convergences are also consequences of our construction.

1. Introduction and statement of results. There are various reasons for the interest in the problem of constructing nonnegative trigonometric polynomials. Among them are the Gibbs phenomenon [**16**, Section 9], univalent functions and polynomials [**7**], positive Jacobi polynomial sums [**1**] and orthogonal polynomials on the unit circle [**15**].

Our study is motivated by a basic fact from the theory of Fourier series and by an intuitive observation which comes from an overview of the variety of known nonnegative trigonometric polynomials. The sequence $\{k_n(\theta)\}$ of even, nonnegative continuous 2π-periodic functions is called an *even positive kernel* if $k_n(\theta)$ are normalized by $(1/2\pi)\int_{-\pi}^{\pi} k_n(\theta)\,d\theta = 1$ and they converge locally uniformly in $(0,2\pi)$ (uniformly on every compact subset of $(0,2\pi)$) to zero. It is a slight modification of the definition in Katznelson's book [**8**]. In what follows we denote by $k_n * f$ the convolution $(1/2\pi)\int_{-\pi}^{\pi} k_n(t)f(\theta-t)\,dt$. It is well known that, for every 2π-periodic function $f \in L_p[-\pi,\pi]$, $1 \le p \le \infty$, the sequence of convolutions $k_n * f$ converges to f in the $L_p[-\pi,\pi]$-norm provided k_n is a sequence of even positive summability kernels. The convolutions converge also pointwise and almost everywhere. We refer to the first chapter of [**8**] for the details.

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On the other hand, most of the classical positive summability kernels are sequences of nonnegative cosine polynomials which obey certain extremal properties. Fejér [**3**] proved that the cosine polynomials

$$F_n(\theta) = 1 + 2\sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)\cos k\theta, \tag{1.1}$$

are nonnegative and established the uniform convergence of the sequence $F_n * f$ to f for any continuous 2π-periodic function f. It is easily seen that Fejér's cosine polynomial (1.1) is the only solution of the extremal problem

$$\max\left\{a_1 + \cdots + a_n : 1 + \sum_{k=1}^{n} a_k \cos k\theta \geq 0\right\}.$$

A basic tool for constructing positive kernels is the Fejér-Riesz representation of nonnegative trigonometric polynomials (see [**4**]). It states that for every nonnegative trigonometric polynomial $T(\theta)$,

$$T(\theta) = a_0 + \sum_{k=1}^{n}(a_k \cos k\theta + b_k \sin k\theta), \tag{1.2}$$

there exists an algebraic polynomial $R(z) = \sum_{k=0}^{n} c_k z^k$ of degree n such that $T(\theta) = |R(e^{i\theta})|^2$, and conversely, for every algebraic polynomial $R(z)$ of degree n, the polynomial $|R(e^{i\theta})|^2$ is a nonnegative trigonometric polynomial of order n. Fejér [**4**] showed that

$$\sqrt{a_1^2 + b_1^2} \leq 2\cos\big(\pi/(n+2)\big) \tag{1.3}$$

for any nonnegative trigonometric polynomial (1.2) with $a_0 = 1$ and that this bound is sharp. As a consequence he obtained the estimate

$$a_1 \leq 2\cos\big(\pi/(n+2)\big) \tag{1.4}$$

for the first coefficient of any nonnegative polynomial of the form

$$1 + \sum_{k=1}^{n} a_k \cos k\theta.$$

Moreover, Fejér determined the nonnegative trigonometric and cosine polynomials for which inequalities in (1.3) and (1.4) are attained. A nice simple proof of Jackson's approximation theorem in Rivlin [**10**, Chapter 1] makes an essential use of the extremal property of this cosine polynomial.

These observations already suggest that many sequences of nonnegative trigonometric polynomials whose coefficients obey certain extremal properties are positive summability kernels.

