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A Weak Maximum Principle for Discrete Optimal Control Problems with Mixed Constraints

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Abstract

In this study, first-order necessary optimality conditions, in the form of a weak maximum principle, are derived for discrete optimal control problems with mixed equality and inequality constraints. Such conditions are achieved by using the Dubovitskii–Milyutin formalism approach. Nondegenerate conditions are obtained under the constant rank of the subspace component (CRSC) constraint qualification, which is an important generalization of both the Mangasarian–Fromovitz and constant rank constraint qualifications. Beyond its theoretical significance, CRSC has practical importance because it is closely related to the formulation of optimization algorithms. In addition, an instance of a discrete optimal control problem is presented in which CRSC holds while other stronger regularity conditions do not.

Keywords Discrete optimal control problems · Mixed constraints · Nondegenerate necessary optimality conditions · Discrete maximum principle · Constant rank of the subspace component constraint qualification

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1 Introduction

In the context of continuous-time optimal control problems, an important tool concerning necessary optimality conditions is the Pontryagin maximum principle (see [17]). Many studies have been made to extend this principle to discrete optimal control (DOC) (see [2, 3, 7, 19], for instance).

In 1959, Rozonoer [19] was the first to prove a discrete version of the Pontryagin maximum principle; he worked on a DOC problem in which both the objective function and the discrete system describing the "dynamics" were linear. Butkovski [3] in 1963, gave an example where the discrete maximum principle is not fulfilled; in this example, nonlinear dynamics were considered. In 1964, Jordan and Polak [7] considered a DOC problem with initial and final state constraints in separated form and abstract control constraints. In such a paper, first-order necessary optimality conditions were obtained, demonstrating that the techniques used in the construction of the maximum principle can be used to obtain a local maximum condition or stationarity. The authors concluded that the necessary conditions of nonlinear DOC problems are related to the conditions of the Pontryagin maximum principle for continuous optimal control problems in the following way: the transversality conditions remain identical, while the requirement of a global maximum of the Hamiltonian function becomes a local maximum or stationarity condition. In 1978, Boltyanskii, one of the authors of the continuous-time maximum principle, was the one who presented in [2] a more detailed study of the discrete optimal control theory for different types of problems. Since then, a great deal of research has been done on this topic, where both theory and methods have been extensively developed.

It is worth mentioning that DOC problems can also be obtained through discretization of continuous optimal control problems (see [2, 5], for example). In [2], Boltyanskii gave some examples of how continuous-time optimal control problems can be transformed into discrete ones. In Cullum [5], an explicit method was presented to discretize continuous nonlinear control problems, where the initial state is fixed and both the final state and control constraints are abstract.

The main purpose of this work is to present the discrete maximum principle for discrete optimal control problems with mixed constraints, which will be obtained via the Dubovitskii–Milyutin formalism. This formalism was given in late 1962 by Dubovitskii and Milyutin (see Girsanov [6]) and provides necessary optimality conditions in the form of an equation defined in the language of functional analysis. One of the main benefits of the Dubovitskii–Milyutin formalism is that it provides a unified framework for analyzing a wide variety of constrained optimization problems. Thereby, the results obtained here can be readily widened to encompass DOC problems posed in other spaces (Banach and Hilbert spaces, for example) or with different kinds of constraints. To obtain nondegenerate conditions, the constant rank of the subspace component (CRSC) constraint qualifications will be used. CRSC was presented in Andreani et al. [1] for nonlinear programming problems and had not been considered yet in the context of discrete optimal control. This constraint qualification has the remarkable property of generalizing several others, including many constant rank-based ones and those involving constant positive linear dependence. For example, CRSC is more general than the relaxed constant positive linear dependence constraint qualification (RCPLD) (which, in turn, is more general than both the constant positive linear dependence (CPLD) and the relaxed constant rank (RCRCQ) constraint qualifications), which is noteworthy. It is also pertinent to mention that constant rank-based constraint qualifications (CQs) (like CRSC) are very interesting from the point of view of model building since it allows the presence of redundancy. Removing redundancy may be simple in linear problems but that is certainly not the case for nonlinear ones. Moreover, CRSC has practical importance because it is closely related to the formulation of optimization algorithms. For example, CRSC implies in the validity of an error bound, which is important in the stability of algorithms and often even in their well-posedness. Despite the fact that there are weaker constraint qualifications in the literature (Abadie and Guignard, for instance), they are mostly merely theoretical and geometrical, and not associated with the convergence (and other properties) of any algorithm. Overall, the CRSC constraint qualification is an important tool in optimization theory and practice and it can be used to derive nondegenerate conditions for a wide range of optimization problems.

We emphasize that studying DOC problems with mixed constraints is of great importance due to their applications, for example, to production planning problems (see Kleindorfer et al. [8]).

In the literature, initially, constrained DOC problems were treated under linear independence-type regularity conditions (see Canon et al. [4] and Magnanti [10], for instance). More recently, we can find cases in which the regularity conditions are of the Abadie or Mangasarian–Fromovitz types, where mixed constraints and state and control constraints were considered separately. See Marinkovíc [12–14], Rojas-Medar et al. [18], Toan et al. [21], Toan and Thuy [22], Toan et al. [23] and Thuy, Thanh and Toan [20], for example.

