On the regularization of mixed complementarity problems

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November 8, 1999

Abstract

A variational inequality problem (VIP) satisfying a constraint qualification can be reduced to a mixed complementarity problem (MCP). Monotonicity of the VIP implies that the MCP is also monotone. Introducing regularizing perturbations, a sequence of strictly monotone mixed complementarity problems are generated. It is shown that, if the original problem is solvable, the sequence of computable inexact solutions of the strictly monotone MCP's is bounded and every accumulation point is a solution. Under an additional condition on the precision used for solving each subproblem, the sequence converges to the minimum norm solution of the MCP.

Keywords. Variational inequalities, complementarity, perturbations, inexact solutions, minimization algorithms, reformulation.

AMS: 90C33, 90C30

1 Introduction

The variational inequality problem was introduced as a tool in the study of partial differential equations [21]. Modern applications of the VIP include

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competitive equilibrium problems arising in economics, operations research and control theory, traffic equilibrium problems, equilibrium points for n-person games, spatial economic equilibrium problems and many others. See [15, 20]. Moreover, the VIP generalizes the optimality conditions of convex minimization and, so, its study enlighten the development of optimization algorithms. Many authors introduced practical algorithms for solving the VIP. See, for example, [1, 14, 17, 21, 24, 25] and references therein.

In this paper we define the variational inequality problem as the problem of finding $x \in \Omega$ such that

$$\langle F(x), z - x \rangle \geq 0 \quad \text{for all} \quad z \in \Omega, \tag{1}$$

where

$$\Omega = \{x \in I\!R^n \ \mid \ g(x) \leq 0, \ Ax = b, \ \ell \leq x \leq u\}, \tag{2}$$

$F : I\!R^n \to I\!R^n$, $g : I\!R^n \to I\!R^p$, $A \in I\!R^{s \times n}$, $\ell \in (I\!R \cup \{-\infty\})^n$, $u \in (I\!R \cup \{\infty\})^n$. As usually, in (2) the inequality $-\infty \leq x_i$ (resp. $u_i \leq \infty$) must be read $-\infty < x_i$ (resp. $u_i < \infty$). We assume that F and g have continuous first partial derivatives and g_i is convex for all $i = 1, \ldots, p$. The problem (1) is said to be *monotone* if F is a monotone operator, i.e.

$$\langle F(y) - F(x), y - x \rangle \geq 0 \quad \text{for all} \quad x, y \in I\!R^n. \tag{3}$$

Condition (3) is equivalent to say that $F'(x)$ is positive semidefinite for all $x \in I\!R^n$ [23].

We say that Ω satisfies a constraint qualification if g is an affine function or if there exists $x \in \Omega$ such that $g(x) < 0$. If a constraint qualification is satisfied, the point $x \in \Omega$ is a solution of the VIP if, and only if, there exist $y \in I\!R^p$, $z \in I\!R^s$ such that

$$[F(x) + g'(x)^T y + A^T z]_i = 0 \quad \text{if} \quad \ell_i < x_i < u_i, \tag{4}$$

$$[F(x) + g'(x)^T y + A^T z]_i \geq 0 \quad \text{if} \quad x_i = \ell_i, \tag{5}$$

$$[F(x) + g'(x)^T y + A^T z]_i \leq 0 \quad \text{if} \quad x_i = u_i, \tag{6}$$

$$y \geq 0, \ \langle g(x), y \rangle = 0. \tag{7}$$

See [16]. Obviously, since $x \in \Omega$ we also have that

$$g(x) \leq 0, \ Ax = b, \ \ell \leq x \leq u. \tag{8}$$

Based on (4–8) the VIP can be reduced to a mixed complementarity problem which is, essentially, a VIP where the domain is a generalized $n-$ dimensional box. See [7, 10]. It is easy to prove that, if the VIP is monotone, the MCP is monotone too.