While there are many results concerning extremal nonnegative cosine polynomials [**2, 12**], only a few results of the same nature about nonnegative sine polynomials are known. Since sine polynomials are odd functions, in what follows we shall call

$$s_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

a nonnegative sine polynomial if $s_n(\theta) \geq 0$ for every $\theta \in [0, \pi]$. It is clear that, if $s_n(\theta)$ is nonnegative, then $b_1 \geq 0$ and $b_1 = 0$ if and only if s_n is identically zero. Rogosinski and Szegő [**11**] considered some extremal problems for nonnegative sine polynomials. Among the other results, they proved that

$$(1.5)\quad s_n'(0) = 1+2b_2+\cdots+nb_n \leq \begin{cases} n(n+2)(n+4)/24 & n \text{ even} \\ (n+1)(n+2)(n+3)/24 & n \text{ odd,} \end{cases}$$

provided b_k are the coefficients of a sine polynomial $s_n(\theta)$ in the space

$$\mathbf{S}_n^+ = \left\{ s_n(\theta) = \sin\theta + \sum_{k=2}^{n} b_k \sin k\theta : s_n(\theta) \geq 0 \quad \text{for } \theta \in [0, \pi] \right\}.$$

However, the sine polynomials for which the above limits are attained were not determined explicitly. The first objective of this paper is to fill this gap.

Theorem 1. *The inequality* (1.5) *holds for every* $s_n(\theta) \in \mathbf{S}_n^+$. *Moreover, if* $n = 2m+2$ *is even, then the equality* $S'_{2m+2}(0) = (m+1)(m+2)(m+3)/3$ *is attained only for the nonnegative sine* *polynomial*

(1.6)

$$S_{2m+2}(\theta) = \sum_{k=0}^{m} \left\{ \left(1 - \frac{k}{m+1}\right)\left(1 - \frac{k}{m+2}\right)\left(2k+1 + \frac{k(k-1)}{m+3}\right) \right.$$
$$\times \sin(2k+1)\theta + (k+1)\left(1 - \frac{k}{m+1}\right)$$
$$\left. \times \left(2 - \frac{k+3}{m+2} - \frac{k(k+1)}{(m+2)(m+3)}\right) \sin(2k+2)\theta \right\},$$

and, if $n = 2m+1$ is odd, the equality

$$S'_{2m+1}(0) = (m+1)(m+2)(2m+3)/6$$

is attained only for the nonnegative sine polynomial

(1.7)

$$S_{2m+1}(\theta) = \sum_{k=0}^{m} \left\{ \left(1 - \frac{k}{m+1}\right)\left(1+2k - \frac{k(k+2)}{m+2} - \frac{k(k+1)(2k+1)}{(m+2)(2m+3)}\right) \right.$$
$$\times \sin(2k+1)\theta + 2(k+1)\left(1 - \frac{k}{m+1}\right)\left(1 - \frac{k+2}{m+2}\right)$$
$$\left. \times \left(1 + \frac{k+1}{2m+3}\right) \sin(2k+2)\theta \right\}.$$

We shall obtain a close form representation of the extremal polynomials (1.6) and (1.7) in terms of the ultraspherical polynomials $P_n^{(2)}(x)$. Recall that, for any $\lambda > -1/2$, $\{P_n^{(\lambda)}(x)\}$ are orthogonal in $[-1,1]$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$ and are normalized by $P_n^{(\lambda)}(1) = (2\lambda)_n/n!$ where $(a)_n$ is the Pochhammer symbol. Section 4.7 of Szegő's book [**13**] provides comprehensive information on the ultraspherical polynomials.