In [12], Marinkovíc obtained first- and second-order necessary optimality conditions for DOC problems with equality constraints in the control and equality boundary constraints; these results were generalized by the same author in [14] for problems with equality and inequality constraints. He used the concept of 2-regularity in the constraints. In [13], Marinkovíc worked with DOC problems with equality and inequality state constraints, where first- and second-order necessary optimality conditions were demonstrated without any *a priori* normality assumptions. In [18], the authors worked with DOC problems with state and control constraints separately, and equality-only boundary constraints. They developed their work using the Dubovitskii–Milyutin formalism but to obtain non-degenerate optimality conditions, they used a miscellaneous Abadie-MFCQ constraint qualification, where the full rankness of the Jacobian of the equality constraints is changed by the requirement that the linearized and the tangent cones do coincide. Such an abstract condition may be difficult to verify in practice. Sufficient optimality conditions are established under generalized convexity assumptions. In [21], second-order necessary conditions for DOC problems with nonconvex objective function and abstract control constraints are obtained. In [22], mixed constraints (without considering abstract control constraints) are added to the problem. The results are developed for cases in which the problem dynamics are linear; using a given regularity condition, they obtained second-order necessary optimality conditions. The regularity condition is somehow similar to the Mangasarian–Fromovitz constraint qualification. It is shown that it implies the Robinson constraint qualification.

When the state equations are nonlinear, the authors in [23] worked with DOC problems with inequality mixed constraints along with inequality boundary constraints. In their approach, mathematical programming tools and the aforementioned regularity condition are used. In [20], DOC problems with linear dynamics and inequality mixed constraints are considered. No-gap second-order necessary optimality conditions are derived under a full rank-type regularity condition. Second-order sufficient optimality conditions are also obtained.

In this paper, we study a general DOC problem with nonlinear dynamics, equality and inequality mixed constraints and equality and inequality endpoint constraints. The objective function depends on the state and control for all the periods and also on the final state. Similar problems have been studied by several authors. For example, in [23] they considered the same kind of objective function and dynamics, but with inequality mixed constraints only. Also, the endpoint inequality constraints do only involve the final state. Endpoint equality constraints are not considered. In [13], the author worked with the same dynamics and endpoint constraints as in this paper but did not consider the final state in the objective function. The equality and inequality state and control constraints were worked in a separate form. Finally, in [18], the same dynamics and equality boundary constraints as in this work are considered, while the objective function and the state and control constraints were the same as in [13].

This article is divided into five sections. In Sect. 2, some concepts that are important for the application of the Dubovitskii–Milyutin formalism are given. In Sect. 3, we present the discrete optimal control problem and the construction of descent, feasible and tangent cones and their respective dual cones for this type of problem. In Sect. 4, we state and prove the discrete maximum principle for DOC problems with mixed constraints. Furthermore, using the CRSC regularity condition, nondegenerate first-order necessary optimality conditions are obtained. This section ends with an illustrative example. The last section is devoted to the conclusions.

2 Preliminaries

This section mainly describes (briefly) the Dubovitskii–Milyutin Formalism, which will be the main tool used to obtain the discrete maximum principle. Details can be found, for example, in Girsanov [6].

We start with some definitions of cones and dual cones that will be important in the description of the Dubovitskii–Milyutin formalism (or simply DM-formalism). Let Y be a topological vector space and Y^* its topological dual.

Definition 2.1 Let K be a subset of Y. We say that K is a cone with a vertex at the origin if for each $d \in K$ it follows that $\lambda d \in K$ for all $\lambda \geq 0$.

Definition 2.2 Let S be a subset of Y. The polar cone $S^\circ \subset Y^*$ is defined by

$$S^\circ = \{l \in Y^* : l(x) \leq 0 \text{ for all } x \in S\}.$$

It is clear from its definition that S° is a convex cone with vertex at the origin.

Definition 2.3 Let K be a cone in Y with a vertex at the origin. The dual cone $K^* \subset Y^*$ is defined by

$$K^* = \{l \in Y^* : l(x) \geq 0 \text{ for all } x \in K\}.$$

Note that K^* is a convex cone with vertex at the origin. Next, we have some technical results involving cones and their duals.

Proposition 2.1 (Girsanov [6]) *Let $K_1, \ldots, K_n$ be open convex cones such that $\bigcap_{i=1}^{n} K_i \neq \emptyset$. Then*

$$\left(\bigcap_{i=1}^{n} K_i\right)^* = \sum_{i=1}^{n} K_i^*.$$

Proposition 2.2 (Girsanov [6]) *If K is a subspace of Y, then*

$$K^* = \{l \in Y^* : l(x) = 0 \ \forall x \in K\}.$$

Proposition 2.3 (Girsanov [6]) *Let $l \in Y^*$. If*

$$K_1 = \{x : l(x) = 0\},\ K_2 = \{x : l(x) \geq 0\},\ K_3 = \{x : l(x) > 0\},$$

then,

$$K_1^* = \{\lambda l,\ -\infty < \lambda < \infty\},\ K_2^* = \{\lambda l,\ 0 \leq \lambda < \infty\},\ K_3^* = \begin{cases} Y^*, & l = 0, \\ K_2^*, & l \neq 0. \end{cases}$$

Next, we state an alternative theorem that will be a useful tool later in the paper.

