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On the natural merit function for solving complementarity problems ⋆

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Abstract Complementarity problems may be formulated as nonlinear systems of equations with non-negativity constraints. The natural merit function is the sum of squares of the components of the system. Sufficient conditions are established which guarantee that stationary points are solutions of the complementarity problem. Algorithmic consequences are discussed.

Keywords Complementarity Problems · Merit functions · Nonlinear Programming

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1 Introduction

The *Complementarity Problem* (CP) considered in this paper consists of finding $x, w \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that

$$H(x, y, w) = 0,\ x^\top w = 0,\ x, w \geq 0, \tag{1}$$

where $H : \mathbb{R}^{n+m+n} \longrightarrow \mathbb{R}^{n+m}$ is continuously differentiable on an open set that contains Ω, and

$$\Omega = \left\{(x, y, w) \in \mathbb{R}^{n+m+n} :\ x, w \geq 0\right\}. \tag{2}$$

The most popular particular case of (1) is the Linear Complementarity Problem (LCP). In this case,

$$H(x, w) = Mx - w + q,$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ are given. Many applications of the LCP have been proposed in science, engineering and economics [9,14,17,22,23,29].

If $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $H(x, w) = G(x) - w$, the CP reduces to the Nonlinear Complementarity Problem [14]. Furthermore, let

$$K = \{x \in \mathbb{R}^n :\ h(x) = 0,\ x \geq 0\}$$

where $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a continuously differentiable function on $\mathbb{R}^n$. We denote $\nabla h(x) = (\nabla h_1(x), \ldots, \nabla h_m(x))$. Then, under a suitable constraint qualification, defining

$$H(x, y, w) = ((G(x) + \nabla h(x)y - w)^\top, h(x)^\top)^\top, \tag{3}$$

the CP problem turns to be equivalent to the Variational Inequality problem defined by the operator G over K [1,14,33]. In particular, if G is the gradient of a continuously differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^1$, then CP with H given by (3) represents the KKT conditions of the optimization problem defined by minimizing f on the set K [14].

The formulation (1) is more general than (3), since there is no restriction on the form of the function H. For example, H may involve the KKT equations of a parametric optimization problem and, additionally, nonlinear conditions involving variables, multipliers and parameters.

Many reformulations of complementarity and variational inequality problems have been discussed in the literature. See, for example, [2,3,11–15,18, 20,21,25,28,35] and references therein. Some reformulations use the fact that $[x_i w_i = 0, x_i \geq 0, w_i \geq 0]$ may be expressed as $\varphi(x_i, w_i) = 0$ by means of the so called NCP functions. The best known one is the Fischer-Burmeister function [19]. As a consequence, complementarity problems may be written as nonlinear systems of equations and Newtonian ideas may be employed for their resolution [18]. In the process os stating nonlinear complementarity problems and variational inequality problems as unconstrained nonlinear systems of equations based on NCP functions, several authors proved equivalence between stationary points of the merit functions and solutions of the original problem under different problem assumptions that go from strict monotonicity to P_0-like conditions. See [11,12,25,15]. Reformulations with simple constraints based on the Fischer-Burmeister function with equivalence results were also proposed in [13].

In the present contribution, we consider the reformulation of CP as the problem of finding a solution $(x, y, w) \in \Omega$ of the square nonlinear system

$$H(x, y, w) = 0, x_1 w_1 = 0, \ldots, x_n w_n = 0. \tag{4}$$

The natural merit function [16,28] is introduced in Section 2, where sufficient conditions are proved ensuring that (first-order) stationary points of the corresponding bound-constrained minimization problem are solutions of (4). Algorithmic consequences of this approach are discussed.

Notation

The set of natural numbers is denoted by $\mathbb{N}$.

The symbol $\|\cdot\|$ denotes the Euclidean norm.