Theorem 2. *For any positive integer m the polynomials* (1.6) *and* (1.7) *are given by*

(1.8)

$$S_{2m+2}(\theta) = \frac{12}{(m+1)(m+2)(m+3)} \sin\theta \left[\cos(\theta/2) P_m^{(2)}(\cos\theta)\right]^2$$

and

(1.9)
$$S_{2m+1}(\theta) = \frac{6}{(m+1)(m+2)(2m+3)}\ \sin\theta\big[P_m^{(2)}(\cos\theta)+P_{m-1}^{(2)}(\cos\theta)\big]^2.$$

Since $2(n+2)P_n^{(2)}(x) = T''_{n+2}(x)$, where $T_n(x)$ denotes the nth Chebyshev polynomial of the first kind, then we can represent $S_{2m+2}(\theta)$ and $S_{2m+1}(\theta)$ in the form

(1.10)
$$S_{2m+2}(\theta) = \frac{3}{(m+1)(m+2)^3(m+3)}\ \sin\theta\big[\cos(\theta/2)T''_{m+2}(\cos\theta)\big]^2$$

and

(1.11)
$$\begin{aligned} S_{2m+1}(\theta) = {} & \frac{3}{2(m+1)^3(m+2)^3(2m+3)} \\ & \times \sin\theta\big[(m+1)T''_{m+2}(\cos\theta) + (m+2)T''_{m+1}(\cos\theta)\big]^2. \end{aligned}$$

Then the well-known representation of the second derivative of the Chebyshev polynomial

$$T''_n(\cos\theta) = \frac{n}{\sin^3\theta}\,\{\cos\theta\sin n\theta + n\sin\theta\cos n\theta\}$$

yields the following equivalent closed-form representations of the above extremal sine polynomials:

$$S_{2m+2}(\theta) = \frac{3\cos^2(\frac{\theta}{2})\big[(m+2)\sin\theta\cos\big((m+2)\theta\big)-\cos\theta\sin\big((m+2)\theta\big)\big]^2}{(m+1)(m+2)(m+3)\sin^5\theta}$$

and

$$\begin{aligned} S_{2m+1}(\theta) = {} & \frac{3}{2(m+1)(m+2)(2m+3)}\,\frac{1}{\sin^3\theta} \\ & \times \Big[(m+2)\cos\big((m+2)\theta\big) + (m+1)\cos\big((m+1)\theta\big) \\ & \quad + \cot\theta\Big(\sin\big((m+2)\theta\big) + \sin\big((m+1)\theta\big)\Big)\Big]^2. \end{aligned}$$

Set $K_n(\theta) = (1/2\pi)\sin\theta S_n(\theta)$ and, for any function $f(x)$ which is 2π-periodic and integrable in $[-\pi, \pi]$, define the trigonometric polynomial

$$K_n(f; x) = \int_{-\pi}^{\pi} K_n(\theta) f(x-\theta)\, d\theta.$$

Observe that $K_n(\theta)$ is a cosine polynomial of order $n+1$. In Section 4 we shall prove that $\{K_n(\theta)\}$ is a positive summability kernel and then the following results on L_p, pointwise and almost everywhere convergence of $K_n(f; x)$ will immediately hold.

Theorem 3. *For any p, $1 \le p \le \infty$, and for every 2π-periodic function $f \in L_p[-\pi, \pi]$, the sequence $K_n(f; x)$ converges to f in $L_p[-\pi, \pi]$.*

Theorem 4. *Let f be a 2π-periodic function which is integrable in $[-\pi, \pi]$. If, for $x \in [-\pi, \pi]$, the limit $\lim_{h\to 0}(f(x+h) + f(x-h))$ exists, then*

$$K_n(f; x) \to (1/2) \quad \lim_{h\to 0}\big(f(x+h) + f(x-h)\big) \quad \textit{as } n \textit{ diverges.}$$

Theorem 5. *Let f be a 2π-periodic function which is integrable in $[-\pi, \pi]$. Then $K_n(f; x)$ converges to f almost everywhere in $[-\pi, \pi]$.*

It is worth mentioning that, while the sequences $\{k_n(\theta)\}$ of classical summability kernels, namely, Fejér's, de la Vallée Poussin's and Jackson's one, converge to infinity at the origin, in our case $K_n(0)$ vanishes for any positive integer n.

2. Preliminary results. The above-mentioned Fejér-Riesz's theorem and a result of Szegő [**13**, p. 4] imply a representation of nonnegative cosine polynomials.

