Generalized second-order complementarity problems: theory and numerical experiments*

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Abstract

The generalized second-order cone complementarity problem (GSOCCP) is reformulated via bound-constrained minimization, preserving differentiability of the original data. Four reformulations are proposed, which are tested in five low dimensional instances. A thorough presentation and discussion of the numerical experiments is provided, to illustrate the performance of the reformulations. In a companion paper, equivalence results relating global minimizers of two of the reformulations with zero objective function value and solutions to GSOCCP are proved, together with sufficient conditions for ensuring correspondence between stationary points of one of the reformulations and solutions to GSOCCP. Reference to these results are included, with additional theoretical insight into the second reformulation that has been previously addressed.
Keywords. complementarity problems, minimization algorithms, reformulation, numerical experiments.

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1 Introduction

Given $F, G : \mathbb{R}^n \to \mathbb{R}^n$, consider the generalized second-order cone complementarity problem GSOCCP$(F, G, \mathcal{K})$ of finding $x \in \mathbb{R}^n$ such that

$$G(x) \in \mathcal{K}, F(x) \in \mathcal{K}^\circ, F(x)^T G(x) = 0 \tag{1}$$

where $\mathcal{K}$ is the convex cone

$$\mathcal{K} = \{x \in \mathbb{R}^n \mid x_1^2 \geq \sum_{j=2}^{n} a_j^2 x_j^2, x_1 \geq 0\}$$

and its polar cone $\mathcal{K}^\circ$ is defined by[1]

$$\mathcal{K}^\circ = \{x \in \mathbb{R}^n \mid \forall y \in \mathcal{K}, \langle x, y\rangle \geq 0\}.$$

If we let, without loss of generality, $A = \mathrm{diag}(1, -a_2^2, -a_3^2, \ldots, -a_p^2, 0, \ldots, 0)$ (where $a_i \neq 0$ for $2 \leq i \leq p$), $\bar{A} = \mathrm{diag}(1, -1/a_2^2, \ldots, -1/a_p^2, 0, \ldots, 0)$ and $M = \mathrm{diag}(m_1, \ldots, m_n)$ where $m_i = 0$ for $i \in \mathcal{P} = \{1, \ldots, p\}$ and $m_i = 1$ for $i > p$, the convex cones considered may be expressed in matrix notation as

$$\mathcal{K} = \left\{x \in \mathbb{R}^n \mid \frac{1}{2}x^T A x \geq 0,\ x_1 \geq 0\right\}$$

and

$$\mathcal{K}^\circ = \left\{x \in \mathbb{R}^n \mid \frac{1}{2}x^T \bar{A} x \geq 0,\ x_1 \geq 0,\ Mx = 0\right\}.$$

Note that $A\bar{A} = \mathrm{diag}(\mathbf{1}_p, 0 \ldots, 0)$ and $AM = MA = \bar{A}M = M\bar{A} = 0$, where $\mathbf{1}_p = (1, \ldots, 1) \in \mathbb{R}^p$.

The GSOCCP$(F, G, \mathcal{K})$ is a difficult problem because, although $\mathcal{K}$ is a convex set, the constraints that define it are not convex. Moreover, they do not verify any constraint qualification at the origin, so the task of reformulating this problem via nonlinear programming becomes particularly challenging.

In this paper we propose four reformulations of the GSOCCP as nonlinear minimization problems with bound constraints, which preserve the differentiability of the original data in the objective function. As a result, any efficient bound-constrained minimization algorithm for large scale problems can be used to solve the reformulated problems.

The paper is organized as follows. In Section 2 the first reformulation is introduced, with reference to the global equivalence result. In Section 3 some computational aspects are discussed and other reformulations are proposed. Section 4 provides conditions to relate stationary points of the first reformulation with a solution of the original GSOCCP, together with a theoretical insight into the fourth reformulation. In Section 5 the numerical experiments are reported, and Section 6 presents final remarks.

[1] The set $\mathcal{K}^\circ$ considered here equals the negative of the polar cone defined by Rockafellar [8].

2 An equivalent formulation

Let

$$
\begin{aligned}
f(x,\lambda,\mu,z,y) &= \|F(x)-\lambda AG(x)-\mu e_1\|^2+\left(\frac{1}{2}G(x)^TAG(x)-z\right)^2 \\
&\quad +(G_1(x)-y)^2+(\lambda z)^2+(\mu y)^2 \qquad (2)
\end{aligned}
$$

and

$$
\begin{aligned}
g(x,\xi,\nu,\zeta,w,s) &= \|G(x)-\xi\bar{A}F(x)-\nu e_1-M\zeta\|^2+\left(\frac{1}{2}F(x)^T\bar{A}F(x)-w\right)^2 \\
&\quad +(F_1(x)-s)^2+\|MF(x)\|^2+(\xi w)^2+(\nu s)^2, \qquad (3)
\end{aligned}
$$

where $x,\zeta\in\mathbb{R}^n$, $\lambda,\mu,z,y,\xi,\nu,w,s\in\mathbb{R}$ and $e_1=(1,0,\ldots,0)^T\in\mathbb{R}^n$.

If the origin were a regular point of the constraints that represent the set $\mathcal{K}$, any of these two functions could constitute a merit function in the reformulation of the GSOCCP as an optimization problem. We circumvent the lack of regularity by combining the two functions as follows: Letting $v_f=(\lambda,\mu,z,y)$ and $v_g=(\xi,\nu,w,s)$, we define the convex combination $\phi(x,v_f,v_g,\zeta,r)=r\,f(x,v_f)+(1-r)\,g(x,v_g,\zeta)$ and the optimization problem

$$
\begin{aligned}
\min \quad & \phi(x,v_f,v_g,\zeta,r) \\
\text{s.t.} \quad & 1\ \ge\ r\ \ge\ 0 \\
& v_f,\,v_g\ \ge\ 0.
\end{aligned} \qquad (4)
$$

To the global minimizer of reformulation (4) it corresponds a solution to GSOCCP$(F,G,\mathcal{K})$ and vice versa, that is, a solution to GSOCCP$(F,G,\mathcal{K})$ can be completed to produce a minimizer of problem (4) with zero objective value. This result is proved in [2].

It is worth noticing that, for the popular special case $G(x)=x$, we could make up for the non-regularity of the origin by testing beforehand whether $F(0)\in\mathcal{K}^\circ$, in which case $x=0$ is a solution of the GSOCCP$(F,G,\mathcal{K})$. On the other hand, if $F(0)\notin\mathcal{K}^\circ$, then clearly $x=0$ is not a solution and we could do without the g part of the convex combination, and use only $f(x,v_f)$ defined in (2) as a merit function.

3 A simple instance and alternative formulations

The formulation (4) was tested for the following instance of GSOCCP$(F,G,\mathcal{K})$:

$$
F(x)=\begin{pmatrix}x_1+1\\ x_2+2\end{pmatrix},\quad G(x)=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\quad\text{and}\quad A=\bar{A}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \qquad (5)
$$

Figure 1 below gives a graphical representation of this instance. The lighter cone identifies the region $\{x\mid F(x)\in\mathcal{K}^\circ\}$, the region $\{x\mid G(x)\in\mathcal{K}\}$ corresponds to the intermediate shade of gray and their overlapping is indicated by the darkest shade. The circle with center $(-1/2, -1)$ and radius $\sqrt{5}/2$ is the loci of points satisfying the orthogonality condition $F(x)^T G(x) = 0$. Clearly, the unique solution to this instance is the vertex of the darkest cone: $(1/2, -1/2)$.

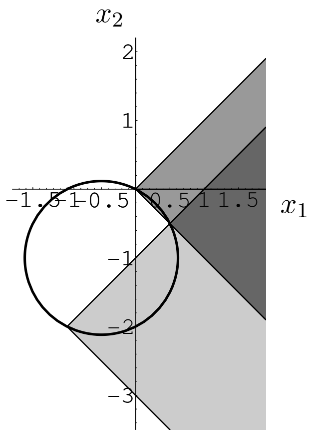


Figure 1: Instance (5)

	ITBOX	FE	ITQUA	MVP	ϕ^*
minimum	7	9	35	56	0
average	284.2	409.5	2366.4	3193.2	
maximum	1020	1470	24658	29207	2E-6

Table 1: Results for instance (5) - formulation (4).