2 The Complementarity Problem and the natural merit function

Let us define $F : \mathbb{R}^{n+m+n} \longrightarrow \mathbb{R}^{n+m+n}$ and $f : \mathbb{R}^{n+m+n} \longrightarrow \mathbb{R}$ by

$$F(x, y, w) = (H(x, y, w)^\top, x_1 w_1, \ldots, x_n w_n)^\top$$

and

$$f(x, y, w) = \|F(x, y, w)\|^2. \tag{5}$$

We consider the problem

$$\text{Minimize } f(x, y, w) \text{ subject to } (x, y, w) \in \Omega, \tag{6}$$

where Ω is given by (2).

We denote $z = (x, y, w)$ from now on. Next we show that, if a stationary point $(\overline{x}, \overline{y}, \overline{w})$ of (6) is not a solution of the complementarity problem (1), then the Jacobian matrix $F'(\overline{x}, \overline{y}, \overline{w})$ is singular. When one tries to solve a nonlinear system $F(x, y, w) = 0$ with $(x, y, w) \in \Omega$ using a standard bound-constraint minimization solver, the main reason for possible failure is the convergence to “bad” stationary points of (6) (generally local minimizers). Therefore, it is interesting to characterize the set of stationary points that *are not* solutions of the system. In the following theorem, we show that this set is reasonably small in the sense that all its elements have singular Jacobians. Note that this property is not true for general nonlinear systems. For example, for the system given by $x + 1 = 0, w - 1 = 0$, the point $(0, 1)$ is stationary but the Jacobian is obviously nonsingular. Therefore, the property proved below is a peculiarity of the complementarity structure of F.

Theorem 1 *Suppose that $\overline{z} = (\overline{x}, \overline{y}, \overline{w})$ is a stationary point of (6). Then, if $\|F(\overline{x}, \overline{y}, \overline{w})\| \neq 0$, the Jacobian $F'(\overline{x}, \overline{y}, \overline{w})$ is singular.*

Proof Since the existence of the variables y_i does not introduce any complication to the proof, in order to simplify the notation, we only consider the case in which $z = (x, w)$ and $F(z) = F(x, w)$.

Let $(\overline{x}, \overline{w})$ be a stationary point of f over Ω. If $\overline{x}_i = \overline{w}_i = 0$, for some $i \in \{1, \dots, n\}$, then row $n + i$ of the Jacobian is null, and the Jacobian is singular. Therefore the theorem is trivial in this case.

Assume that $\overline{x}_{i_k}, \overline{w}_{i_k} > 0$ for p indices i_k, $k = 1, \dots, p$ belonging to $\{1, \dots, n\}$. Then there are three possible cases:

Case 1: $p = n$;
Case 2: $p = 0$;
Case 3: $1 \le p < n$.

In Case 1, the derivatives of f with respect to all the variables must vanish. Since

$$\nabla f(z) = 2F'(z)^\top F(z),$$

this implies the desired result.

Let us now consider Case 2. Since $x_i + w_i > 0$ for all $i = 1, \dots, n$, we may assume without loss of generality that

$$x_i = 0, \ w_i > 0$$

for all $i = 1, \dots, n$. The Jacobian may be written as follows:

$$F'(x, w) = \begin{bmatrix} \frac{\partial H}{\partial x_1} & \cdots & \frac{\partial H}{\partial x_n} & \frac{\partial H}{\partial w_1} & \cdots & \frac{\partial H}{\partial w_n} \\ w_1 & \dots & 0 & x_1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & w_n & 0 & \dots & x_n \end{bmatrix},$$

where

$$\frac{\partial H}{\partial x_j}, \frac{\partial H}{\partial w_j} \in \mathbb{R}^n$$

for $j = 1, \dots, n$. Therefore at $(\overline{x}, \overline{w})$ we have

$$F'(\overline{x}, \overline{w}) = \begin{bmatrix} \frac{\partial H}{\partial x_1}(\overline{x}, \overline{w}) & \dots & \frac{\partial H}{\partial x_n}(\overline{x}, \overline{w}) & \frac{\partial H}{\partial w_1}(\overline{x}, \overline{w}) & \dots & \frac{\partial H}{\partial w_n}(\overline{x}, \overline{w}) \\ \overline{w}_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \overline{w}_n & 0 & \dots & 0 \end{bmatrix}. \quad (7)$$