Lemma 1. *Let*

$$c_n(\theta) = a_0 + 2\sum_{k=1}^{n} a_k \cos k\theta$$

be a cosine polynomial of order n which is nonnegative for every real θ. Then an algebraic polynomial $R(z) = \sum_{k=0}^{n} c_k z^k$ exists of degree n with real coefficients, such that $c_n(\theta) = |R(e^{i\theta})|^2$. Thus, the cosine polynomial $c_n(\theta)$ of order n is nonnegative if and only if there exist real numbers c_k, $k = 0, 1, \dots, n$, such that

$$a_0 = \sum_{k=0}^{n} c_k^2 \quad \text{and} \quad a_k = \sum_{\nu=0}^{n-k} c_{k+\nu} c_\nu \quad \text{for } k = 1, \dots, n. \tag{2.12}$$

The following relation between nonnegative sine polynomials $s_n(\theta)$ and nonnegative cosine polynomials $c_{n-1}(\theta)$ is an immediate consequence of the relation $s_n(\theta) = \sin\theta c_{n-1}(\theta)$ (see [**11**]).

Lemma 2. *The sine polynomial of order n*

$$s_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is nonnegative in $[0, \pi]$ if and only if the cosine polynomial of order $n - 1$

$$c_{n-1}(\theta) = a_0 + 2\sum_{k=1}^{n-1} a_k \cos k\theta,$$

where

$$\begin{aligned} b_k &= a_{k-1} - a_{k+1} \quad \text{for } k = 1, \dots, n-2, \\ b_{n-1} &= a_{n-2}, \\ b_n &= a_{n-1}, \end{aligned} \tag{2.13}$$

is nonnegative.

These two lemmas imply a parametric representation for the coefficients of the nonnegative sine polynomials.

Lemma 3. *The sine polynomial of order n*

$$s_n(\theta) = \sum_{k=1}^{n} b_k \sin k\theta$$

is nonnegative if and only if there exist real numbers $c_0, \dots, c_{n-1}$ *such that*

$$\begin{aligned}
b_1 &= \sum_{\nu=0}^{n-1} c_\nu^2 - \sum_{\nu=0}^{n-3} c_\nu c_{\nu+2}, \\
b_k &= \sum_{\nu=0}^{n-k} c_{k+\nu-1} c_\nu - \sum_{\nu=0}^{n-k-2} c_{k+\nu+1} c_\nu, \quad \text{for } k = 2, \dots, n-2, \qquad (2.14) \\
b_{n-1} &= c_0 c_{n-2} + c_1 c_{n-1}, \\
b_n &= c_0 c_{n-1}.
\end{aligned}$$

Set $q_k = (k+1)/(2k)$. It can be verified that, if n is even, $n = 2m+2$, then b_1 is given by

(2.15)

$$\begin{aligned}
b_1 = &\sum_{k=0}^{m-1} \Big\{ q_{k+1} \big((c_{2k} - c_{2k+2})/(2q_{k+1}) \big)^2 + q_{k+1} \big((c_{2k+1} - c_{2k+3})/(2q_{k+1}) \big)^2 \Big\} \\
&+ q_{m+1} c_{2m}^2 + q_{m+1} c_{2m+1}^2,
\end{aligned}$$

and, if n is odd, $n = 2m+1$, then

(2.16)

$$\begin{aligned}
b_1 = &\sum_{k=0}^{m-1} \Big\{ q_{k+1} \big((c_{2k} - c_{2k+2})/(2q_{k+1}) \big)^2 + q_{k+1} \big((c_{2k+1} - c_{2k+3})/(2q_{k+1}) \big)^2 \Big\} \\
&+ q_m \big((c_{2m-2} - c_{2m})/(2q_m) \big)^2 + q_m c_{2m-1}^2 + q_{m+1} c_{2m}^2.
\end{aligned}$$