Proposition 2.4 (Mangasarian [11], Motzkin Alternative Theorem) *For any matrices $A_i \in \mathbb{R}^{m_i \times n}$, $i = 0, 1, 2$, $A_0 \neq 0$, one, and only one, of the following two systems has a solution:*

$$A_0 x > 0, \quad A_1 x \geq 0, \quad A_2 x = 0, \quad x \in \mathbb{R}^n,$$

or

$$A_0^\top y^0 + A_1^\top y^1 + A_2^\top y^2 = 0, \quad (y^0, y^1, y^2) \in (\mathbb{R}_+^{m_0} \backslash \{0\} \times \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}).$$

We now present the theory developed by Dubovitskii and Milyutin, which is the main tool in obtaining the discrete maximum principle. The DM-formalism provides necessary optimality conditions for the following optimization problem:

$$\begin{aligned} &\text{minimize } \Psi(x) \\ &\text{subject to } x \in \bigcap_{i=1}^{l+1} Q_i, \end{aligned} \tag{P}$$

where $\Psi : X \to \mathbb{R}$, X is a locally convex topological linear space, $Q_i \subset X$, $i = 1, \ldots, l$, has a non-empty interior and $Q_{l+1} \subset X$ has an empty interior.

Our interest in this paper is to work with problems of type (P) where X is a Banach space, Q_i is described through inequality constraints for each $i = 1, \ldots, l$, and Q_{l+1} is given by a system of equality constraints. We assume, from now on in this section, that X is a Banach space.

In the following, we have some definitions and results associated to the Dubovitskii–Milyutin formalism. Then, we close the section by stating the necessary optimality conditions for (P), known as the Euler–Lagrange equation. All definitions and results can be found in Girsanov [6].

Definition 2.4 A direction of decrease of the functional $\Psi : X \to \mathbb{R}$ at the point $x^* \in X$ is defined as a direction $d \in X$ for which there exist a scalar $\epsilon^* > 0$, a neighborhood V of the vector d, and a number $\alpha = \alpha(\Psi, x^*, d)$ with $\alpha < 0$, such that, for every $\epsilon \in (0, \epsilon^*)$ and any $\bar{d} \in V$, we have

$$\Psi(x^* + \epsilon\bar{d}) \leq \Psi(x^*) + \epsilon\alpha.$$

The set of all directions of decrease of the functional Ψ at the point x^* is denoted by $D(\Psi, x^*)$. The functional Ψ is said to be regularly decreasing if $D(\Psi, x^*)$ is convex.

It is easy to see that $D(\Psi, x^*)$ is an open cone with a vertex at the origin. Moreover,

Proposition 2.5 *If Ψ is Fréchet differentiable at x^* and $\Psi'(x^*) \neq 0$, then $D(\Psi, x^*)$ is an open convex cone given as $D(\Psi, x^*) = \{d : \Psi'(x^*) \cdot d < 0\}$.*

Definition 2.5 Let $Q \subset X$, $\mathrm{int}(Q) \neq \emptyset$. A direction $d \in X$ is defined as a feasible direction for Q at the point x^* if there exist a neighborhood U of the vector d and a scalar $\epsilon^* > 0$ such that, for every $\epsilon \in (0, \epsilon^*)$ and all $\bar{d} \in U$, we have

$$x^* + \epsilon\bar{d} \in Q.$$

The set of all feasible directions for Q at x^* is denoted by $V(Q, x^*)$. The set Q is said to be regular at x^* when $V(Q, x^*)$ is convex.

Clearly, $V(Q, x^*)$ is an open cone with a vertex at the origin.

Proposition 2.6 *We have that*

i. *If $x^* \in \mathrm{int}(Q)$, then $V(Q, x^*) = X$;*
ii. *If $Q = \{x \in X : f(x) \leq f(x^*)\}$, where $f : X \to \mathbb{R}$ is a continuous functional, then $D(f, x^*) \subset V(Q, x^*)$;*
iii. *If $Q = \{x \in X : f(x) \leq 0\}$, where $f : X \to \mathbb{R}$ is a Fréchet differentiable functional with $f'(x^*) \neq 0$, $x^* \in X$, then*

$$V(Q, x^*) = \{d \in X : f'(x^*) \cdot d < 0\} \text{ when } f(x^*) = 0$$

and

$$V(Q, x^*) = X \text{ when } f(x^*) < 0;$$

Moreover, it follows that $V(Q, x^*)$ *is open and convex.*

iv. *Let* $Q_1, Q_2 \subset X$ *and* $x^* \in Q_1 \cap Q_2$*. It is easy to see from the definition that*

$$V(Q_1 \cap Q_2, x^*) = V(Q_1, x^*) \cap V(Q_2, x^*).$$

Definition 2.6 Let $Q \subset X$. A vector $d \in X$ is defined as a tangent direction to Q at the point x^* if there exist $\epsilon^* > 0$ and a map $r : (0, \epsilon^*) \to X$ such that for every $\epsilon \in (0, \epsilon^*)$, we have

$$x^* + \epsilon d + r(\epsilon) \in Q, \quad \|r(\epsilon)\| = o(\epsilon).$$

The set of all tangent directions to Q at x^* is denoted by $T(Q, x^*)$. The set Q is said to be regular at x^* if $T(Q, x^*)$ is convex.

From the definition of tangent directions, we can see that $T(Q, x^*)$ is a cone with a vertex at 0. However, this cone, in general, is neither closed nor open. We also note that if d is a feasible direction, then d is a tangent direction. The converse is false.