By stationarity, the derivatives of f with respect to w_j must vanish. Hence, by (7),

$$H(\overline{x}, \overline{w})^\top \frac{\partial H}{\partial w_j}(\overline{x}, \overline{w}) = 0$$

for all $j = 1, \dots, n$. Therefore, either $H(\overline{x}, \overline{w}) = 0$ or the vectors $\frac{\partial H}{\partial w_1}(\overline{x}, \overline{w}), \dots, \frac{\partial H}{\partial w_n}(\overline{x}, \overline{w})$ are linearly dependent. By (7), the latter case implies the singularity of the Jacobian. On the other hand, if $H(\overline{x}, \overline{w}) = 0$ then $F(\overline{x}, \overline{w}) = 0$, due to the complementarity assumption ($\overline{x}_i \overline{w}_i = 0$ for all $i = 1, \dots, n$).

Let us now consider Case 3. Suppose, without loss of generality, that

$$\overline{x}_i, \overline{w}_i > 0 \text{ for } i = 1, \ldots, p < n$$

and

$$\overline{x}_i = 0,\ \overline{w}_i > 0 \text{ for } i = p+1, \ldots, n.$$

Then

$$F'(\overline{x}, \overline{w}) = \begin{bmatrix} \frac{\partial H}{\partial x_1} & \cdots & \frac{\partial H}{\partial x_p} & \frac{\partial H}{\partial x_{p+1}} & \cdots & \frac{\partial H}{\partial x_n} & \frac{\partial H}{\partial w_1} & \cdots & \frac{\partial H}{\partial w_p} & \frac{\partial H}{\partial w_{p+1}} & \cdots & \frac{\partial H}{\partial w_n} \\ \overline{w}_1 & \ldots & 0 & 0 & \ldots & 0 & \overline{x}_1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \overline{w}_p & 0 & \ldots & 0 & 0 & \ldots & \overline{x}_p & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \overline{w}_{p+1} & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \overline{w}_n & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \quad (8)$$

where the partial derivatives $\frac{\partial H}{\partial w_j}$ are computed at $(\overline{x}, \overline{w})$, for all $j = 1, \ldots, n$. Clearly

$$F(\overline{x}, \overline{w}) = \left(H(\overline{x}, \overline{w})^\top, \overline{x}_1\overline{w}_1, \ldots, \overline{x}_p\overline{w}_p, 0, \ldots, 0\right)^\top \in \mathbb{R}^{n+n}. \quad (9)$$

Consider the function $\overline{F} : \mathbb{R}^{n+n} \longrightarrow \mathbb{R}^{n+p}$ given by

$$\overline{F}(\overline{x}, \overline{w}) = \left(H(\overline{x}, \overline{w})^\top, \overline{x}_1\overline{w}_1, \ldots, \overline{x}_p\overline{w}_p\right)^\top \quad (10)$$

and

$$F'(\overline{x}, \overline{w}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ O_{n-p,p} & W & O_{n-p,p} & O_{n-p,n-p} \end{bmatrix}, \quad (11)$$

where $C_{11}, C_{13} \in \mathbb{R}^{(n+p)\times p}$, $C_{12}, C_{14} \in \mathbb{R}^{(n+p)\times(n-p)}$, O_{jk} is the null matrix in $\mathbb{R}^{j\times k}$, $j, k \in \mathbb{N}$ and $W \in \mathbb{R}^{(n-p)\times(n-p)}$ is a diagonal matrix whose diagonal elements are $\overline{w}_{p+1}, \ldots, \overline{w}_n$.