3. Proof of Theorem 1. We need to maximize $s_n'(0)$ subject to the conditions $s_n(\theta) \geq 0$ in $[0, \pi]$ and $b_1 = 1$. Apply Lemmas 1 and 2 to represent the derivative of the nonnegative sine polynomial $s_n(\theta)$ at the origin in terms of the parameters c_k. We obtain

$$\begin{aligned}
s_n'(0) &= \sum_{k=1}^{n} k b_k = \sum_{k=1}^{n-2} k(a_{k-1} - a_{k+1}) + (n-1) a_{n-2} + n a_{n-1} \\
&= a_0 + 2 \sum_{k=1}^{n-1} a_k = \sum_{j=0}^{n-1} c_j^2 + 2 \sum_{0 \leq j < k \leq n-1} c_j c_k = \Big(\sum_{k=0}^{n-1} c_k \Big)^2.
\end{aligned}$$

Set

$$G(c) = G(c_0, \dots, c_{n-1}) = \left(\sum_{k=0}^{n-1} c_k \right)^2 .$$

Then the extremal problem to be solved is

$$\max\{G(c) : b_1(c) = 1\}, \tag{3.17}$$

where $b_1 = b_1(c) = b_1(c_0, \dots, c_{n-1})$ is defined by (2.15) or (2.16) depending on the parity of n.

We employ the Lagrange multipliers approach (see [**6**, Section 9.1]) to solve this problem. The necessary conditions for $c = (c_0, \dots, c_{n-1})$ to be a point of extremum for (3.17) are

$$\nabla G(c) = \lambda \nabla b_1(c) \quad \text{and} \quad b_1(c) = 1, \tag{3.18}$$

where ∇ denotes the gradient operator and λ is the Lagrange multiplier.

It is obvious that every solution c of (3.18) which corresponds to $\lambda = 0$ minimizes $G(c)$. Indeed, in this case we have $\sum_{i=0}^{n-1} c_i = 0$ and these are the only points where the nonnegative function $G(c)$ vanishes. It is worth mentioning that this observation means that a nonnegative sine polynomial has a derivative which vanishes at the origin if and only if its coefficients are given by (2.14) and the parameters satisfy $\sum_{i=1}^{n-1} c_i = 0$.

Thus, for the points of maximum the Lagrange multiplier λ is nonzero. Define

$$r_e(r_o) = \lambda^{-1} \sum_{j=1}^{n-1} c_j \quad \text{if } n \text{ even (odd)},$$

and

$$\xi_{2k} = c_{2k} - (2q_{k+1})^{-1} c_{2k+2},$$

and

$$\xi_{2k+1} = c_{2k+1} - (2q_{k+1})^{-1} c_{2k+3}, \quad k = 0, \dots, m,$$

where we set $c_{2m+2} = c_{2m+3} = 0$.

Consider first the case $n = 2m+2$. The conditions $\nabla G(c) = \lambda \nabla b_1(c)$ reduce to the parametric system of linear equations

$$\begin{gathered} q_1\xi_0 = q_1\xi_1 = r_e \\ -\xi_{2k} + 2q_{k+2}\xi_{2k+2} = -\xi_{2k+1} + 2q_{k+2}\xi_{2k+3} = 2r_e, \\ k = 0, \dots, m-1 \end{gathered}$$

for the unknowns ξ_k, $k = 0, \dots, 2m+1$ and the parameter r_e. The explicit form of q_k yields

$$\frac{1}{q_{k+1}}\left(1 + \frac{1}{2q_k}\left(1 + \cdots + \frac{1}{2q_2}\left(1 + \frac{1}{2q_1}\right)\right)\right) = k+1.$$

Then we obtain the solution ξ_k, $k = 0, \dots, 2m+1$, of the above linear system explicitly in terms of r_e:

$$\begin{aligned} \xi_{2k} = \xi_{2k+1} &= \frac{1}{q_{k+1}}\left(1 + \frac{1}{2q_k}\left(1 + \cdots + \frac{1}{2q_2}\left(1 + \frac{1}{2q_1}\right)\right)\right) r_e \\ &= (k+1) r_e, \quad k = 0, \dots, m. \end{aligned}$$