Important particular cases of the MCP are the classical nonlinear complementarity problem, where the domain is the positive orthant, and the problem of solving a nonlinear system of equations, where the domain is $\mathbb{R}^n$. An attractive strategy for solving the MCP is to define reformulations by means of which the MCP is transformed onto a global optimization problem. See [4, 5, 8]. This amounts to define a suitable nonnegative merit function that vanishes at solutions of the MCP. In practice, optimization algorithms with clever global strategies are able to find global minimizers (solutions of the MCP) in many cases. Moreover, monotonicity-like conditions usually ensure that stationary points of the optimization problem are global minimizers. In this case, if it is possible to guarantee that the generated sequence is bounded, standard minimization algorithms necessarily find solutions of the reformulated MCP . Different solutions for the boundedness problem can be found in [4, 5, 18] and [26, 27].

The solution for the boundedness problem suggested in [4, 5] involves the introduction of artificial bounds. For each variable that is unbounded above in the original problem, an artificial slack variable must be introduced. Also, slack variables must be introduced for each unbounded below variable. So, in the worst possible case, the number of variables could be triplicated. In this paper, the boundedness problem associated to reformulations is solved in a different way. We consider regularized problems, in which the operator is modified by a diagonal perturbation. By means of such a perturbation, a monotone MCP becomes strongly monotone. We will consider inexact solutions of the perturbed MCP, which will be defined in terms of an error measure strongly connected with the merit function used in the reformulations [2, 3, 5, 9, 12, 13, 14]. See, also, the survey [22]. We will prove that, if the perturbation of a solvable monotone MCP tends to zero, the resulting sequence is bounded and all its limit points are solutions of the original MCP. Moreover, if the subproblems are solved up to a precision that tends to zero faster than the perturbation, the sequence converges to the minimum norm solution of the MCP.

This work is organized as follows. In Section 2 we show how to transform the VIP into the MCP and we prove that monotone VIP's generate monotone MCP's. In Section 3 we define the perturbation of the MCP and we prove the convergence results. Some conclusions are given in Section 4.

2 Reduction of VIP to MCP

Consider the VIP defined by (1) and (2). Let us define $G : \mathbb{R}^{n+p+s} \rightarrow \mathbb{R}^{n+p+s}$ by

$$G(x, y, z) = (F(x) + g'(x)^T y + A^T z,\ -g(x),\ b - Ax). \tag{9}$$

Moreover, consider the generalized box in $I\!R^{n+p+s}$ given by

$$\Omega_{aux} = \{(x, y, z) \in I\!R^{n+p+s} \mid \ell \leq x \leq u, \ \ y \geq 0\}. \tag{10}$$

Theorem 1. *If the VIP defined by (1–2) satisfies a constraint qualification, then $x \in I\!R^n$ is one of its solutions if, and only if, there exist $y \in I\!R^p, z \in I\!R^s$ such that (x, y, z) is a solution of the MCP defined by (9–10).*

Proof. Conditions (4–8) can be written

$$[G(x, y, z)]_i = 0 \quad \text{if} \quad \ell_i < x_i < u_i, i = 1, \ldots, n; \tag{11}$$

$$[G(x, y, z)]_i \geq 0 \quad \text{if} \quad \ell_i = x_i, i = 1, \ldots, n; \tag{12}$$

$$[G(x, y, z)]_i \leq 0 \quad \text{if} \quad x_i = u_i, i = 1, \ldots, n; \tag{13}$$

$$[G(x, y, z)]_{n+i} = 0 \quad \text{if} \quad y_i > 0, i = 1, \ldots, p; \tag{14}$$

$$[G(x, y, z)]_{n+i} \geq 0 \quad \text{if} \quad y_i = 0, i = 1, \ldots, p; \tag{15}$$

$$[G(x, y, z)]_{n+p+i} = 0 \ \ , i = 1, \ldots, s. \tag{16}$$

Clearly, (11–16) are the conditions that characterize a solution of the MCP defined by (9) on the set (10). □