By the optimality condition, we have

$$\frac{\partial f}{\partial x_j}(\overline{x}, \overline{w}) = 0,\ j = 1, \ldots, p,$$
$$\frac{\partial f}{\partial w_j}(\overline{x}, \overline{w}) = 0,\ j = 1, \ldots, n$$

But

$$\nabla f(\overline{x}, \overline{w}) = 2F'(\overline{x}, \overline{w})^\top F(\overline{x}, \overline{w}),$$

Hence the optimality condition states that $\overline{F}(\overline{x}, \overline{w})$ is orthogonal to the columns of C_{11}, C_{13} and C_{14}. Since these are $n + p$ columns and $\overline{F}(\overline{x}, \overline{w}) \in \mathbb{R}^{n+p}$, then either $\overline{F}(\overline{x}, \overline{w}) = 0$ or the columns of C_{11}, C_{13} and C_{14} are linearly dependent. In the first case, by (9) and (10), $F(\overline{x}, \overline{w}) = 0$. Otherwise, by (8) and (11), the Jacobian $F'(\overline{x}, \overline{w})$ is singular. This completes the proof. □

To establish the second result concerning stationary points of (6), we first consider the Variational Inequality Problem over a convex set

$$\begin{array}{l} \text{Find } \overline{x} \in \mathcal{K} \text{ such that} \\ G(\overline{x})^\top (x - \overline{x}) \geq 0, \forall x \in \mathcal{K}, \end{array} \tag{12}$$

where $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuously differentiable mapping, $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}^1$, $i = 1, \ldots, l$ are convex twice smooth functions on $\mathbb{R}^n$ and

$$\mathcal{K} = \{x \in \mathbb{R}^n : \; Ax = b, x \geq 0, \; g_i(x) \leq 0, \; i = 1, \ldots, l\}.$$

If $\mathcal{K}$ satisfies a constraint qualification, this problem is equivalent to the following CP problem:

$$\begin{array}{l} G(x) = A^\top y - \nabla g(x)\mu + w \\ Ax = b \\ g(x) + \alpha = 0 \\ x \geq 0, \; \mu \geq 0, \; w \geq 0, \; \alpha \geq 0 \\ x^\top w = 0 \\ \mu^\top \alpha = 0, \end{array}$$

where $g(x) = (g_1(x), \ldots, g_p(x))$, $\nabla g(x) = (\nabla g_1(x), \ldots, \nabla g_p(x)) \in \mathbb{R}^{n \times l}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\mu \in \mathbb{R}^l$, $w \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^l$ is a vector of slack variables for the constraints $g(x) \leq 0$. The natural merit function for this CP takes the form:

$$\begin{aligned} \Phi(x, w, y, \beta, \alpha) = {} & \left\| G(x) + \nabla g(x)\mu - A^\top y - w \right\|^2 + \|Ax - b\|^2 + \|g(x) + \alpha\|^2 \\ & + \sum_{i=1}^{n} (x_i w_i)^2 + \sum_{i=1}^{l} (\alpha_i \mu_i)^2. \end{aligned}$$

Observe that, replacing x by (x, μ) and w by (w, α), the merit function Φ coincides with the merit function f defined in (5).

Moreover, we may write:

$$\Omega = \left\{ (x, y, w, \mu, \alpha) \in \mathbb{R}^{2n+m+2l} : \; x \geq 0, \; w \geq 0, \; \mu \geq 0, \; \alpha \geq 0 \right\}$$

.

Theorem 2 *If G is monotone on the nullspace of A, $\mathcal{K} \neq \emptyset$ and $\mathcal{K}_1 = \{x \in \mathbb{R}^n : \; Ax = b, \; x \geq 0\}$ is bounded, then every stationary point of Φ over Ω is a solution of (12).*

Proof Let (x, y, w, μ, α) be a stationary point of the merit function over Ω. Then,

$$\begin{aligned} & \left[G'(x)^\top + \sum_{i=1}^{l} \mu_i \nabla^2 g_i(x) \right] p + A^\top (Ax - b) \\ & \quad + \nabla g(x) \left(g(x) + \alpha \right) + (XW)w = v \perp x \end{aligned} \tag{13}$$