Hence, for the parameters c_k, $k = 0, \dots, 2m$, we obtain

$$c_{2k} = c_{2k+1} = (k+1)(m-k+1) r_e, \quad k = 0, \dots, m.$$

Now the condition $b_1(c) = 1$ gives

$$r_e^2 = 3/\big((m+1)(m+2)(m+3)\big). \tag{3.19}$$

In the case when $n = 2m+1$ similar observations yield

$$\begin{aligned} c_{2k} &= (k+1)(m-k+1) r_o, \quad k = 0, \dots, m, \\ c_{2k+1} &= (k+1)(m-k) r_o, \quad k = 0, \dots, m-1, \end{aligned}$$

and

$$r_o^2 = 6/\big((m+1)(m+2)(2m+3)\big). \tag{3.20}$$

Now the relations (2.14) and straightforward calculations imply the explicit form of the coefficients b_k.

4. Proofs of Theorems 2, 3, 4 and 5.

Proof of Theorem 2. Observe that the statement of Lemma 2 is equivalent to the equality $S_n(\theta) = \sin\theta C_{n-1}(\theta)$, where

$$C_{n-1}(\theta) = a_0 + 2\sum_{k=1}^{n-1} a_k \cos k\theta,$$

and the coefficients of $C_{n-1}(\theta)$ are given by (2.12) for the parameters $c_0, \dots, c_{n-1}$ determined in the proof of Theorem 1. Then Lemma 1 and the explicit form of c_k yield

$$S_n(\theta) = r^2 \sin\theta \left| R_{n-1}(e^{i\theta}) \right|^2, \tag{4.21}$$

where $r = r_e$ or $r = r_o$ depending on the parity of n and

$$R_{2m+1}(z) = (1+z)\sum_{k=0}^{m}(k+1)(m-k+1)z^{2k}$$

and

$$R_{2m}(z) = \sum_{k=0}^{m}(k+1)(m-k+1)z^{2k} + z\sum_{k=0}^{m-1}(k+1)(m-k)z^{2k}.$$

Following Fejér and Szegő [**5**], consider the Cesàro sums $\mathcal{S}_m^j(\zeta)$ of order j of the geometric series,

$$\mathcal{S}_m^j(\zeta) = \sum_{k=0}^{m} \binom{m+j-k}{j} \zeta^k.$$

Then

$$R_{2m+1}(z) = (1+z)\,\frac{d\mathcal{S}_{m+1}^1(\zeta)}{d\zeta}\Big|_{\zeta=z^2}$$

and

$$R_{2m}(z) = \frac{d\mathcal{S}_{m+1}^1(\zeta)}{d\zeta}\Big|_{\zeta=z^2} + z\,\frac{d\mathcal{S}_{m}^1(\zeta)}{d\zeta}\Big|_{\zeta=z^2}.$$

On the other hand, Turán [**14**] proved that, for any pair n, j of positive integers,

$$(d/d\zeta)^j \mathcal{S}^j_{n+j}(\zeta) = j!\zeta^{n/2} P_n^{(j+1)}\big((\zeta^{1/2} + \zeta^{-1/2})/2\big),$$

where $P_n^{(\lambda)}$ denotes the ultraspherical polynomial. Applying this result for $j = 1$, we obtain

$$R_{2m+1}(z) = (1+z) z^m P_m^{(2)}\big((z+z^{-1})/2\big)$$

and

$$R_{2m}(z) = z^m \big\{ P_m^{(2)}\big((z+z^{-1})/2\big) + P_{m-1}^{(2)}\big((z+z^{-1})/2\big)\big\}.$$

Substitute $z = \exp(i\theta)$ in these representations and use (4.21) to complete the proof. $\square$

The next is a basic technical result.