Usually, to characterize the cone of tangent directions, Lyusternik Theorem is used, which demands for a complete rank type regularity condition. In this paper, the constant rank type regularity condition is used to characterize the tangent cone. Next, we prove a proposition that will be of great use in the construction of the tangent cone.

Proposition 2.7 *Let* $f : \mathbb{R}^n \to \mathbb{R}^q$ *be a given application. If* $\nabla f(x)$ *has the same rank in a neighborhood of* $x^* \in Q := \{x \in \mathbb{R}^n : f(x) = 0\}$*, then*

i. $T(Q, x^*) = \ker(\nabla f(x^*))$*;*
ii. *For any* $d \in \ker(\nabla f(x^*))$ *there exists* $\epsilon^* > 0$ *and a continuously differentiable map* $r(\epsilon)$ *such that*

$$\lim_{\epsilon \to 0} \frac{r(\epsilon)}{\epsilon} = 0, \quad r(0) = 0, \quad f(x^* + \epsilon d + r(\epsilon)) = 0, \quad \epsilon \in [-\epsilon^*, \epsilon^*].$$

Proof See Minchenko and Stakhovsk [16], Theorem 1. □

As already mentioned, the Dubovitskii–Milyutin theorem, stated next, is used to derive the maximum principle for discrete-time optimal control problems.

Theorem 2.1 (Dubovitskii–Milyutin) *Let* $x^* \in Q := \cap_{i=1}^{l+1} Q_i$ *be a local minimum for (P). Suppose that the objective function* Ψ *is regularly decreasing at* x^**, the inequality constraints* Q_i*,* $i = 1, \ldots, l$*, are regular at* x^**, and the equality constraint* Q_{l+1} *is regular at* x^**. Then, there exist continuous linear functionals* $\chi_0 \in D(\Psi, x^*)^*$*,* $\chi_i \in V(Q_i, x^*)^*$*,* $i = 1, \ldots, l$*, and* $\chi_{l+1} \in T(Q_{l+1}, x^*)^*$*, not all null, which satisfy the Euler Lagrange equation:*

$$\sum_{i=0}^{l+1} \chi_i = 0.$$

Remark 2.1 Under the conditions of the previous theorem, we have that a sufficient condition to ensure $\chi_0 \neq 0$ is that there exists at least one direction which is simultaneously feasible and tangent, that is, $\bigcap_{i=1}^{l} V(Q_i, x^*) \bigcap T(Q_{l+1}, x^*) \neq \emptyset$.

Let us consider a particular case of (P) in which X has finite dimension and the sets Q_i are given as

$$Q_i = \{x \in X : f_i(x) \leq 0\},\ i = 1, \ldots, l,$$

and

$$Q_{l+1} = \{x \in X : F(x) = 0\},$$

where $f_i : X \to \mathbb{R}$, $i = 1, \ldots, l$ and $F : X \to \mathbb{R}^m$ are continuously differentiable maps.

Definition 2.7 The linearized cone of set Q at a point $x^* \in Q$, denoted by $L(Q, x^*)$, is defined as

$$L(Q, x^*) = \{d \in X : \nabla f_i(x^*) \cdot d \leq 0,\ i \in I(x^*)\} \cap \{d \in X : \nabla F(x^*) d = 0\},$$

where $I(x^*) = \{i \in \{1, \ldots, l\} : f_i(x^*) = 0\}$.

Remark 2.2 It is well known that, in finite dimension, (after applying the Farkas Lemma, for example—see [11]) the polar cone of $L(Q, x^*)$ is given by

$$L(Q, x^*)^\circ = \left\{ \sum_{I(x^*)} \mu_i \nabla f_i(x^*) + \nabla F(x^*)^\top \lambda :\ \mu_i \geq 0,\ i \in I(x^*),\ \lambda \in \mathbb{R}^m \right\}.$$

3 The discrete optimal control problem with mixed constraints

We are concerned with discrete optimal control problem (DOCP) with mixed constraints posed as follows:

$$\begin{aligned}
\text{minimize } & \sum_{k=0}^{N} \psi(x(k), u(k), k) + \zeta(x(N+1)) \\
\text{subject to } & x(k+1) = f(x(k), u(k), k),\ k = 0, \ldots, N, \\
& b(x(k), u(k), k) = 0,\ k = 0, \ldots, N, \\
& g(x(k), u(k), k) \leq 0,\ k = 0, \ldots, N, \\
& \varphi(x(0), x(N+1)) = 0, \\
& \phi(x(0), x(N+1)) \leq 0,
\end{aligned} \tag{DOCP}$$

where, for each $k = 0, \ldots, N$,

$$\begin{aligned}
\psi(\cdot, \cdot, k) &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \\
f(\cdot, \cdot, k) &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \\
b(\cdot, \cdot, k) &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{r_k}, \\
g(\cdot, \cdot, k) &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{q_k},
\end{aligned}$$

and $\zeta : \mathbb{R}^n \to \mathbb{R}$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r_\varphi}$ and $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{r_\phi}$ are continuously differentiable functions. It should be noted that the values of r_k and q_k may vary depending on the period k.

A feasible process is comprised of a pair $(x, u) \in \mathbb{R}^{n(N+2)} \times \mathbb{R}^{m(N+1)}$ with

$$\begin{aligned}
x &= (x(0), x(1), \ldots, x(N), x(N+1)),\ x(k) \in \mathbb{R}^n, \\
u &= (u(0), u(1), \ldots, u(N)),\ u(k) \in \mathbb{R}^m,
\end{aligned}$$

that satisfy all the constraints of the problem. The first component in the pair (x, u) is termed state variable while the second one is called control variable.