Table 1 contains the outcomes obtained running `easy`[2] for formulation (4) of problem (5) with 200 initial points x^0 ramdomly generated with components in the box $[-10, 10] \times [-10, 10]$ and remaining variables set to 0.5. The stopping criteria was norm of projected gradient less than 10^{-8}, achieved for 82% of the tests. The remaining 18% stopped with too small a step (less than 10^{-8}), meaning that possibly the end point is close to a local minimizer. The number of iterations of the trust-region algorithm for simple-bounded minimization, functional evaluations performed, iterations of the inner quadratic solver and matrix-vector products computed are given, respectively, by `ITBOX`, `FE`, `ITQUA` and `MVP`. Column with header ϕ^* contains the final objective function value of (4).

The three distinct end points obtained were I $(0.5, -0.5)$, II $(-1.0021, -1.9958)$, and III $(0.0000, -0.0387)$, with ϕ^*_I, ϕ^*_{II} and ϕ^*_{III} of order 10^{-20}, 10^{-6} and 10^{-7}, resp. The unique solution (point I) was obtained in 59% of the tests. The remaining ended in points close to $(-1, -2)$ (17% to point II) or to $(0, 0)$ (25% to point III). It is worth mentioning that of the 13% of tests that stopped with too small a step, 4.5% ended at the solution, whereas the remaining 13.5% ended at point III.

Despite the very small objective function values reached, the corresponding points II and III are not feasible for GSOCCP$(F, G, \mathcal{K})$. At these points we have orthogonality and the membership in exactly one (but not both) of the cones. In fact, they are far from the feasible set (the darkest cone in Figure 1).

[2] Fortran double-precision code for solving nonlinear programming problems, based on augmented Lagrangian [6], trust region [5] and projected gradients combined with a mild active set strategy [3]. Available at `http://www.ime.unicamp.br/~martinez`.

Consider, for instance, point II obtained in 17% of the tests. We observed that for this point, the corresponding final value $r^* = 0$ was always achieved, and so the objective value came to $g(-1.0021, -1.9958) \approx 10^{-6}$. The second and third terms in the expression (3) of $g(-1.0021, -1.9958)$ enforce the membership of $F(x)$ in the polar cone. Orthogonality is enforced by the terms $(\xi w)^2$ and $(\nu s)^2$. But membership of $G(x)$ in $\mathcal{K}$ is enforced only indirectly by the first term, $\|G(x) - \xi \bar{A} F(x) - M\zeta\|^2$. We can drive this term to zero (this was achieved by lowering the parameter used as stopping criterion in `easy`) and still have $G(x)$ outside $\mathcal{K}$. For point III, since the corresponding value $r^* = 1$ was obtained and $f(0.0000, -0.0387) \approx 10^{-7}$, the previous reasoning holds with g and G replaced by f and F, respectively.

It is not so bad to end in a point that is not a solution, but it is very bad to end at a point that looks as if it is a solution, because the objective value is very small. After all, algorithms for nonlinear problems will seldom arrive at the exact solution. In this case, solutions with objective values close to zero may seem to be close to the global solution. It would be preferable to have higher objective function values associated with solutions II and III.

This phenomenon motivated the following equivalent formulations:

(i) Remove variables μ and ν from (4), obtaining

$$\phi_{\text{mod}}(x, \lambda, z, y, \xi, w, s, \zeta, r) = r f_{\text{mod}}(x, \lambda, z, y) + (1-r) g_{\text{mod}}(x, \xi, w, s, \zeta)$$

and

$$\begin{array}{ll} \min & \phi_{\text{mod}}(x, \lambda, z, y, \xi, w, s, \zeta, r) \\ \text{s.t.} & 1 \ \geq \ r \ \geq \ 0 \\ & \lambda, z, y, \xi, w, s \ \geq \ 0. \end{array} \tag{6}$$

(ii) Define

$$\begin{aligned} \Phi(x, \lambda, z, y, w, s, r) \ = \ & \tfrac{1}{2} \Big(r(\|F(x) - \lambda A G(x)\|^2 + (\lambda z)^2) \\ & + (1-r)(\|G(x) - \lambda \bar{A} F(x)\|^2 + (\lambda w)^2) \\ & + \left(\tfrac{1}{2} G(x)^T A G(x) - z\right)^2 + (G_1(x) - y)^2 \\ & + \left(\tfrac{1}{2} F(x)^T \bar{A} F(x) - w\right)^2 + (F_1(x) - s)^2 + \|M F(x)\|^2 \Big) \end{aligned}$$

and

$$\begin{array}{ll} \min & \Phi(x, \lambda, z, y, w, s, r) \\ \text{s.t.} & 1 \ \geq \ r \ \geq \ 0 \\ & \lambda, z, y, w, s \ \geq \ 0. \end{array} \tag{7}$$

Notice that membership in both cones is directly enforced in the objective function.

(iii) Assuming the F and G were scaled so that $A = \bar{A} = (1, -1, \ldots, -1, 0, \ldots, 0)$ (notice that the relation $A\bar{A} = (\mathbf{1}_p, 0, \ldots, 0)$ remains true), define

$$\begin{aligned}\Upsilon(x, \lambda, \alpha, z, y, w, s) \;=\; & \tfrac{1}{2}\Big(\|\lambda F(x) - \alpha AG(x)\|^2 + (\lambda w)^2 + (\alpha z)^2 \\ & + \left(\tfrac{1}{2}G(x)^T AG(x) - z\right)^2 + (G_1(x) - y)^2 \\ & + \left(\tfrac{1}{2}F(x)^T \bar{A}F(x) - w\right)^2 + (F_1(x) - s)^2 + \|MF(x)\|^2\Big)\end{aligned}$$

and

$$\begin{aligned} \min \quad & \Upsilon(x, \lambda, \alpha, z, y, w, s) \\ \text{s.t.} \quad & \lambda + \alpha = 1 \\ & \lambda, \alpha, z, y, w, s \geq 0. \end{aligned} \tag{8}$$

(iv) Same as (iii) above, but the equality constraint is eliminated:

$$\begin{aligned}\Xi(x, \lambda, z, y, w, s) \;=\; & \tfrac{1}{2}\Big(\|\lambda F(x) - (1-\lambda)AG(x)\|^2 + (\lambda w)^2 + ((1-\lambda)z)^2 \\ & + \left(\tfrac{1}{2}G(x)^T AG(x) - z\right)^2 + (G_1(x) - y)^2 \\ & + \left(\tfrac{1}{2}F(x)^T \bar{A}F(x) - w\right)^2 + (F_1(x) - s)^2 + \|MF(x)\|^2\Big)\end{aligned}$$

and

$$\begin{aligned} \min \quad & \Xi(x, \lambda, z, y, w, s) \\ \text{s.t.} \quad & 1 \geq \lambda \geq 0 \\ & z, y, w, s \geq 0. \end{aligned} \tag{9}$$

Equivalence results can be proved to problems (6), (7), (8) and (9), in the sense that to each reformulation it corresponds a solution to GSOCCP$(F, G, \mathcal{K})$ and vice versa, that is, a solution to GSOCCP$(F, G, \mathcal{K})$ can be completed to produce a minimizer of each reformulation, with zero objective value. In [2] a result related to formulation (9) is proved.

4 Conditions on stationary points

From a practical point of view, once a solution is attained, one can easily check whether the x-part of it is (close to) a solution to the original problem (1). Now, most algorithms for nonlinear programming, when successful, end in stationary points. Hence it is interesting, from the theoretical point of view, to establish conditions under which a stationary point of the reformulation will have objective function zero, and thus be a global solution thereof, containing a solution of (1). The next theorem tackles this problem, regarding formulation (4).

Theorem 1 *Let $(x^*, v_f^*, v_g^*, \zeta^*, r^*)$ be a stationary point of (4), and define*

$$H_g = \nabla G(x^*)^T (\nabla F(x^*)^T)^{-1} - \xi^* \bar{A} \quad \textit{and} \quad H_f = \nabla F(x^*)^T (\nabla G(x^*)^T)^{-1} - \lambda^* A.$$

(a) If $r^* = 1$ and H_f is positive definite then x^* is a solution to GSOCCP.