Theorem 2. *If F is monotone, then G is monotone on the set $\{(x, y, z) \in I\!R^{n+p+s} \mid y \geq 0\}$.*

Proof. Take $(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z}) \in I\!R^{n+p+s}$ with $y, \tilde{y} \geq 0$. Direct calculation yields

$$\langle G(x, y, z) - G(\tilde{x}, \tilde{y}, \tilde{z}), (x, y, z) - (\tilde{x}, \tilde{y}, \tilde{z})\rangle$$
$$= \langle F(x) - F(\tilde{x}), x - \tilde{x}\rangle + \langle g'(x)^T y - g'(\tilde{x})^T \tilde{y}, x - \tilde{x}\rangle + +\langle g(\tilde{x}) - g(x), y - \tilde{y}\rangle$$

Since F is monotone, we conclude that

$$\langle G(x, y, z) - G(\tilde{x}, \tilde{y}, \tilde{z}), (x, y, z) - (\tilde{x}, \tilde{y}, \tilde{z})\rangle$$
$$\geq \langle g'(x)^T y - g'(\tilde{x})^T \tilde{y}, x - \tilde{x}\rangle + \langle g(\tilde{x}) - g(x), y - \tilde{y}\rangle$$

Simple manipulation on the right-hand side of the above inequality gives

$$\langle g'(x)^T y - g'(\tilde{x})^T \tilde{y}, x - \tilde{x}\rangle + \langle g(\tilde{x}) - g(x), y - \tilde{y}\rangle$$
$$= \langle y, g'(x)(x - \tilde{x}) + g(\tilde{x}) - g(x)\rangle + \langle \tilde{y}, g'(\tilde{x})(\tilde{x} - x) + g(x) - g(\tilde{x})\rangle$$
$$= \sum_{i=1}^{p} \Big(g_i(\tilde{x}) - (g(x) + \nabla g_i(x)(\tilde{x} - x))\Big)[y]_i + \sum_{i=1}^{p} \Big(g_i(x) - (g(\tilde{x}) + \nabla g_i(\tilde{x})(x - \tilde{x}))\Big)[\tilde{y}]_i$$

Since the components of g are convex and $y, \tilde{y} \geq 0$, the desired result follows. □

3 Convergence under positive perturbations

By Theorems 1 and 2, monotone variational inequalities (with constraint qualification) can be reduced to monotone mixed complementarity problems. As we mentioned in the introduction, the MCP is the variational inequality problem on a generalized box. In this section, the generalized box will be given by

$$\Omega_{box} = \{x \in \mathbb{R}^n \mid \ell \leq x \leq u\}, \tag{17}$$

where $-\infty \leq \ell_i < u_i \leq \infty$ for all $i = 1, \ldots, n$ and $-\infty \leq x_i$ (resp. $x_i \leq \infty$) means $-\infty < x_i$ (resp. $x_i < \infty$).

If $F = (f_1, \ldots, f_n)$, the variational inequality problem defined by F on Ω_{box} reduces to finding $x \in \Omega_{box}$ such that

$$f_i(x) = 0 \quad \text{if} \quad \ell_i < x_i < u_i, \tag{18}$$

$$f_i(x) \geq 0 \quad \text{if} \quad \ell_i = x_i, \tag{19}$$

$$f_i(x) \leq 0 \quad \text{if} \quad x_i = u_i. \tag{20}$$

We will consider the case in which F is monotone on Ω_{box}, that is:

$$\langle F(y) - F(x), y - x \rangle \geq 0 \quad \text{for all} \quad x, y \in \Omega_{box}. \tag{21}$$

We say, as usually, that F is *strongly monotone* on Ω_{box} if there exists $\mu > 0$ such that

$$\langle F(y) - F(x), y - x \rangle \geq \mu \|y - x\|^2 \quad \text{for all} \quad x, y \in \Omega_{box}. \tag{22}$$