Lemma 4. *For every positive integer n and for any real θ, the inequality*

$$\sin^2(\theta/2) K_n(\theta) \leq c(1/n), \tag{4.22}$$

holds with an absolute constant c.

Proof. We provide three different proofs of the lemma depending on the representation of the extremal sine polynomials. It suffices to prove the above inequality for $\theta \in [0, \pi]$.

The first proof uses representations (1.8) and (1.9) of $S_n(\theta)$. First we establish (4.22) in the case $n = 2m+2$. It follows from the definition of the kernel $K_n(\theta)$ and from (1.8) that

$$\begin{aligned} \sin^2(\theta/2) K_{2m+2}(\theta) &= \frac{1}{2\pi} \sin^2(\theta/2) \sin\theta S_{2m+2}(\theta) \\ &= \frac{r_e^2}{2\pi} \sin^2(\theta/2) \sin^2\theta \big[2\cos(\theta/2) P_m^{(2)}(\cos\theta)\big]^2 \\ &= \frac{r_e^2}{2\pi} \big[\sin^2\theta P_m^{(2)}(\cos\theta)\big]^2. \end{aligned} \tag{4.23}$$

Kogbetliantz [**9**] proved that, for any $\lambda \ge 0$,

$$(\sin\theta)^\lambda \big|P_n^{(\lambda)}(\cos\theta)\big| \le \frac{2\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)} \quad \text{for } 0 \le \theta \le \pi. \tag{4.24}$$

An application of this result for $\lambda = 2$ yields

$$\sin^2(\theta/2)K_{2m+2}(\theta) \le \frac{6}{\pi}\,\frac{m+1}{(m+2)(m+3)}.$$

When $n = 2m+1$, we need a little more effort. We have

$$\sin^2(\theta/2)K_{2m+1}(\theta) = \frac{r_o^2}{2\pi}\sin^2(\theta)\sin^2(\theta/2)\big[P_m^{(2)}(\cos\theta) + P_{m-1}^{(2)}(\cos\theta)\big]^2.$$

The recurrence relation for the ultraspherical polynomials (see (4.7.17) in [**13**]) implies

$$P_m^{(2)}(x)+P_{m-1}^{(2)}(x) = (1+x)P_{m-1}^{(2)}(x) + \frac{m+2}{m}\left(xP_{m-1}^{(2)}(x) - P_{m-2}^{(2)}(x)\right)$$

and formulae (4.7.14) and (4.7.28) in [**13**] yield

$$xP_{m-1}^{(2)}(x)-P_{m-2}^{(2)}(x) = \frac{1}{2}\left(x\,\frac{d}{dx}P_m^{(1)}(x) - \frac{d}{dx}P_{m-1}^{(1)}(x)\right) = \frac{m}{2}P_m^{(1)}(x).$$

Therefore,

$$\begin{aligned}&\sin^2(\theta/2)K_{2m+1}(\theta)\\ &= \frac{r_o^2}{2\pi}\sin^2(\theta)\sin^2(\theta/2)\left[2\cos^2(\theta/2)P_{m-1}^{(2)}(\cos\theta) + \frac{m+2}{2}P_m^{(1)}(\cos\theta)\right]^2.\end{aligned}$$

Then we use (4.24) for $\lambda = 1$ and for $\lambda = 2$ to obtain

$$\begin{aligned}\sin^2(\theta/2)&K_{2m+1}(\theta)\\ &\le \frac{r_o^2}{2\pi}\bigg\{\cos^2(\theta/2)\big[\sin^2\theta P_{m-1}^{(2)}(\cos\theta)\big]^2\\ &\qquad + \frac{m+2}{2}\,\sin\theta\big[\sin^2\theta\big|P_{m-1}^{(2)}(\cos\theta)\big|\big]\big[\sin\theta\big|P_m^{(1)}(\cos\theta)\big|\big]\\ &\qquad + \frac{(m+2)^2}{4}\,\sin^2(\theta/2)\big[\sin\theta P_m^{(1)}(\cos\theta)\big]^2\bigg\}\\ &\le \frac{r_o^2}{2\pi}\left\{4m^2\cos^2\frac{\theta}{2} + 2m(m+2)\sin\theta + (m+2)^2\sin^2\frac{\theta}{2}\right\}\\ &= \frac{r_o^2}{2\pi}\,\big|2m\cos(\theta/2) + (m+2)\sin(\theta/2)\big|^2\\ &\le \frac{1}{2\pi}\,\frac{6}{(m+1)(m+2)(2m+3)}\,\frac{(5m^2+4m+4)^2}{4m^2}.\end{aligned}$$