(b) If $r^* = 0$, and H_g is positive definite then x^* is a solution to GSOCCP.

(c) If $0 < r^* < 1$ then $T(x^*, v_f^*, v_g^*, \zeta^*, r)$ is constant for $0 \leq r \leq 1$ and

(i) if H_g or H_f are positive definite then x^* is a solution to GSOCCP.

or

(ii) $(\nabla_x f(x^*, v_f^*), 0, \ldots, 0)$ is a descent direction for T from $(x^*, v_f^*, v_g^*, \zeta^*, 1)$ and $(\nabla_x g(x^*, v_g^*, \zeta^*), 0, \ldots, 0)$ is a descent direction for T from $(x^*, v_f^*, v_g^*, \zeta^*, 0)$.

Proof. See [2]. ∎

Regarding reformulation (9), we explain in the following the role played by the variable λ, which somehow acts as a convex combination parameter at the same time that it ensures complementarity between F and G. Due to these features, we address (9) as a *two-in-one* reformulation, or shortly, as TO.

Let $(\hat{x}, \hat{\lambda}, \hat{z}, \hat{y}, \hat{w}, \hat{s})$ be a stationary point of (9). The eight terms that compose the objective function of problem (9) are easily recalled by introducing the next auxiliary quantities

$$\begin{aligned}
\hat{l}_1 &= \hat{\lambda} F(\hat{x}) - (1-\hat{\lambda}) A G(\hat{x}) \\
\hat{l}_2 &= \hat{\lambda} \hat{w} \\
\hat{l}_3 &= (1-\hat{\lambda}) \hat{z} \\
\hat{l}_4 &= \frac{1}{2} G(\hat{x})^T A G(\hat{x}) - \hat{z} \\
\hat{l}_5 &= G_1(\hat{x}) - \hat{y} \\
\hat{l}_6 &= \frac{1}{2} F(\hat{x})^T \bar{A} F(\hat{x}) - \hat{w} \\
\hat{l}_7 &= F_1(\hat{x}) - \hat{s} \\
\hat{l}_8 &= M F(\hat{x}).
\end{aligned}$$

Clearly, if $\hat{l}_1 \neq 0$ then $\hat{x}$ is not a solution to GSOCCP$(F, G, \mathcal{K})$. So, what would the assumption $\hat{l}_1 = 0$ bring about?

The next example illustrates that

$$\hat{l}_1 = \hat{\lambda} F(\hat{x}) - (1-\hat{\lambda}) A G(\hat{x}) = 0 \tag{10}$$

is not a sufficient condition to ensure correspondence between a stationary point of formulation (9) and a solution to GSOCCP$(F, G, \mathcal{K})$. Let

$$F(x) = \begin{pmatrix} x_1 - 1 \\ x_2 + \sqrt{3} \end{pmatrix}, \quad G(x) = \begin{pmatrix} x_1 - 1 \\ x_2 - \sqrt{3} \end{pmatrix} \quad \text{and} \quad A = \bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

The unique solution of this GSOCCP$(F,G,\mathcal{K})$ is $x^* = (\sqrt{3}+1,0)^T$.

The point $(\hat{x},\hat{\lambda},\hat{z},\hat{y},\hat{w},\hat{s})$ with $\hat{x}=(0,0)^T$, $\hat{\lambda}=1/2$, $\hat{z}=\hat{y}=\hat{w}=\hat{s}=0$ is stationary for (9), but since $\hat{\ell}_1=(0,0)^T$, $\hat{\ell}_2=\hat{\ell}_3=0$, and $\hat{\ell}_4=\hat{\ell}_5=\hat{\ell}_6=\hat{\ell}_7=-1$, it is not a solution to the corresponding GSOCCP$(F,G,\mathcal{K})$.

Although we may have stationary points that are not solutions, a further analysis is possible. Assuming that (10) holds, if $\hat{\lambda}=0$, together with the linear independency of the set $\{\nabla F_j(\hat{x})\}_{j=1}^n$, then it can be proved that $\hat{x}$ is a solution to GSOCCP$(F,G,\mathcal{K})$.

Likewise, if $\hat{\lambda}=1$ and the set $\{\nabla G_j(\hat{x})\}_{j=1}^p$ is linearly independent then one can show that $\hat{x}$ is a solution to GSOCCP$(F,G,\mathcal{K})$.

Now, suppose that $0<\hat{\lambda}<1$. From (10), $F(\hat{x})=\dfrac{1-\hat{\lambda}}{\hat{\lambda}}AG(\hat{x})$ and so

$$F(\hat{x})^T\bar{A}F(\hat{x})=\left(\frac{1-\hat{\lambda}}{\hat{\lambda}}\right)^2 G(\hat{x})^T AG(\hat{x}), \tag{12}$$

that is, $F(\hat{x})^T\bar{A}F(\hat{x})$ and $G(\hat{x})^TAG(\hat{x})$ have the same sign. Moreover, $\hat{l}_8=0$.

If $F(\hat{x})^T\bar{A}F(\hat{x})<0$, from (12) it follows that $G(\hat{x})^TAG(\hat{x})<0$. Since $\hat{z}\geq 0$ and $\hat{w}\geq 0$ then $\hat{\ell}_4\neq 0$, $\hat{\ell}_6\neq 0$, and thus $\hat{x}$ is not a solution to GSOCCP$(F,G,\mathcal{K})$.

If $F(\hat{x})^T\bar{A}F(\hat{x})\geq 0$ then $F_1(\bar{x})\geq 0$ and so $\hat{s}=F_1(\bar{x})\geq 0$, that is $\hat{\ell}_7=0$. Also, from (12) it follows that $G(\hat{x})^TAG(\hat{x})\geq 0$ and so $G_1(\bar{x})\geq 0$, which implies that $\hat{y}=G_1(\bar{x})\geq 0$, and hence $\hat{\ell}_5=0$.

Let $F(\hat{x})^T\bar{A}F(\hat{x})>0$. If $\hat{w}=0$ then $\hat{\ell}_6\neq 0$, and if $\hat{w}>0$ then $\hat{\ell}_2\neq 0$, so $\hat{x}$ is not a solution to GSOCCP$(F,G,\mathcal{K})$.

Finally, if $F(\hat{x})^T\bar{A}F(\hat{x})=0$, from (12) we have $G(\hat{x})^TAG(\hat{x})=0$, and then $\hat{z}=\hat{w}=0$, which implies that $\hat{\ell}_2=\hat{\ell}_3=\hat{\ell}_4=\hat{\ell}_6=0$. Therefore, in this case $\hat{x}$ is a solution to GSOCCP$(F,G,\mathcal{K})$.

5 Numerical experiments

Five sets of low dimensional instances were tested to provide geometric insight into the GSOCCP and into the performance of the reformulations: (1) affine functions with unique solution (2-D and 3-D); (2) nonlinear functions (2-D) with singular Jacobian at (unique) solution; (3) nonlinear functions (2-D) with four isolated solutions; (4) affine functions (2-D) with infinitely many solutions; (5) affine functions (5-D) with unique solution, from [7].

For all the test problems, four versions of the code `easy` were run. The first version, `easycc`, addresses problems with the formulation (4), with the *convex combination* of f and g. The second, `eccmod`, is a *modification* of formulation (4), where variables μ and ν were eliminated. The third, `easymx`, solves problems with the *mixed* equivalent formulation (7), and the fourth, `easyto`, addresses problems with the *two-in-one* formulation (9).

5.1 Affine functions

• Bidimensional case

The first instance of GSOCCP$(F, G, \mathcal{K})$ tested was (5). Results are given in previous Table 1 and in Tables 2, 3 and 4. The same notation and initialization used to prepare Table 1 is adopted. For all the tests, the run ended either with norm of projected gradient smaller than 10^{-8} or with a step with norm smaller than 10^{-8}.

The Jacobians of functions F and G given in (5) are the identity matrix. Analyzing matrices H_f and H_g at the end points, one can see that, for $r^* = 1$, H_f is never positive definite, whereas for $r^* = 0$, H_g is positive definite for 83 runs out of 200. Therefore, the sufficient conditions of Theorem 1 hold for 42% of these tests. It is worth mentioning that there were 34 runs (17%) for which convergence to point I was obtained and $r^* = 0$, without positive definiteness of matrix H_g.