Given $\varepsilon > 0$, we define $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_\varepsilon(x) = F(x) + \varepsilon x \quad \text{for all} \quad x \in \Omega_{box}. \tag{23}$$

The mapping F_ε is strongly monotone, since

$$\langle F_\varepsilon(x) - F_\varepsilon(y), x - y \rangle \geq \langle F(x) - F(y), x - y \rangle + \varepsilon \|x - y\|^2 \geq \varepsilon \|x - y\|^2.$$

The MCP defined by F_ε on Ω_{box}, has unique solution [16] and can be solved using reliable algorithms [1, 5, 6, 11, 8, 19]. On the other hand, the MCP problem (18-20) with monotone (but not strongly monotone) F can have infinitely many solutions or no solution at all. We will see that, if (18-20) is solvable, a solution can be obtained as a limit of inexact solutions of perturbed MCP problems defined by F_ε and Ω_{box}.

Let ε_k, $k = 0, 1, 2, \ldots$ be such that $\varepsilon_k > 0$ for all k and $\lim_{k \to \infty} \varepsilon_k = 0$. We define, for all $k = 0, 1, 2, \ldots$,

$$F_k(x) = F_{\varepsilon_k}(x) = F(x) + \varepsilon_k x \quad \text{for all } x \in \Omega_{box}. \tag{24}$$

Let us call MCP(k) the MCP defined by F_k on Ω_{box}. We denote, from now on, $\|\cdot\| = \|\cdot\|_2$. We define

$$I_1 = \{i \in \{1,\dots,n\} \mid \ell_i > -\infty\}$$

and

$$I_2 = \{i \in \{1,\dots,n\} \mid u_i < \infty\}$$

By (18-20), $x \in I\!R^n$ is (the unique) solution of MCP(k) if and only if, there exist $v, w \in I\!R^n$ such that

$$\|F_k(x) - v + w\|^2 + [\sum_{i\in I_1}(x_i - \ell_i)v_i]^2 + [\sum_{i\in I_2}(x_i - u_i)w_i]^2 = 0, \tag{25}$$

with

$$\ell \le x \le u, \;\; v, w \ge 0, \;\; v_i = 0 \text{ if } \ell_i = -\infty, \;\; w_i = 0 \text{ if } u_i = \infty \tag{26}$$

for $i = 1,\dots,n$.

Clearly, if (26) holds, (25) is equivalent to

$$\|F_k(x) - v + w\|^2 + 2\varepsilon_k \left(\sum_{i\in I_1}(x_i - \ell_i)v_i + \sum_{i\in I_2}(u_i - x_i)w_i \right) = 0. \tag{27}$$

Let K be a strictly positive scalar. Expressions (25–26) motivate us to define an inexact solution of MCP(k) as a point $x_k \in I\!R^n$ that satisfies, for some $v_k, w_k \in I\!R^n$,

$$\sqrt{\|F_k(x_k) - v_k + w_k\|^2 + [\sum_{i\in I_1}([x_k]_i - \ell_i)[v_k]_i]^2 + [\sum_{i\in I_2}([x_k]_i - u_i)[w_k]_i]^2} \le K\varepsilon_k, \tag{28}$$

and

$$\ell \le x_k \le u, \;\; v_k, w_k \ge 0, \;\; [v_k]_i = 0 \text{ if } \ell_i = -\infty, \;\; [w_k]_i = 0 \text{ if } u_i = \infty \tag{29}$$

for $i = 1,\dots,n$.