Here the latter inequality follows from the fact that the maximal value in $[0,\pi]$ of the function $2m\cos(\theta/2)+(m+2)\sin(\theta/2)$, which is positive in $[0,\pi]$, is attained at the point θ_0 for which $2m\sin(\theta_0/2) = (m+2)\cos(\theta_0/2)$. This completes the first proof of the lemma.

The second proof is based on the representations (1.10) and (1.11) of $S_n(\theta)$ in terms of the Chebyshev polynomials. It is essentially equivalent to the first one. We only sketch the proof for $n=2m+2$ because, for $n=2m+1$, it is similar. The representation (4.23) and the relation between $P_n^{(2)}(x)$ and $T''_{m+2}(x)$ yield

$$\sin^2(\theta/2)K_{2m+2}(\theta) = \frac{r_e^2}{8\pi(m+2)^2}\left[\sin^2\theta T''_{m+2}(\cos\theta)\right]^2.$$

Then the second order differential equation for the Chebyshev polynomials

$$(1-x^2)T''_{m+2}(x) = xT'_{m+2}(x) - (m+2)^2T_{m+2}(x)$$

and the inequalities $|T_{m+2}(x)| \le 1$ and $|T'_{m+2}(x)| \le (m+2)^2$ for $x\in[-1,1]$ imply

$$\sin^2(\theta/2)K_{2m+2}(\theta) \le \frac{r_e^2}{8\pi(m+2)^2}\,4(m+2)^4.$$

The third proof is straightforward, and it was the one we obtained before discovering the nice relation between the extremal polynomials $S_n(\theta)$ and ultraspherical and Chebyshev polynomials.

The coefficients β_k in the representation

$$\sin^2(\theta/2)K_n(\theta) = \frac{1}{8\pi}\sum_{k=0}^{n+2}\beta_k\cos k\theta$$

are given by

$$\begin{aligned}
\beta_0 &= 3/(m+2),\\
\beta_{2k} &= -12k/\big((m+1)(m+2)(m+3)\big), \quad k=1,\dots,m+1,\\
\beta_{2k+1} &= 0, \quad k=0,\dots,m+1,\\
\beta_{2m+4} &= 3(m+1)/\big((m+2)(m+3)\big),
\end{aligned}$$

when $n = 2m + 2$, and by

$$\begin{aligned}
\beta_0 &= 6/(2m+3), \\
\beta_{2k} &= -12k/\big((m+1)(m+2)(2m+3)\big), \quad k = 1, \dots, m+1, \\
\beta_{2k+1} &= -6(2k+1)/\big((m+1)(m+2)(2m+3)\big), \quad k = 0, \dots, m, \\
\beta_{2m+3} &= 6(m+1)/\big((m+2)(2m+3)\big),
\end{aligned}$$

when $n = 2m + 1$. It is easy to see that the sum of the modulus of these coefficients is less than c/n where c is an absolute constant. More precisely, we have

$$\big|\sin^2(\theta/2) K_{2m+2}(\theta)\big| \le \frac{3}{2\pi(m+3)}, \quad \big|\sin^2(\theta/2) K_{2m+1}(\theta)\big| \le \frac{3}{2\pi(m+2)}.$$

As it has already been mentioned, Lemma 4 implies the truth of Theorems 3, 4 and 5.

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