	ITBOX	FE	ITQUA	MVP	ϕ^*_{mod}
minimum	6	7	21	33	0
average	191.9	274.6	524.4	903.8	
maximum	1031	1485	2648	4601	0.57381

Table 2: Results for instance (5) - formulation (6).

	ITBOX	FE	ITQUA	MVP	Φ^*
minimum	7	8	23	34	1E-31
average	11.1	14.2	61.3	90.4	
maximum	24	31	185	291	0.95677

Table 3: Results for instance (5) - formulation (7).

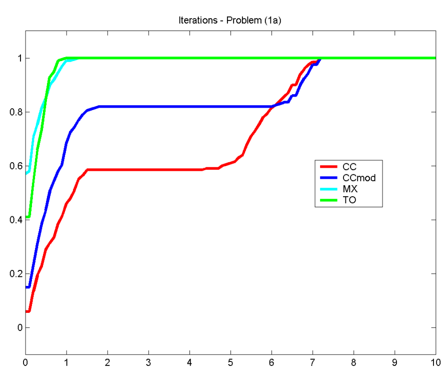
	ITBOX	FE	ITQUA	MVP	Ξ^*
minimum	6	7	26	34	1E-32
average	11.3	15.4	70.2	102.8	
maximum	18	27	161	225	1E-17

Table 4: Results for instance (5) - formulation (9).

Three additional end points appeared with formulations (6) and (7): IV $(-0.648, -1.146)$, V $(-0.614, -1.202)$ and VI $(-1.232, -1.5849)$. The convergence occurred as follows: formulation (6) converged to point I (unique solution) in 62.5% of the tests, to point II in 18% of the tests and to point IV in the remaining 19.5% of the tests. It is worth mentioning that $\left(\phi^*_{\text{mod}}\right)_{\text{II}} \approx 2\text{E}{-}6$ and $\left(\phi^*_{\text{mod}}\right)_{\text{IV}} = 0.57381$. Moreover, 98% of the tests ended with norm of the projected gradient less than 10^{-8}. The four tests (2%) that stopped with a too small step converged to point I, with norm of the projected gradient between 2×10^{-8} and 5×10^{-7}.

Formulation (7) converged to point I for 63% of the tests, to point V in 33.5% of the tests and to point VI in the remaining 3.5% of the tests. The corresponding objective function values were $\Phi^*_{\text{V}} = 0.61052$ and $\Phi^*_{\text{VI}} = 0.95677$. Formulation (9) converged to point I in 100% of the tests. All the runs stopped with norm of the projected gradient less than 10^{-8} for both formulations (7) and (9).

When it comes to computational effort, from Tables 1-4, we notice that formulation (4) is the most expensive, followed by (6), which manages to distinguish better than (4) the nature of end points by means of the merit function values. Formulations (7) and (9) are by far cheaper than (4) and (6). Formulation (7) also has shown to be prone to separate well the solutions, but no doubt formulation (9) is the most effective regarding this instance. This behaviour stands out in the plots depicted in Figure 2, where 'outer iterations' (`ITBOX`) and 'matrix vector products' (`MVP`) were elected as measures of global effort. The plots were prepared using the performance profile technique of Dolan & Moré [4], scaling the ratios with $\log_2$ to accommodate the curves in a single chart.



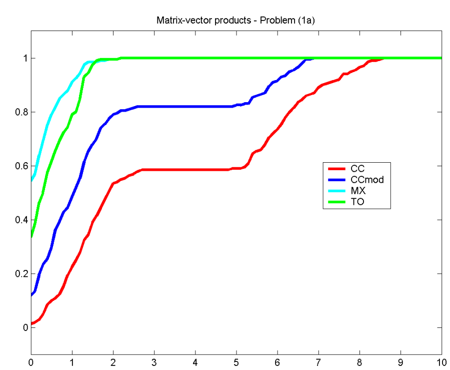


Figure 2: Performance profile plots of outer iterations (left) and matrix vector products (right) of instance (5).

- **Tridimensional case**

The previous instance was extended to $\mathbb{R}^3$ as

$$F(x) = \begin{pmatrix} x_1 + 1 \\ x_2 + 2 \\ x_3 + 3 \end{pmatrix}, G(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad A = \bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{13}$$

and it is graphically represented in Figure 3(b). The two cones correspond to points $x \in \mathbb{R}^3$ such that $G(x) \in \mathcal{K}$ and $F(x) \in \mathcal{K}^\circ$, whereas the sphere is the loci of points verifying $F(x)^T G(x) = 0$.

Two hundred initial points x^0 were randomly generated in the box $[-10, 10]^3$, and remaining variables set to 0.5. Results are given in Tables 5, 6, 7 and 8, for which the same notation of previous Tables is adopted. The unique solution x^* was denoted by I $(1.302, -0.723, -1.083)$, but there were six additional end points:

II	$(-1.02, -1.96, -2.94)$;	III	$(0.0507, -0.0920, -0.138)$;
IV	$(4\text{E}{-6}, -0.067, -0.101)$;	V	$(-1.59, -1.16, -1.74)$;
VI	$(11.2, 13.0, -3.32)$;	VII	$(-7.01, -8.56, -0.345)$.

Three were the reasons for stopping: norm of projected gradient smaller than 10^{-8}, step with norm smaller than 10^{-8} and more than 10000 outer iterations performed.

Formulation (4) converged to point I in 61.5%, to II in 28%, to III in 2.5%, and to IV in 8% of tests. As far as the merit function values at these end points, ϕ^*_I is always below 10^{-12} (36 tests, out the 200, reached exactly zero), whereas ϕ^*_{II}, ϕ^*_{III} and ϕ^*_{IV} are between 10^{-6} and 10^{-4}. Analyzing the end points, 43.5% had norm of projected gradient below 10^{-8}, 55% stopped with too small a step, and 1.5% exceeded the maximum number of iterations.

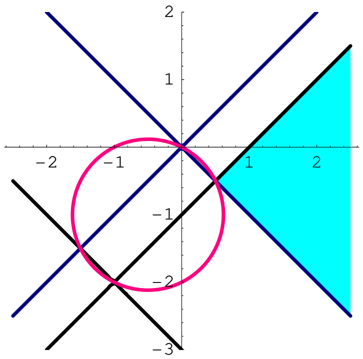
Formulation (6) ended at point I in 69% runs, at II in 26.5%, and at III in 4.5% of tests. The end value $\left(\phi^*_{\text{mod}}\right)_{\text{I}}$ is always below 10^{-12}, with 57 stops, out the 200, reaching exactly zero, whereas $\left(\phi^*_{\text{mod}}\right)_{\text{II}}$ and $\left(\phi^*_{\text{mod}}\right)_{\text{III}}$ varies between 10^{-6} and 10^{-4}. The reasons for stopping were: 43.5% with norm of projected gradient below 10^{-8}, 52.5% with too small a step, and 4% reached the maximum number of iterations.

Formulation (7) converged to the solution I in 69.5%, and to V in the remaining 30.5% of tests. At the end points, $10^{-20} < \Phi^*_I < 10^{-15}$, whereas $\Phi^*_V > 2.5$. From the 200 tests, 173 (86.5%) stopped with norm of projected gradient less than 10^{-8}, and 27 (13.5%) stopped with too small a step.

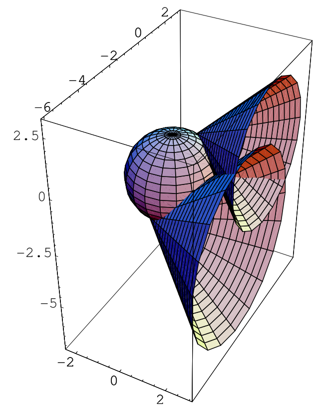
Formulation (9) converged to the solution I in 68.5%, to V in 30.5%, to VI in 0.5% and to VII in 0.5% of tests. At the end points, $10^{-20} < \Xi^*_I < 10^{-14}$, $\Xi^*_V \approx 2.36$, $\Xi^*_{VI} \approx 185$, and $\Xi^*_{VII} \approx 1361$. The reasons for stopping were: 169 out of 200 test (84.5%) ended with norm of projected gradient smaller than the given tolerance, and 31 (15.5%) with too small a step.