We will denote the space of state variables $\mathbb{R}^{n(N+2)}$ by X and the space of control variables $\mathbb{R}^{m(N+1)}$ by U.

The set of all feasible processes of (DOCP) will be denoted by $Q = Q_F \cap Q_G$, where Q_F and Q_G are the sets with empty interior and nonempty interior, respectively, defined as

$$\begin{aligned}
Q_F = \Big\{(x, u) \in X \times U \ : \ & x(k+1) = f(x(k), u(k), k),\ k = 0, \ldots, N; \\
& b(x(k), u(k), k) = 0,\ k = 0, \ldots, N; \\
& \varphi(x(0), x(N+1)) = 0\Big\}
\end{aligned}$$

and

$$\begin{aligned}
Q_G = \Big\{(x, u) \in X \times U \ : \ & g(x(k), u(k), k) \leq 0,\ k = 0, \ldots, N; \\
& \phi(x(0), x(N+1)) \leq 0\Big\}.
\end{aligned}$$

Let $(x^*, u^*) \in Q$ be a feasible process. We denote the index sets of active inequality constraints in the k-th period as

$$I_g(x^*(k), u^*(k), k) = \Big\{j \in \{1, \ldots, q_k\} : g_j(x^*(k), u^*(k), k) = 0\Big\},\ k = 0, \ldots, N,$$

and

$$I_\phi(x^*(0), x^*(N+1)) = \Big\{j \in \{1, \ldots, r_\phi\} : \phi_j(x^*(0), x^*(N+1)) = 0\Big\}.$$

When it is clear from the context, we will denote $I_g^*(k)$ instead of $I_g(x^*(k), u^*(k), k)$, $k = 0, \ldots, N$, and I_ϕ^* instead of $I_\phi(x^*(0), x^*(N+1))$.

We can rewrite (DOCP) as the following mathematical programming problem:

$$\begin{aligned}\text{minimize } & \Psi(x,u)\\ \text{subject to } & F(x,u)=0,\\ & G(x,u)\leq 0,\end{aligned}$$

where

- $\Psi : X \times U \to \mathbb{R}$ is defined by

$$\Psi(x,u)=\sum_{k=0}^{N}\psi(x(k),u(k),k)+\zeta(x(N+1));$$

- $F : X \times U \to \mathbb{R}^{n(N+1)} \times \mathbb{R}^{(r_0+\cdots+r_N)} \times \mathbb{R}^{r_\varphi}$ is given by

$$F(x,u)=(\bar{F}(x,u),\hat{F}(x,u),\tilde{F}(x,u)),$$

  with

$$\begin{aligned}&\bar{F}_k(x,u)=x(k+1)-f(x(k),u(k),k),\ k=0,\ldots,N,\\ &\hat{F}_k(x,u)=b(x(k),u(k),k),\ k=0,\ldots,N,\\ &\tilde{F}(x,u)=\varphi(x(0),x(N+1));\end{aligned}$$

- $G : X \times U \to \mathbb{R}^{(q_0+\cdots+q_N)} \times \mathbb{R}^{r_\phi}$ is determined by

$$G(x,u)=(\hat{G}(x,u),\tilde{G}(x,u)),$$

  with

$$\begin{aligned}&\hat{G}_k(x,u)=g(x(k),u(k),k),\ k=0,\ldots,N,\\ &\tilde{G}(x,u)=\phi(x(0),x(N+1)).\end{aligned}$$

We will now set the notation related to the gradients and Jacobians of the functions Ψ, F, and G. Let (x^*,u^*) be a feasible process. We have

$$\nabla\Psi(x^*,u^*)=\begin{bmatrix}\nabla_x\Psi(x^*,u^*)\\ \nabla_u\Psi(x^*,u^*)\end{bmatrix},$$

where

$$\nabla_x\Psi(x^*,u^*)=\begin{bmatrix}\nabla_{x(0)}\psi(x^*(0),u^*(0),0)\\ \vdots\\ \nabla_{x(N)}\psi(x^*(N),u^*(N),N)\\ \nabla\zeta(x^*(N+1))\end{bmatrix} \tag{1}$$

and

$$\nabla_u \Psi(x^*, u^*) = \begin{bmatrix} \nabla_{u(0)} \psi(x^*(0), u^*(0), 0) \\ \vdots \\ \nabla_{u(N)} \psi(x^*(N), u^*(N), N) \end{bmatrix}. \tag{2}$$

Given $(s, t) \in X \times U$ in which

$$s = (s(0), \ldots, s(N+1)) \quad \text{and} \quad t = (t(0), \ldots, t(N)),$$

we, then, have

$$\begin{aligned} \nabla \Psi(x^*, u^*) \cdot (s, t) = & \sum_{k=0}^{N} \nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \\ & + \sum_{k=0}^{N} \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k) \\ & + \nabla \zeta(x^*(N+1))^\top s(N+1). \end{aligned} \tag{3}$$