$$-Ap = 0 \tag{14}$$
$$-p + (XW)x = z \perp w \tag{15}$$
$$(g(x) + \alpha) + (\Lambda\Upsilon)\mu = \beta \perp \alpha \tag{16}$$
$$\nabla g(x)^\top p + (\Lambda\Upsilon)\alpha = \gamma \perp \mu \tag{17}$$
$$x, v, z, w, \alpha, \beta, \gamma, \mu \geq 0,$$

where $\nabla^2 g_i(x)$ is the Hessian of g_i at x, $\Lambda = \mathrm{diag}(\alpha_i) \in \mathbb{R}^{l\times l}$, $\Upsilon = \mathrm{diag}(\mu_i) \in \mathbb{R}^{l\times l}$ and $X = \mathrm{diag}(x_i) \in \mathbb{R}^{n\times n}$, $W = \mathrm{diag}(w_i) \in \mathbb{R}^{n\times n}$, and

$$p = G(x) + \nabla g(x)\mu - A^\top y - w.$$

By (16) and (17),

$$\begin{aligned} p^\top \nabla g(x) \left(g(x) + \alpha\right) &= \left[\gamma - (\Lambda\Upsilon)\alpha\right]^\top \left[\beta - \Lambda\Upsilon)\mu\right] \\ &= \gamma^\top \beta + \sum_{i=1}^{l} (\alpha_i \mu_i)^3. \end{aligned}$$

Furthermore, using (13) and (15), we have that

$$\begin{aligned} p^\top (XW)w &= -z^\top (XW)w + x^\top (XWXW)w \\ &= \sum_{i=1}^{n} z_i w_i^2 x_i + \sum_{i=1}^{n} x_i^3 w_i^3 \\ &= \sum_{i=1}^{n} (x_i w_i)^3, \end{aligned}$$

and, by (14),

$$p^\top v = v^\top (XW)x - z^\top v = \sum_{i=1}^{n} x_i^2 w_i v_i - z^\top v = -z^\top v.$$

Since $Ap = 0$, then $p = Zd$, for some $d \in \mathbb{R}^{n-m}$, where $Z \in \mathbb{R}^{n\times(n-m)}$ is a matrix whose columns form a basis of the nullspace of A. The above inequalities and (14) yield:

$$d^\top Z^\top \left[G'(x)^\top + \sum_{i=1}^{l} \mu_i \nabla^2 g_i(x)\right] Zd =$$
$$-\left(v^\top z + \gamma^\top \beta + \sum_{i=1}^{l} (\alpha_i \mu_i)^3 + \sum_{i=1}^{n} (x_i w_i)^3\right) \leq 0.$$

Since $Z^\top \left[G'(x)^\top + \sum_{i=1}^{l} \mu_i \nabla^2 g_i(x)\right] Z$ is positive semi–definite, it follows that

$$d^\top Z^\top \left[G'(x)^\top + \sum_{i=1}^{l} \mu_i \nabla^2 g_i(x)\right] Zd = 0$$

and

$$\begin{cases} v^\top z &= 0 \\ \beta^\top \gamma &= 0 \\ \alpha^\top \mu &= 0 \\ x^\top w = 0. \end{cases}$$

Since $(XW)x = 0$, it follows from (15) that

$$p = -z \leq 0.$$

Hence,

$$Ap = 0, \ p \leq 0.$$

Since $\mathcal{K}_1 = \{x \in R^n : \ Ax = b, \ x \geq 0\}$ is bounded, then $p = 0$. Therefore, using $x^\top w = \alpha^\top \mu = 0$ in the equations (13) − (17), we obtain:

$$\begin{array}{rl} A^\top(Ax-b) + \nabla g(x)(g(x)+\alpha) & = v \\ g(x)+\alpha & = \beta \\ x \geq 0, \ v \geq 0, \ \alpha \geq 0, \ \beta \geq 0 & \\ x^\top v = \alpha^\top \beta = 0. & \end{array} \tag{18}$$

Define:

$$\mathcal{I} = \{i \in \{1,2,\ldots,m\} : \ g_i(x) \geq 0\}.$$

If $i \notin \mathcal{I}$, we have that $g_i(x) < 0$, so, by (18), $\alpha_i \geq \beta_i$. Thus, since $\alpha_i\beta_i = 0$, we obtain that $\beta_i = 0$. Therefore, $g_i(x) + \alpha_i = 0$.