	ITBOX	FE	ITQUA	MVP	ϕ^*
minimum	17	25	987	1116	0
average	982.0	1480.7	134079.7	137000.4	
maximum	10000	15536	1442551	1471472	2E-4

Table 5: Results for instance (13) - formulation (4).



(a)



(b)

Figure 3: (a) 2-D instance (5) and (b) 3-D instance (13).

	ITBOX	FE	ITQUA	MVP	ϕ^*_{mod}
minimum	15	22	738	831	0
average	962.5	1396.2	118981.2	121747.5	
maximum	10000	15758	1358550	1388081	2E-4

Table 6: Results for instance (13) - formulation (6).

	ITBOX	FE	ITQUA	MVP	Φ^*
minimum	11	12	44	60	1E-20
average	52.5	83.8	6392.8	6776.7	
maximum	350	541	50183	51855	2.5769

Table 7: Results for instance (13) - formulation (7).

	ITBOX	FE	ITQUA	MVP	Ξ^*
minimum	0	19	7	40	1E-20
average	48.1	77.6	5573.1	5948.4	
maximum	142	224	20186	21243	1361.4

Table 8: Results for instance (13) - formulation (9).

Besides the good separations provided by formulations (7) and (9) as far as the merit function values at the end points, from the figures of Tables 5-8 we also observe the smaller computational effort demanded to solve problems (7) and (9), compared to formulations (4) and (6). The performance profile charts of Figure 4 corroborate these observations.

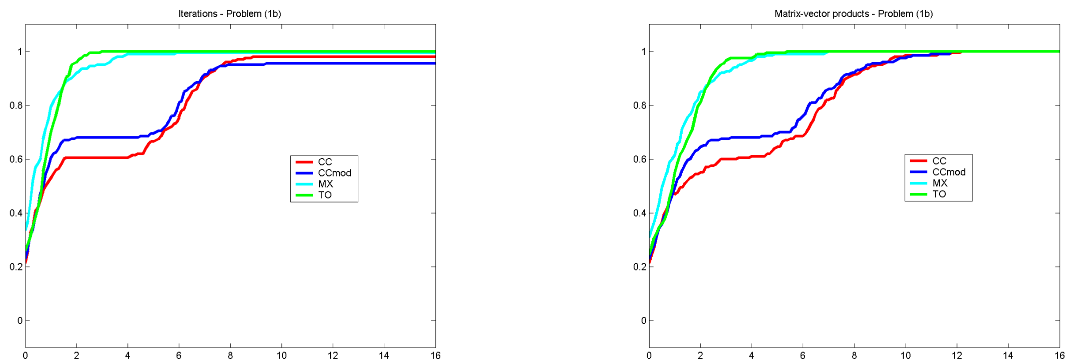


Figure 4: Performance profile plots of outer iterations (left) and matrix vector products (right) of instance (13).

5.2 Nonlinear function (singular Jacobian at solution)

The second instance of GSOCCP$(F, G, \mathcal{K})$ tested was

$$F(x) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, G(x) = \begin{pmatrix} (x_1 - 1)^2 \\ x_2^2 \end{pmatrix} \quad \text{and} \quad A = \bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{14}$$

illustrated in Figure 5, where the shaded region represents the points $x \in \mathbb{R}^2$ such that $G(x) \in \mathcal{K}$ and $F(x) \in \mathcal{K}^\circ$. The complementarity $F(x)^T G(x) = 0$ is satisfied by the points at the lighter curve, constituted by the horizontal axis and the rotated hyperbole, so the unique solution to problem GSOCCP$(F, G, \mathcal{K})$ at the origin is easily seen.

Two hundred initial points x^0 were randomly generated in the box $[-10, 10] \times [-10, 10]$, and remaining variables set to 0.5. Results are given in Tables 9, 10, 11 and 12. Besides the solution I (0,0), six additional end points were obtained, namely

II	$(-2\text{E}{-5}, 3\text{E}{-2})$;	III	$(3\text{E}{-3}, 0)$;
IV	$(-3\text{E}{-3}, 0)$;	V	$(0.631, 0.282)$;
VI	$(0.614, 0)$;	VII	$(0.604, 0.252)$.

For this instance,

$$F'(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad G'(x) = \begin{pmatrix} 2(x_1 - 1) & 0 \\ 0 & 2x_2 \end{pmatrix}$$

and so $G'(0,0)$ is singular. This singularity seems to deteriorate the results, as we describe next.

Formulation (4) converged to the solution in 64% of runs, to II in 3%, to III in 13%, and to IV in 20% of the tests, with $\phi_I^* = 0$, $\phi_{II}^* \approx 10^{-8}$, $\phi_{III}^*, \phi_{IV}^* \approx 10^{-12}$. The reasons for stopping were norm of projected gradient less than 10^{-8} (95% of runs) and too small a step (5% of runs).

Formulation (6) ended at the solution I in 39% of tests, at II in 17%, at V in 22.5%, and at VI in 21.5% of the tests, with $\left(\phi_{\text{mod}}^*\right)_{\text{I}} = 0$, $\left(\phi_{\text{mod}}^*\right)_{\text{II}} \approx 10^{-8}$, $\left(\phi_{\text{mod}}^*\right)_{\text{V}} \approx 0.025$, and $\left(\phi_{\text{mod}}^*\right)_{\text{VI}} \approx 0.029$. Two were the reasons for stopping: 97% of tests reached the prescribed tolerance to the norm of projected gradient, and 3% ended with too small a step.

Formulation (7) stopped at the solution I in 19.5% of tests, at II in 15%, at V in 56%, and at VI in 9.5% of the tests, with $10^{-29} < \Phi_I^* < 10^{-17}$, $\Phi_{II}^* \approx 10^{-8}$, $\Phi_V^* \approx 0.025$, and $\Phi_{VI}^* \approx 0.029$. All the runs stopped with norm of projected gradient smaller than 10^{-8}.

Formulation (9) converged to solution I in 42%, and to VII in the remaining 58% of tests, with $\Xi_I^* < 10^{-17}$, and $\Xi_{VII}^* \approx 0.025$. The norm of the projected gradient was smaller than the given tolerance for 100% of the tests.

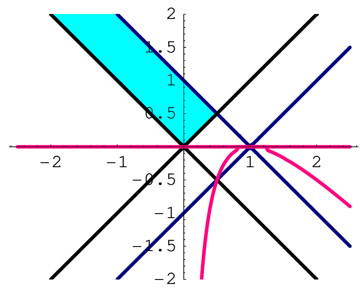


Figure 5: Second instance

	ITBOX	FE	ITQUA	MVP	ϕ^*
minimum	8	11	50	76	0
average	63.6	94.9	322.8	632.9	
maximum	1027	1527	5370	7786	9E-8

Table 9: Results for instance (14) - formulation (4).

As far as the average computational effort, formulations (4) and (7) are comparable, (6) is the most expensive and (9) is the cheapest. The performance profile plots of Figure 6 give a graphical perspective of the global effort.

As occurred with the affine first instance, from formulation (4) the solution is not easily distinguished from the end values of the merit function. On the other hand, by solving formulations (6), (7), and (9), the end points that are not solutions clearly stand out. By observing the end

	ITBOX	FE	ITQUA	MVP	ϕ^*_{mod}
minimum	7	9	21	34	0
average	181.1	269.5	662.7	1148.1	
maximum	1023	1527	3703	6042	0.028865

Table 10: Results for instance (14) - formulation (6).

	ITBOX	FE	ITQUA	MVP	Φ^*
minimum	9	10	22	37	1E-29
average	100.8	145.8	445.6	646.8	
maximum	992	1461	4332	6267	0.02887

Table 11: Results for instance (14) - formulation (7).

	ITBOX	FE	ITQUA	MVP	Ξ^*
minimum	9	11	42	59	0
average	17.9	24.6	87.1	135.8	
maximum	26	39	165	270	0.24719

Table 12: Results for instance (14) - formulation (9).

points reached by the four formulations, we notice that formulation (9) is less prone to be trapped by near solutions, like points II, III and IV.