Since $F(x, u) = (\bar{F}(x, u), \hat{F}(x, u), \tilde{F}(x, u))$, we have

$$\nabla F(x^*, u^*) = \begin{bmatrix} \nabla_x \bar{F}(x^*, u^*) & \nabla_u \bar{F}(x^*, u^*) \\ \nabla_x \hat{F}(x^*, u^*) & \nabla_u \hat{F}(x^*, u^*) \\ \nabla_x \tilde{F}(x^*, u^*) & \nabla_u \tilde{F}(x^*, u^*) \end{bmatrix}, \tag{4}$$

where

$$\nabla_x \bar{F}(x^*, u^*) = \begin{bmatrix} \bar{A}_0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & \bar{A}_1 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & \bar{A}_2 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bar{A}_N & I \end{bmatrix}, \tag{5}$$

$$\nabla_u \bar{F}(x^*, u^*) = \begin{bmatrix} \bar{B}_0 & 0 & 0 & \cdots & 0 \\ 0 & \bar{B}_1 & 0 & \cdots & 0 \\ 0 & 0 & \bar{B}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{B}_N \end{bmatrix}, \tag{6}$$

$$\nabla_x \hat{F}(x^*, u^*) = \begin{bmatrix} \hat{A}_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \hat{A}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \hat{A}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \hat{A}_N & 0 \end{bmatrix}, \tag{7}$$

$$\nabla_u \hat{F}(x^*, u^*) = \begin{bmatrix} \hat{B}_0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{B}_1 & 0 & \cdots & 0 \\ 0 & 0 & \hat{B}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{B}_N \end{bmatrix}, \tag{8}$$

$$\nabla_x \tilde{F}(x^*, u^*) = \left[\tilde{A}_0 \; 0 \; 0 \; 0 \cdots 0 \; \tilde{A}_{N+1}\right], \tag{9}$$

and

$$\nabla_u \tilde{F}(x^*, u^*) = \left[0 \; 0 \; 0 \cdots 0\right], \tag{10}$$

with

$$\bar{A}_k = -\nabla_{x(k)} f(x^*(k), u^*(k), k) \in \mathbb{R}^{n\times n}, \; k = 0, \cdots, N, \tag{11}$$

$$\bar{B}_k = -\nabla_{u(k)} f(x^*(k), u^*(k), k) \in \mathbb{R}^{n\times m}, \; k = 0, \cdots, N, \tag{12}$$

$$\hat{A}_k = \nabla_{x(k)} b(x^*(k), u^*(k), k) \in \mathbb{R}^{r_k\times n}, \; k = 0, \ldots, N, \tag{13}$$

$$\hat{B}_k = \nabla_{u(k)} b(x^*(k), u^*(k), k) \in \mathbb{R}^{r_k\times m}, \; k = 0, \ldots, N, \tag{14}$$

$$\tilde{A}_0 = \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) \in \mathbb{R}^{r_\varphi\times n}, \tag{15}$$

$$\tilde{A}_{N+1} = \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) \in \mathbb{R}^{r_\varphi\times n}; \tag{16}$$

$$I \in \mathbb{R}^{n\times n} \text{ is the identity matrix.} \tag{17}$$

Remark 3.1 It is easy to see that the Jacobian matrix

$$\nabla \bar{F}(x^*, u^*) = \left[\nabla_x \bar{F}(x^*, u^*) \; \nabla_u \bar{F}(x^*, u^*)\right]$$

has full rank $n(N+1)$.

Given $(s, t) \in X \times U$, we can write

$$\nabla F(x^*, u^*) \cdot (s, t) = \begin{bmatrix} \nabla_x \bar{F}(x^*, u^*) \cdot s + \nabla_u \bar{F}(x^*, u^*) \cdot t \\ \nabla_x \hat{F}(x^*, u^*) \cdot s + \nabla_u \hat{F}(x^*, u^*) \cdot t \\ \nabla_x \tilde{F}(x^*, u^*) \cdot s + \nabla_u \tilde{F}(x^*, u^*) \cdot t \end{bmatrix}$$

with, for each $k = 0, \ldots, N$,

$$\begin{aligned} &\nabla_x \bar{F}_k(x^*, u^*) \cdot s + \nabla_u \bar{F}_k(x^*, u^*) \cdot t \\ &\quad = s(k+1) - \nabla_{x(k)} f(x^*(k), u^*(k), k) s(k) \\ &\qquad - \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k), \end{aligned} \tag{18}$$

$$\begin{aligned} &\nabla_x \hat{F}_k(x^*, u^*) \cdot s + \nabla_u \hat{F}_k(x^*, u^*) \cdot t \\ &\quad = \nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) \\ &\qquad + \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k), \end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\nabla_x \tilde{F}(x^*, u^*) \cdot s + \nabla_u \tilde{F}(x^*, u^*) \cdot t \\
&= \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) \\
&\quad + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1). \qquad (20)
\end{aligned}$$

Given that $G(x, u) = (\hat{G}(x, u), \tilde{G}(x, u))$, we have that

$$\nabla G(x^*, u^*) = \begin{bmatrix} \nabla_x \hat{G}(x^*, u^*) & \nabla_u \hat{G}(x^*, u^*) \\ \nabla_x \tilde{G}(x^*, u^*) & \nabla_u \tilde{G}(x^*, u^*) \end{bmatrix}, \qquad (21)$$

where

$$\nabla_x \hat{G}(x^*, u^*) = \begin{bmatrix} \hat{C}_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \hat{C}_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \hat{C}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{C}_N & 0 \end{bmatrix}, \qquad (22)$$

$$\nabla_u \hat{G}(x^*, u^*) = \begin{bmatrix} \hat{D}_0 & 0 & 0 & \cdots & 0 \\ 0 & \hat{D}_1 & 0 & \cdots & 0 \\ 0 & 0 & \hat{D}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \hat{D}_N \end{bmatrix}, \qquad (23)$$