If $i \in \mathcal{I}$ we have that $g_i(x) + \alpha_i = \beta_i > 0$. Thus, since $\alpha_i\beta_i = 0$, we have that $\alpha_i = 0$. Therefore, the first equation of (18) may be written:

$$A^\top(Ax-b) + \sum_{i\in\mathcal{I}} \nabla g_i(x) g_i(x) = v. \tag{19}$$

Since $\mathcal{K} \neq \emptyset$, there exists $\widetilde{x}$ such that $A\widetilde{x} = b, \widetilde{x} \geq 0$. Pre-multiplying (19) by $(x - \tilde{x})^\top$, we obtain:

$$(x-\widetilde{x})^\top [A^\top(Ax-b) + \sum_{i\in\mathcal{I}} \nabla g_i(x) g_i(x)] = (x-\widetilde{x})^\top v.$$

Then, since $x^\top v = 0$,

$$\|Ax-b\|^2 + \sum_{i\in\mathcal{I}} \nabla g_i(x) g_i(x)] = -\widetilde{x}^\top v \leq 0. \tag{20}$$

By the convexity of g_i, we have:

$$g_i(x) - g_i(\widetilde{x}) \leq \nabla g_i(x)^\top (x-\widetilde{x}), \ i = 1,\ldots,p. \tag{21}$$

By (20) and (21), using that $g_i(x) \geq 0$ for $i \in \mathcal{I}$, we get:

$$\|Ax-b\|^2 + \sum_{i\in\mathcal{I}} g_i(x)(g_i(x) - g_i(\widetilde{x}) \leq 0.$$

Since $\widetilde{x}$ is feasible, this implies that:

$$\|Ax - b\|^2 + \sum_{i \in \mathcal{I}} g_i(x)^2 \leq 0.$$

Therefore, $Ax = b$ and $g_i(x) = 0$ for all $i \in \mathcal{I}$. This completes the proof. $\square$

Remark. Equivalence results based on Fischer-Burmeister reformulations usually do not need the compactness of the polytope $\mathcal{K}_1$. Compactification of the domain may be obtained adding the equality $\sum_{i=1}^{n+1} x_i = M$ for large M, a transformation that does not alter the structure of problem (12). In [4] the conditions under which this type of transformation preserve the correct solutions of the problem have been analyzed.

Next we show that, for some Affine Variational Inequality Problems, a stationary point of the natural merit function over Ω either gives a solution or shows that the problem is infeasible. Consider again the set

$$\mathcal{K} = \{x \in \mathbb{R}^n : \ Ax = b, \ x \geq 0\} \tag{22}$$

with $A \in \mathbb{R}^{m \times n}$ full rank and $m < n$. Let the columns of $Z \in \mathbb{R}^{n \times (n-m)}$ be a basis of the nullspace of A. Let us consider the problem

$$\begin{array}{l} \text{Compute } \overline{x} \in \mathcal{K} \\ \text{such that } (M\overline{x} + q)^\top (x - \overline{x}) \geq 0, \ \forall x \in \mathcal{K} \end{array} \tag{23}$$

where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. As before, $\overline{x}$ is a solution of (23) if and only if $(\overline{x}, \overline{y}, \overline{w})$ is a solution of the problem:

$$\begin{aligned} w &= q + Mx - A^\top y \\ 0 &= Ax - b \\ x^\top w &= 0 \\ x, w &\geq 0. \end{aligned} \tag{24}$$

Theorem 3 *Let $(\overline{x}, \overline{y}, \overline{w})$ be a stationary point over Ω of the merit function*

$$f(x, y, w) = \|q + Mx - A^\top y - w\|^2 + \|Ax - b\|^2 + \sum_{i=1}^{n} (x_i w_i)^2 \tag{25}$$

If the columns of Z form a basis of the nullspace of A and $Z^\top M Z$ is a positive semi–definite matrix, then

(i) If $f(\bar{x}, \bar{y}, \bar{w}) = 0$, then $(\bar{x}, \bar{y}, \bar{w})$ is a solution of (24).
(ii) If $f(\bar{x}, \bar{y}, \bar{w}) > 0$, then the problem (24) is infeasible.