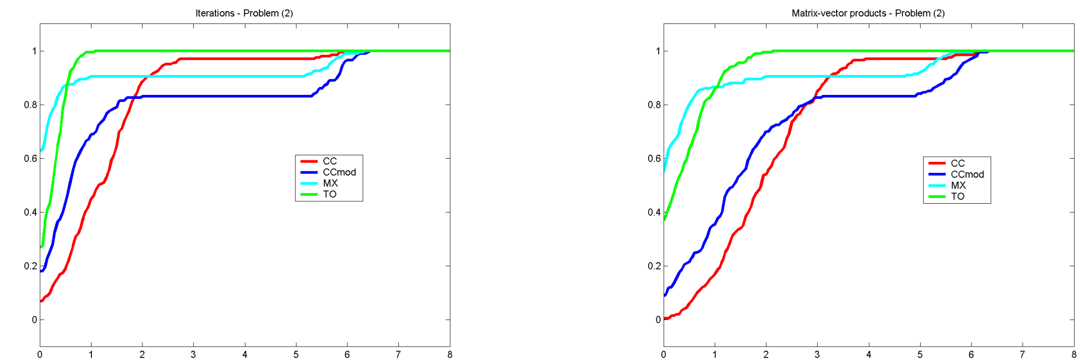


Figure 6: Performance profile plots of outer iterations (left) and matrix vector products (right) of instance (14).

5.3 Nonlinear functions (four isolated solutions)

The third instance of GSOCCP$(F, G, \mathcal{K})$ tested was

$$F(x) = \begin{pmatrix} x_1 x_2 \\ x_2 - x_1 \end{pmatrix}, G(x) = \begin{pmatrix} x_1^2 - 1 \\ x_1 - x_2 \end{pmatrix} \quad \text{and} \quad A = \bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Figure 7 depicts its solution set. The shaded region corresponds to points $x \in \mathbb{R}^2$ for which simultaneously $G(x) \in \mathcal{K}$ and $F(x) \in \mathcal{K}^\circ$. The lighter curve is the loci of points x verifying $F(x)^T G(x) = 0$, so the four isolated solutions can be clearly visualized.

Beisdes the four solutions I $(1, 1)$, II $(1.32472, 0.56984)$, III $(-1, -1)$, and IV $(-1.32472, -0.56984)$, other end points were obtained as well: V $(1.0147, 1.0147)$, VI $(-1.0147, -1.0147)$, VII $(0.8225, 0.4911)$, and VIII $(-0.8225, -0.4911)$.

Two hundred initial points were generated as in the previous instances. Results are reported in Tables 13, 14, 15 and 16.

For this instance,

$$F'(x) = \begin{pmatrix} x_2 & x_1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad G'(x) = \begin{pmatrix} 2x_1 & 0 \\ 1 & -1 \end{pmatrix}$$

and so $F'(x_1, x_2)$ is singular for $x_1 = -x_2$ and $G'(x_1, x_2)$ is singular for $x_1 = 0$. At the eight reported end points, the Jacobians are invertible.

Convergence of formulation (4) occurred as follows: 95.5% of runs reached one of the four solutions (29.5% converged to I, 18% to II, 32% to III and 16% to IV), whereas 3% of the remaining

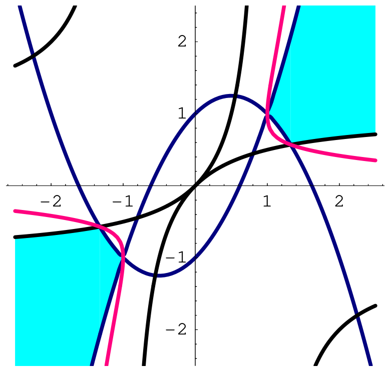


Figure 7: Third instance

runs ended in point V, and 1.5% in point VI. The optimal values ϕ^*_I, ϕ^*_{II}, ϕ^*_{III} ϕ^*_{IV} were always below 10^{-11}, and $\phi^*_V, \phi^*_{VI} \approx 10^{-8}$. Two were the reasons for stopping: norm of projected gradient less than 10^{-8} (91.5%) and too small a step (8.5%).

Formulation (6) ended at one of the four solutions in 65.5% of tests (8% at I, 21.5% at II, 12% at III and 24% at IV), at V in 10%, at VI in 10.5%, at VII in 6.5%, and at VIII in 7.5% of the tests. The optimal values $(\phi^*_{\text{mod}})_{\text{I}}$, $(\phi^*_{\text{mod}})_{\text{II}}$, $(\phi^*_{\text{mod}})_{\text{III}}$, and $(\phi^*_{\text{mod}})_{\text{IV}}$ were always below 10^{-15}, whereas $(\phi^*_{\text{mod}})_{\text{V}}, (\phi^*_{\text{mod}})_{\text{VI}} \approx 10^{-8}$ and $(\phi^*_{\text{mod}})_{\text{VII}}, (\phi^*_{\text{mod}})_{\text{VIII}} = 0.189$. The reasons for stopping were achieving the prescribed tolerance to the norm of projected gradient (95%), and too small a step (5%).

Formulation (7) stopped at one of the four solutions in 98% of tests (18.5% at I, 29.5% at II, 21% at III and 29% at IV), and at point V in the remaining 2% of the tests, with $10^{-33} < \Phi^*_I, \Phi^*_{II}, \Phi^*_{III}, \Phi^*_{IV} < 10^{-16}$, and $\Phi^*_V \approx 10^{-8}$. All the runs stopped with norm of projected gradient smaller than 10^{-8}.

Formulation (9) converged to one of the four solutions in 100% of tests (17% at I, 31% at II, 21% at III and 31% at IV), with $0 \geq \Xi^*_I, \Xi^*_{II}, \Xi^*_{III}, \Xi^*_{IV} < 10^{-16}$. The norm of the projected gradient was smaller than the given tolerance for 100% of the tests.

	ITBOX	FE	ITQUA	MVP	ϕ^*
minimum	9	10	33	46	0
average	51.5	78.2	392.1	596.6	
maximum	672	1025	5919	7668	1E-8

Table 13: Results for instance (15) - formulation (4).

When it comes to computational effort and efficiency in obtaining a solution to the GSOCCP$(F, G, \mathcal{K})$, the formulation (9) is by far the best, followed by a practical tie between

	ITBOX	FE	ITQUA	MVP	ϕ^*_{mod}
minimum	7	10	32	44	0
average	142.5	214.3	801.4	1148.0	
maximum	660	1011	3754	5297	0.1888

Table 14: Results for instance (15) - formulation (6).

	ITBOX	FE	ITQUA	MVP	Φ^*
minimum	9	10	36	54	1E-33
average	34.3	49.4	228.9	328.6	
maximum	734	1109	5567	7171	1E-8

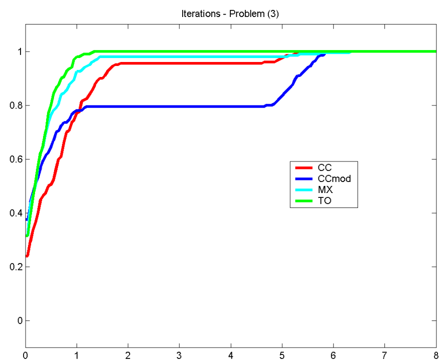
Table 15: Results for instance (15) - formulation (7).

	ITBOX	FE	ITQUA	MVP	Ξ^*
minimum	8	10	42	53	0
average	19.6	27.0	96.7	156.5	
maximum	30	43	220	321	1E-17

Table 16: Results for instance (15) - formulation (9).

the other three. The charts of Figure 8 provide visual support to these observations.

The formulations (4), (6) and (7) are somehow trapped by end points with very small objective values, but that are poor approximations to the solutions. Formulation (6) is the only one that reached numerically stationary but safely not solutions to the complementarity problem.