Let us define $\delta_k \in I\!R^n, \alpha_k \in I\!R, \beta_k \in I\!R$ by

$$F_k(x_k) - v_k + w_k = \delta_k \tag{30}$$

$$\sum_{i\in I_1}([x_k]_i - \ell_i)[v_k]_i = \alpha_k \tag{31}$$

and

$$\sum_{i\in I_2}(u_i - [x_k]_i)[w_k]_i = \beta_k. \tag{32}$$

By (28), (30), (31) and (32),

$$\sqrt{\delta_k^2 + \alpha_k^2 + \beta_k^2} \leq K\varepsilon_k.$$

Hence,

$$\|\delta_k\| \leq K\varepsilon_k, \;\; \alpha_k \leq K\varepsilon_k, \;\; \beta_k \leq K\varepsilon_k. \tag{33}$$

Define

$$\theta_k = \sqrt{\delta_k^2 + 2\varepsilon_k(\alpha_k + \beta_k)}.$$

By (33), we have that

$$\theta_k \leq \varepsilon_k \sqrt{K^2 + 4K}. \tag{34}$$

Reciprocally, it is easy to see that, if

$$\theta_k \leq K'\varepsilon_k, \tag{35}$$

then the inequalities (33) and (28) hold for different values of K. This means that inexact solutions, which have been defined here in terms of (28), can be defined, equivalently, as points that satisfy (35) or (33).

Now, we are ready to prove that the sequence of inexact solutions of the MCP is bounded and that limit points are solutions of the original MCP.

Theorem 3. *Assume that F is monotone and continuous and a solution $\hat{x}$ of the MCP (18-20) exists. Then, the sequence $\{(x_k, v_k, w_k)\}$ is bounded and any limit point of this sequence defines a solution of the MCP.*

Moreover, if

$$\lim_{k\to\infty} \theta_k/\varepsilon_k = 0$$

then x_k converges to the minimum norm solution of the MCP.

Proof. Since F_k is strongly monotone we have that

$$\varepsilon_k \|x_k - \hat{x}\|^2 \leq \langle F(x_k) + \varepsilon_k x_k - F(\hat{x}) - \varepsilon_k \hat{x}, x_k - \hat{x}\rangle. \tag{36}$$

So, by (30),

$$\varepsilon_k \|x_k - \hat{x}\|^2 \leq \langle \delta_k + v_k - w_k - \varepsilon_k \hat{x}, x_k - \hat{x}\rangle - \langle F(\hat{x}), x_k - \hat{x}\rangle. \tag{37}$$

Since $\hat{x}$ is a solution of (21) and $\ell \leq \hat{x} \leq u$, (37) implies that

$$\varepsilon_k \|x_k - \hat{x}\|^2 \leq \langle \delta_k, x_k - \hat{x}\rangle + \langle v_k, x_k - \hat{x}\rangle - \langle w_k, x_k - \hat{x}\rangle - \varepsilon_k \langle \hat{x}, x_k - \hat{x}\rangle. \tag{38}$$

Now, by (26) and (29),

$$\langle v_k, x_k - \hat{x}\rangle = \sum_{i\in I_1} [v_k]_i [x_k - \ell]_i + \sum_{i\in I_1} [v_k]_i [\ell - \hat{x}]_i$$

and

$$\sum_{i\in I_1}[v_k]_i[\ell-\hat{x}]_i \leq 0.$$

Therefore, by (31),

$$\langle v_k, x_k-\hat{x}\rangle \leq \alpha_k. \tag{39}$$

Analogously, we can prove that

$$-\langle w_k, x_k-\hat{x}\rangle \leq \beta_k. \tag{40}$$

By (38), (39) and (40), we obtain:

$$\varepsilon_k\|x_k-\hat{x}\|^2 \leq \langle \delta_k, x_k-\hat{x}\rangle - \varepsilon_k\langle \hat{x}, x_k-\hat{x}\rangle + \alpha_k+\beta_k.$$

Multiplying by $2/\varepsilon_k$ and rearranging this inequality, we obtain:

$$2\|x_k-\hat{x}\|^2 - (2/\varepsilon_k)\langle \delta_k, x_k-\hat{x}\rangle + 2\langle \hat{x}, x_k-\hat{x}\rangle \leq (2/\varepsilon_k)(\alpha_k+\beta_k). \tag{41}$$