Proof Let $(\overline{x}, \overline{y}, \overline{w})$ be a stationary point of the merit function (25) over Ω. By a proof similar to the one presented in Theorem 2, $(\overline{x}, \overline{y}, \overline{w})$ satisfies

$$\begin{array}{lll} M^\top p & + A^\top (Ax - b) & = v \perp x \\ -Ap & & = 0 \\ \ \ -p & & = z \perp w \end{array} \tag{26}$$

$$x \perp w, x, v, z, w \geq 0,$$

where

$$p = q + Mx - A^\top y - w.$$

These are the KKT conditions of the following convex quadratic program

$$\begin{array}{l} \min_{x,y,w} \|q + Mx - A^\top y - w\|^2 + \|Ax - b\|^2 \\ \text{subject to } x \geq 0,\ w \geq 0. \end{array}$$

The result follows since the objective function is convex. □

Let us now consider a Quadratic Program (QP):

$$\begin{array}{c} \text{Minimize } q^\top x + \frac{1}{2} x^\top M x = f(x) \\ \text{subject to } Ax = b \\ x \geq 0. \end{array} \tag{27}$$

The KKT conditions for this Program consist of an LCP of the form of (24). By Theorem 3, if f is convex over the nullspace of A, then a stationary point $(\overline{x}, \overline{y}, \overline{w})$ of the merit function (25) over Ω solves the convex QP, in the sense that:

1. If $f(\overline{x}, \overline{y}, \overline{w}) = 0$, then $\overline{x}$ is a global minimum of the quadratic program;
2. If $f(\overline{x}, \overline{y}, \overline{w}) > 0$, then the quadratic program is primal or dual infeasible.

In particular, this result applies to Linear Programming problems, which have the form (27) with $M = 0$.

3 Conclusions

In this paper we considered the Complementarity Problem CP in the form (4), as a general square nonlinear system that includes complementarity constraints. This form is more general than the ones that represent optimality conditions and variational inequality problems. We used the natural squared norm of the residual as merit function, and we showed that stationary points of this function over Ω are solutions of the problem or possess singular Jacobians (Theorem 1). Furthermore, for the variational inequality problem (VI) with a mapping F and a convex domain $\mathcal{K}$, under a weak monotonicity assumption and a boundedness condition on $\mathcal{K}$, stationary points are solutions of the problem (Theorem 2). When F is affine and $\mathcal{K}$ is a polyhedron, a stationary point of the natural merit function either gives a solution to the VI or shows that the VI has no solution (Theorem 3). In particular, for linear and convex quadratic programs, a stationary point of the associated natural merit function either provides an optimal solution or establishes that the problem is primal or dual infeasible.

Nonlinear systems of equations with bounds on the variables have been considered in [5]. Many other papers (see, por example, [6–8]) deal with convex-constrained optimization and are able to handle large scale problems.

The results presented here help to predict what can be expected from those general convex-constraint approaches, when applied to the complementarity problem using the natural sum-of-squares merit function. Moreover, the employment of simply-constrained instead of unconstrained reformulations is advantageous to avoid possible convergence to stationary points that do not satisfy positivity constraints and do not fulfill the monotonicity-like assumptions required for equivalence results.

Interior point methods applied to specific complementarity problems were defined in [24, 27, 31, 32, 34, 36], among others. The effect of degeneracy was analyzed in [10, 26]. A large number of numerical experiments using a method that combines interior points and projected gradients was given in [30].

In spite of the large number of specific methods for different complementarity problems, many users prefer to use well established bound-constraint minimization algorithms for solving practical problems. Since these solvers usually generate sequences that converge to first-order stationary points, the relations between stationary points and solutions will continue to be algorithmically relevant.

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