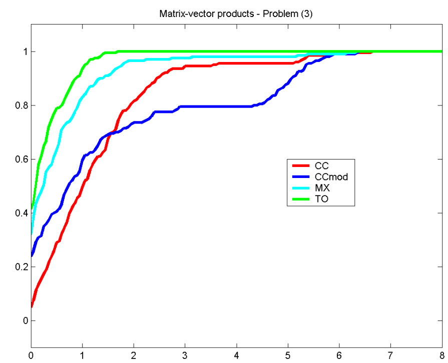


Figure 8: Performance profile plots of outer iterations (left) and matrix vector products (right) of instance (15).

5.4 Affine function with infinitely many solutions

The fourth instance of GSOCCP$(F, G, \mathcal{K})$ tested was

$$F(x) = \begin{pmatrix} x_1 \\ x_2 - x_1 \end{pmatrix}, G(x) = \begin{pmatrix} x_1 \\ x_1 - x_2 \end{pmatrix} \quad \text{and} \quad A = \bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

The solutions of this problem lies in the rays $(t, 0), t \geq 0$ and $(t, 2t), t \geq 0$, illustrated in Figure 9(a), where the shaded regions corresponds to $x \in \mathbb{R}^2$ such that $G(x) \in \mathcal{K}$ and $F(x) \in \mathcal{K}^\circ$. The pair of lines $x_2 = 0$ and $x_2 = 2x_1$ are the loci where $F(x)^T G(x) = 0$ holds.

The starting with 200 initial points also followed the methodology previously described. All the end points belong to a strip encompassing the solution rays, as depicted in Figure 9(b). The precision ε_x is such that the end pair (x_1^*, x_2^*) satisfies $|x_2^*| < \varepsilon_x$ and $|x_2^* - 2x_1^*| < \varepsilon_x$, respectively. The precision attained for each formulation is reported in Table 17, where (7) is the most precise, followed closely by (9).

Formulation	(4)	(6)	(7)	(9)
ε_x	0.003316	0.002347	0.001840	0.001916

Table 17: Precision of obtained solution for each formulation.

Results are reported in Tables 18, 19, 20 and 21. All the formulations and every run stopped with norm of projected gradient less than the tolerance 10^{-8}.

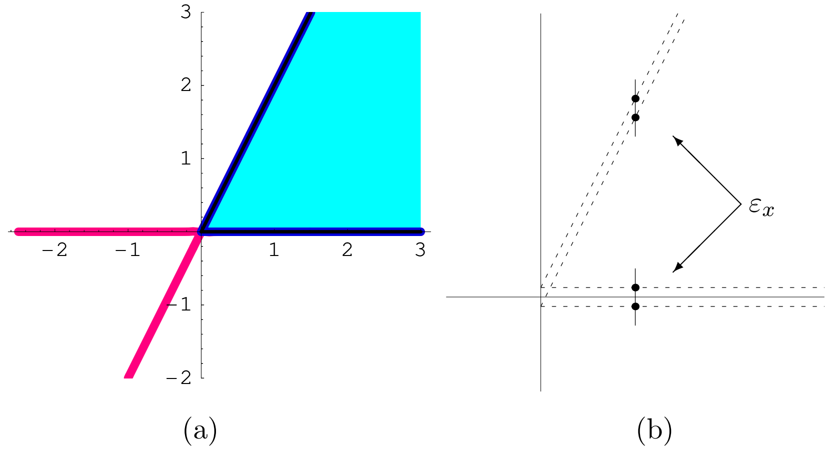


Figure 9: Fourth instance (a) and width of strip that represents the precision of obtained solution (b).

The similar low figures achieved by the four formulations corroborated the easy nature of this instance. The charts of Figure 10 illustrate this behaviour. From the plot on the right we can see a slight advantage to formulation (9) as far as matrix-vector products are concerned.

	ITBOX	FE	ITQUA	MVP	ϕ^*
minimum	4	5	10	17	0
average	10.8	14.5	40.9	67.2	
maximum	27	37	133	199	1E-11

Table 18: Results for instance (16) - formulation (4).

	ITBOX	FE	ITQUA	MVP	ϕ^*_{mod}
minimum	4	5	10	17	0
average	10.4	14.0	36.1	59.6	
maximum	23	31	125	193	1E-12

Table 19: Results for instance (16) - formulation (6).

	ITBOX	FE	ITQUA	MVP	Φ^*
minimum	4	5	11	18	0
average	10.6	14.0	35.7	62.3	
maximum	20	34	113	182	1E-12

Table 20: Results for instance (16) - formulation (7).

	ITBOX	FE	ITQUA	MVP	Ξ^*
minimum	4	5	10	16	0
average	11.1	14.1	26.9	47.7	
maximum	24	30	61	117	1E-12

Table 21: Results for instance (16) - formulation (9).

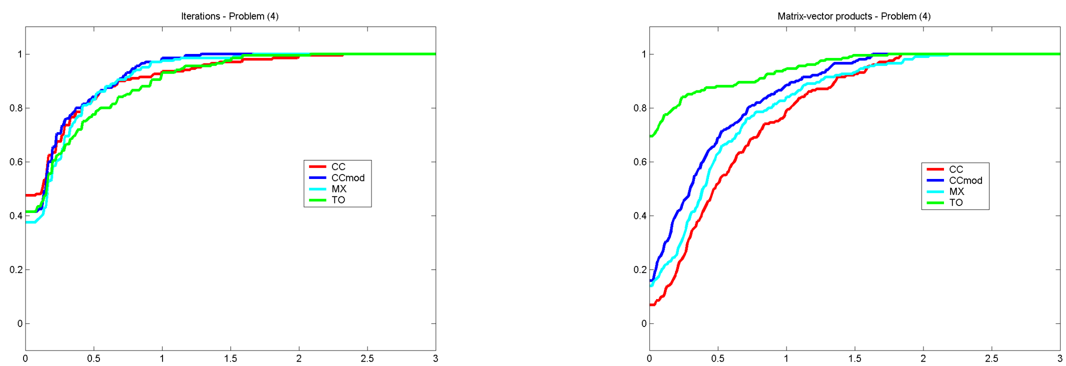


Figure 10: Performance profile plots of outer iterations (left) and matrix vector products (right) of instance (16).

5.5 Affine functions of Peng & Yuan

The fifth instance of GSOCCP$(F, G, \mathcal{K})$ tested was obtained from [7, Problem 3]:

$$F(x) = \begin{pmatrix} 15x_1 - 5x_2 - x_3 + 4x_4 - 5x_5 \\ 5x_2 + x_5 \\ -x_1 - 3x_2 + 8x_3 + 2x_4 - 3x_5 \\ 2x_1 - 4x_2 + 2x_3 + 9x_4 - 4x_5 \\ -5x_2 + 10x_5 - 1 \end{pmatrix}, \; G(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \tag{17}$$

with $A = \bar{A} = \operatorname{diag}(1, -1, -1, -1, -1)$. Three types of end points were obtained, namely

$$\begin{aligned}\text{(unique solution) I}\,&(0.049185, -0.0030997, 0.0096024, 0.0031883, 0.048033),\\ \text{II}\,&(0.048919, -0.0031088, 0.0096054, 0.0032299, 0.048093),\\ \text{and III}\,&(0.020424, -0.016125, 0.022867, 0.021211, 0.0879190).\end{aligned}$$

Tables 22-25 contain the results of 200 runs obtained from randomly generated $x^0 \in [-10, 10]^5$ and remaining variables set to 0.5.

For reformulation (4), 96% of tests ended with norm of projected gradient less than 10^{-8}, whereas 4% stopped with too small a step (infinity norm smaller than 10^{-8}). In terms of quality of results, 93.5% of tests converged to the unique solution, with $\phi_I^* < 10^{-11}$, although only 56.5% reached $\phi_I^* < 10^{-13}$, the threshold obtained by reformulation (9). Point II was reached for 2.5% of tests ($\phi_{II}^* \approx 10^{-10}$) and point III for the remaining 4%, with $\phi_{III}^* \approx 10^{-8}$.

Formulation (6) converged to the solution I in 65.5% of the runs, to II in 27% and to III in the remaining 7.5%, with $(\phi_{\text{mod}}^*)_{\text{I}} \approx 10^{-16}$, $(\phi_{\text{mod}}^*)_{\text{I}} \approx 10^{-10}$, and $(\phi_{\text{mod}}^*)_{\text{I}} \approx 10^{-8}$. All the runs stopped with norm of projected gradient below the prescribed tolerance.