$$\nabla_x \tilde{G}(x^*, u^*) = \begin{bmatrix} \tilde{C}_0 \; 0 \; 0 \cdots 0 \; \tilde{C}_{N+1} \end{bmatrix}, \qquad (24)$$

and

$$\nabla_u \tilde{G}(x^*, u^*) = \begin{bmatrix} 0 \; 0 \; 0 \cdots 0 \end{bmatrix} \qquad (25)$$

with

$$\hat{C}_k = \nabla_{x(k)} g(x^*(k), u^*(k), k) \in \mathbb{R}^{q_k \times n}, \; k = 0, \ldots, N, \qquad (26)$$

$$\hat{D}_k = \nabla_{u(k)} g(x^*(k), u^*(k), k) \in \mathbb{R}^{q_k \times m}, \; k = 0, \ldots, N, \qquad (27)$$

$$\tilde{C}_0 = \nabla_{x(0)} \phi(x^*(0), x^*(N+1)) \in \mathbb{R}^{r_\phi \times n}, \qquad (28)$$

$$\tilde{C}_{N+1} = \nabla_{x(N+1)} \phi(x^*(0), x^*(N+1)) \in \mathbb{R}^{r_\phi \times n}. \qquad (29)$$

Let $(s, t) \in X \times U$. Then,

$$\nabla G(x^*, u^*) \cdot (s, t) = \begin{bmatrix} \nabla_x \hat{G}(x^*, u^*) \cdot s + \nabla_u \hat{G}(x^*, u^*) \cdot t \\ \nabla_x \tilde{G}(x^*, u^*) \cdot s + \nabla_u \tilde{G}(x^*, u^*) \cdot t \end{bmatrix}$$

with, for each $k = 0, \ldots, N$,

$$\nabla_x \hat{G}_k(x^*, u^*) \cdot s + \nabla_u \hat{G}_k(x^*, u^*) \cdot t = \nabla_{x(k)} g(x^*(k), u^*(k), k)s(k) + \nabla_{u(k)} g(x^*(k), u^*(k), k)t(k),$$

and

$$\nabla_x \tilde{G}(x^*, u^*) \cdot s + \nabla_u \tilde{G}(x^*, u^*) \cdot t = \nabla_{x(0)} \phi(x^*(0), x^*(N+1))s(0) + \nabla_{x(N+1)} \phi(x^*(0), x^*(N+1))s(N+1).$$

Remark 3.2 According to Definition 2.7, the linearized cone $L(Q, (x^*, u^*))$, associated to problem (DOCP) at a feasible point $(x^*, u^*) \in Q$ is, then, given as all pairs $(s, t) \in X \times U$ such that,

$$s(k+1) = \nabla_{x(k)} f(x^*(k), u^*(k), k)s(k) + \nabla_{u(k)} f(x^*(k), u^*(k), k)t(k), \quad (30)$$

$$0 = \nabla_{x(k)} b(x^*(k), u^*(k), k)s(k) + \nabla_{u(k)} b(x^*(k), u^*(k), k)t(k), \quad (31)$$

for $k = 0, \ldots, N$,

$$0 = \nabla_{x(0)} \varphi(x^*(0), x^*(N+1))s(0) + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1))s(N+1), \quad (32)$$

and

$$0 \geq \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^\top t(k), \quad (33)$$

for $j \in I_g^*(k)$, $k = 0, \ldots, N$,

$$0 \geq \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0) + \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1), \quad (34)$$

for $j \in I_\phi^*$.

Remark 3.3 According to Remark 2.2, if $(l_X, l_U) \in L(Q, x^*, u^*))^\circ \subset X \times U$, there exist $\gamma \in \mathbb{R}^{r_\varphi}$, $\eta_j \geq 0$, $j \in I_\phi^*$, and for each $k = 0, \ldots, N$, $p(k+1) \in \mathbb{R}^n$, $\lambda(k) \in \mathbb{R}^{r_k}$ and $\mu_j(k) \geq 0$, $j \in I_g^*(k)$, such that

$$l_X(0) = -\nabla_{x(0)} f(x^*(0), u^*(0), 0)^\top p(1) + \nabla_{x(0)} b(x^*(0), u^*(0), 0)^\top \lambda(0) + \nabla_{x(0)} \varphi(x^*(0), x^*(N+1))^\top \gamma + \sum_{j \in I_g^*(0)} \mu_j(0) \nabla_{x(0)} g_j(x^*(0), u^*(0), 0) + \sum_{I_\phi^*} \eta_j \nabla_{x(0)} \phi(x^*(0), x^*(N+1)),$$

$$l_X(k) = p(k) - \nabla_{x(k)} f(x^*(k), u^*(k), k)^\top p(k+1) + \nabla_{x(k)} b(x^*(k), u^*(k), k)^\top \lambda(k)$$

$$
\begin{aligned}
&+\sum_{j\in I_g^*(k)} \mu_j(k)\nabla_{x(k)}g_j(x^*(k),u^*(k),k),\ k=1,\dots,N,\\
l_X(N+1) &= p(N+1)+\nabla_{x(N+1)}\varphi(x^*(0),x^*(N+1))^\top\gamma\\
&\quad+\sum_{j\in I_\phi^*}\eta_j\nabla_{x(N+1)}\phi_j(x^*(0),x^*(N+1)),\\
l_U(k) &= -\nabla_{u(k)}f(x^*(k),u^*(k),k)^\top p(k+1)+\nabla_{u(k)}b(x^*(k),u^*(k),k)^\top\lambda(k)\\
&\quad+\sum_{j\in I_g^*(k)}\mu_j(k)\nabla_{u(k)}g_j(x^*(k),u^*(k)),\ k=0,\dots,N.
\end{aligned}
$$