Elementary algebraic manipulations on the left-hand side of (41) give

$$\begin{aligned} &2\|x_k-\hat{x}\|^2 - (2/\varepsilon_k)\langle \delta_k, x_k-\hat{x}\rangle + 2\langle \hat{x}, x_k-\hat{x}\rangle \\ &= \|x_k-\hat{x}\|^2 - (2/\varepsilon_k)\langle \delta_k, x_k-\hat{x}\rangle + \|x_k-\hat{x}\|^2 + 2\langle \hat{x}, x_k-\hat{x}\rangle \\ &= \|x_k-\hat{x}-(1/\varepsilon_k)\delta_k\|^2 - \|(1/\varepsilon_k)\delta_k\|^2 + \|x_k\|^2 - \|\hat{x}\|^2. \end{aligned}$$

Therefore, by (41),

$$\|x_k\|^2 + \|x_k-\hat{x}-(1/\varepsilon_k)\delta_k\|^2 \leq \|\hat{x}\|^2 + (1/\varepsilon_k^2)(\|\delta_k\|^2 + 2\varepsilon_k(\alpha_k+\beta_k)).$$

So, by (34),

$$\|x_k\|^2 \leq \|\hat{x}\|^2 + (\theta_k/\varepsilon_k)^2 \leq \|\hat{x}\|^2 + K^2 + 4K. \tag{42}$$

Thus, $\{x_k\}$ is bounded. By (28) and the continuity of F, this implies that $\{-v_k+w_k\}$ is also bounded. Therefore, if $[v_k]_i$ is unbounded for some i, $[w_k]_i$ must be unbounded too. By (28) this implies that, for some subsequence, $[x_k]_i$ tends to ℓ_i and $[x_k]_i$ tends to u_i simultaneously. This is impossible, since $\ell_i < u_i$. Therefore, the sequence $\{(x_k, v_k, w_k)\}$ is bounded. To prove that the limits points of this sequence are solutions of the MCP (18-20), take the limit (for a convergent subsequence) on both sides of (28).

To prove the second part of the theorem, remember that the solution set of the MCP (18-20) is *close* and *convex*. Since, by hypothesis, this set is nonempty, there exists a minimum norm element, say x_*. Inequality (42) holds for any $\hat{x}$ solution of the MCP. In particular,

$$\|x_k\|^2 \leq \|x_*\|^2 + (\theta_k/\varepsilon_k)^2.$$

Let $\bar{x}$ be a limit point of $\{x_k\}$. Taking the limit for a subsequence that converges to $\bar{x}$, we get

$$\|\bar{x}\|^2 \leq \|x_*\|^2.$$

So, $\bar{x}$ is a minimum norm solution of the problem. By the uniqueness of projections on convex sets, $\bar{x} = x_*$. Since $\{x_k\}$ is bounded, this implies that $\{x_k\}$ converges to x_*. $\square$

4 Final remarks

In real-life ill-conditioned problems in which the data are subject to errors, the exact solution is meaningless and useful approximations are those obtained by regularization of the operator. See [28]. This amounts to choose a reasonable regularizing parameter $\varepsilon > 0$ and to solve the corresponding perturbed MCP. Many strategies for choosing a good regularization parameter for different problems can be found in the literature. See, for example, [29] and references therein. In this paper we showed that the regularizing strategy has a good theoretical support when one deals with monotone mixed complementarity problems.

Moreover, a common practice in the resolution of nonlinear systems of equations is to merge the original problem into a family of better conditioned perturbed problems, in such a way that the solution of each of them helps to define a good initial point for solving the following. In this way, good initial approximations compensate the harmful effect of ill-conditioning and increasing nonlinearity. This is the spirit of the so called "regularizing homotopy", which has been largely and successfully used in the resolution of nonlinear systems coming from Physics and Engineering. See [30]. Further research is necessary in order to define suitable algorithms for tracing the homotopy path in the MCP case.

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