Formulation (7) ended at the solution I in 54.5% of runs, and at point III in the remaining 45.5% runs. The attained objective values were $\Phi_I^* \approx 10^{-20}$ and $\Phi_{III}^* \approx 10^{-6}$. The reason for stopping was always norm of projected gradient smaller than 10^{-8}.

As far as reformulation (9) is concerned, 100% of tests stopped with norm of projected gradient less than 10^{-8}, 72% ended at point I ($\Xi_I^* < 10^{-13}$) and 28% at point III ($\Xi_{III}^* > 10^{-7}$).

	ITBOX	FE	ITQUA	MVP	ϕ^*
minimum	31	44	238	483	1E-25
average	220.3	319.8	2591.4	3504.3	
maximum	348	538	5645	7358	1E-8

Table 22: Results for instance (17) - formulation (4).

	ITBOX	FE	ITQUA	MVP	ϕ_{mod}^*
minimum	22	30	140	250	1E-28
average	195.6	285.0	1518.0	2269.8	
maximum	333	496	4162	5770	1E-8

Table 23: Results for instance (17) - formulation (6).

Sorting the 200 final objective function values in ascending order and ploting the resulting vector produced the graphs in Figures 11 and 12. Notice how reformulations (7) and (9) produce better discrimination between the solutions obtained, comparatively to (4) and (6).

In terms of the average computational effort demanded by this instance, Tables 22-25 evidence

	ITBOX	FE	ITQUA	MVP	Φ^*
minimum	29	39	201	290	3E-24
average	63.1	88.3	654.7	910.5	
maximum	371	562	5645	5199	1E-5

Table 24: Results for instance (17) - formulation (7).

	ITBOX	FE	ITQUA	MVP	Ξ^*
minimum	21	32	189	244	1E-27
average	69.6	96.6	667.3	907.5	
maximum	262	389	2823	3904	1E-5

Table 25: Results for instance (17) - formulation (9).

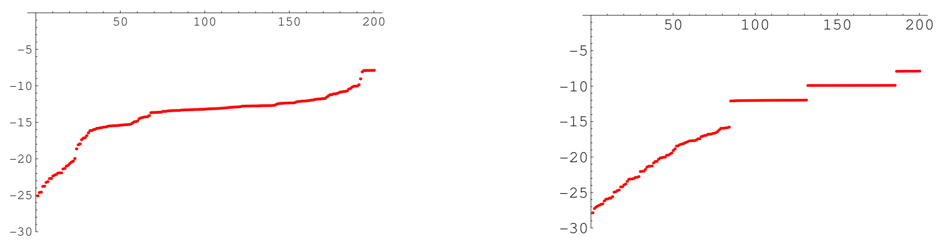


Figure 11: Test problem $\times \log_{10}(\phi^*)$ (left) and test problem $\times \log_{10}(\phi^*_{\text{mod}})$ (right) for the 200 tests of instance (17).

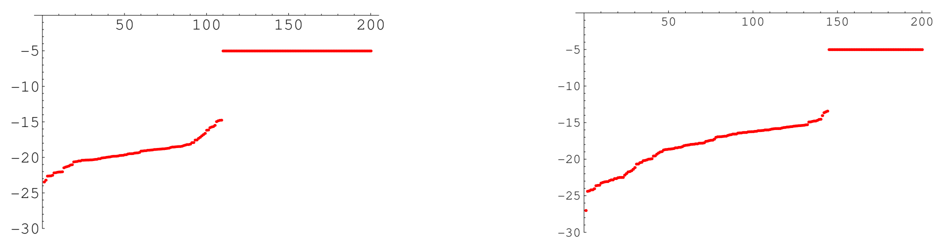


Figure 12: Test problem $\times \log_{10}(\Phi^*)$ (left) and test problem $\times \log_{10}(\Xi^*)$ (right) for the 200 tests of instance (17).

two groups, the most expensive encompass formulations (4) and (6), whereas the most effective is constituted of (7) and (9). The performance profile plots of Figure 13 corroborate this observation.

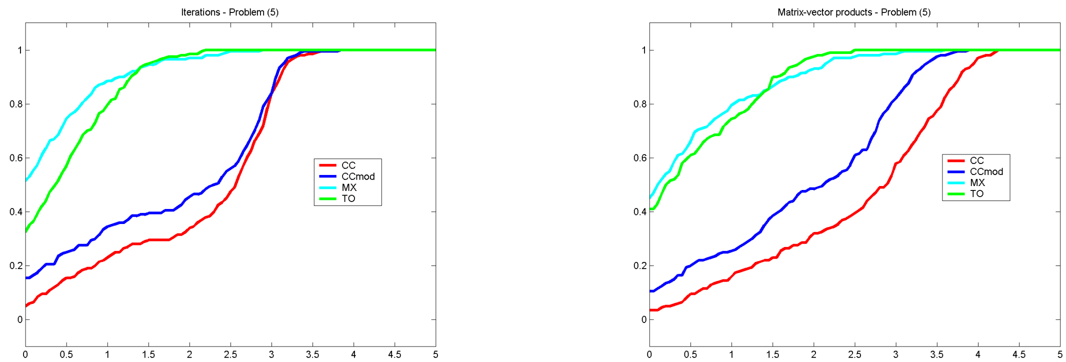


Figure 13: Performance profile plots of outer iterations (left) and matrix vector products (right) of instance (17).

6 Final remarks

Table 26 summarizes the numerical results presented in the previous section, as far as convergence to solution of GSOCCP is concerned. From the second column, we notice that the elimination of variable μ and ν does not improve significantly the performance of formulation (4). In fact, it is deteriorated for nonlinear instances (14) and (15). Comparing the third and fourth columns, corresponding to formulations (7) and (9) resp., an advantage to the latter is evidenced. On the other hand, by analyzing the first and the fourth columns, that is, formulations (4) and (9) resp., it is not clear which is more effective. Despite the better performance of formulation (9) to solve the first instance (2-D and 3-d), for instance (14), the most difficult to be solved, and for the affine problem (17) of Peng & Yuan, formulation (4) was more successful than (9).

Instance	easycc	eccmod	easymx	easyto
(5)	59%	62.5%	63%	100%
(13)	61.5%	69%	69.5%	68.5%
(14)	64%	39%	39%	42%
(15)	96%	66%	98%	100%
(16)	100%	100%	100%	100%
(17)	93.5%	65.5%	54.5%	72%

Table 26: Percentage of convergence to solution of corresponding instance of each code/formulation.

When it comes to better discrimination of the solutions, no doubt formulation (9) overcomes the drawbacks of formulation (4). However, the theoretical condition under which the stationary points of reformulation (4) correspond to solutions of the GSOCCP is still to be stated.

References

[1] R. Andreani, A. Friedlander, S. A. Santos, On the resolution of generalized nonlinear complementarity problems, *SIAM Journal on Optimization*, **12** (2001) 303-321.

[2] R. Andreani, A. Friedlander, M. P. Mello, S. A. Santos, Complementarity problems in second-order cones, submitted to publication, 2006.

[3] R.H. Bielchowsky, A. Friedlander, F.A.M. Gomes, J.M. Martínez, M. Raydan, An adaptive algorithm for bound constrained quadratic minimization. *Investigación Operativa* **7** (1997) 67-102.

[4] E. D. Dolan, J. J. Moré, Benchmarking optimization software with performance profiles, *Mathematical Programming*, **91** (2002) 201-213.

[5] A. Friedlander, J.M. Martínez, S.A. Santos, A new trust region algorithm for bound constrained minimization, *Applied Mathematics & Optimization* **30** (1994) 235-266.

[6] N. Krejić, J.M. Martínez, M.P. Mello, E.A. Pilotta. Validation of an augmented Lagrangian algorithm with a Gauss-Newton Hessian approximation using a set of hard-spheres problems. *Computational Optimization and Applications* **16** (2000) 247-263.

[7] J.-M. Peng, Y.-X. Yuan, Unconstrained Methods for Generalized Complementarity Problems, *Journal of Computational Mathematics* **15** (1997) 253-264.

[8] R. T. Rockafellar, *Convex Analysis*, Princeton (NJ): Princeton Univ. Press, 1970.