Next, we will define two concepts of regularity that will be used both in constructing the feasible and tangent cones and in obtaining non-degenerate necessary optimality conditions. Namely, the extended Mangasarian–Fromovitz constraint qualification (MFCQ) and the constant rank of the subspace component (CRSC) condition. The extended MFCQ was defined in Minchenko and Stakhovski [15] for nonlinear programming problems and will be used as an auxiliary tool in the process of establishing the nondegenerate discrete maximum principle. That is because it is naturally more adequate in the construction of cones and in the application of the Dubovitskii–Milyutin formalism. The nondegenerate version of the maximum principle will be derived using CRSC.

Definition 3.1 Let (x^*, u^*) be a feasible process of (DOCP). We say that $(x^*, u^*) \in Q$ is a regular process of extended Mangasarian–Fromovitz type if:

1. There exists a neighborhood V of (x^*, u^*) such that $\nabla F(x, u)$ has the same rank for each $(x, u) \in V$;
2. There exists $(s, t) \in \ker\big(\nabla F(x^*, u^*)\big)$ in which

$$s = \big(s(0), \dots, s(N+1)\big) \quad \text{and} \quad t = \big(t(0), \dots, t(N)\big)$$

such that, for each $k = 0, \dots, N$, and $j \in I_g^*(k)$,

$$0 > \nabla_{x(k)}g_j(x^*(k),u^*(k),k)^\top s(k) + \nabla_{u(k)}g_j(x^*(k),u^*(k),k)^\top t(k)$$

and, for $j \in I_\phi^*$,

$$0 > \nabla_{x(0)}\phi_j(x^*(0),x^*(N+1))^\top s(0) + \nabla_{x(N+1)}\phi_j(x^*(0),x^*(N+1))^\top s(N+1).$$

Remark 3.4 Note that, item 2 of Definition 3.1 ensures that

$$
\begin{aligned}
&\nabla g_j(x^*(k),u^*(k),k)\neq 0,\ k=0,\dots,N,\ j\in I_g^*(k),\\
&\nabla\phi_j(x^*(0),x^*(N+1))\neq 0,\ j\in I_\phi^*.
\end{aligned}
$$

Before we give the next definition, let us introduce the following notation. Given $(x^*, u^*) \in Q$, we denote

$$\mathcal{J}_g^*(k) = \{j \in I_g^*(k) : -\nabla \hat{G}_{kj}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ\},$$

for $k = 0, \ldots, N$,

$$\mathcal{J}_\phi^* = \{j \in I_\phi^* : -\nabla \tilde{G}_j(x^*, u^*) \in L(Q, (x^*, u^*))^\circ\},$$

and

$$\mathcal{J}_-^* = \left(\bigcup_{k=0}^{N} \mathcal{J}_g^*(k) \right) \bigcup \mathcal{J}_\phi^*.$$

We will denote by $\nabla G^{\mathcal{J}_-^*}(x^*, u^*)$ the matrix obtained from $\nabla G(x^*, u^*)$ after removing the rows whose indices do not belong to $\mathcal{J}_-^*$.

Definition 3.2 Let (x^*, u^*) be a feasible process of (DOCP). We say that the constraints of (DOCP) satisfy the constant rank of the subspace component (CRSC) regularity condition at $(x^*, u^*) \in Q$ if there exists a neighborhood V of (x^*, u^*) such that the matrix

$$\begin{bmatrix} \nabla F(x, u) \\ \nabla G^{\mathcal{J}_-^*}(x, u) \end{bmatrix}$$

has the same rank for each $(x, u) \in V$.

The constant rank of the subspace component regularity condition was introduced in Andreani et al. [1] for mathematical programming problems. In Kruger, Minchenko and Outrata [9], it is shown that CRSC coincides with the generalized Mangasarian–Fromovitz condition. The authors also mention that the constant rank of the subspace component regularity condition (Definition 3.2) is weaker than the extended Mangasarian–Fromovitz condition (Definition 3.1).

Remark 3.5 Let

$$J_g^*(k) := \left\{ j \in I_g^*(k) : \nabla \hat{G}_{kj}(x^*, u^*) \cdot (s, t) = 0 \ \forall (s, t) \in L(Q, (x^*, u^*)) \right\},$$

for $k = 0, \ldots, N$, and

$$J_\phi^* := \left\{ j \in I_\phi^* : \nabla \tilde{G}_j(x^*, u^*) \cdot (s, t) = 0 \ \forall (s, t) \in L(Q, (x^*, u^*)) \right\}.$$

It is easy to see that

$$J_g^*(k) = \mathcal{J}_g^*(k), \ k = 0, \ldots, N, \quad \text{and} \quad J_\phi^* = \mathcal{J}_\phi^*.$$

Moreover,

$$J_g^*(k) = \Big\{ j \in I_g^*(k) : \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^\top t(k) = 0 \quad \forall (s,t) \in L(Q, (x^*, u^*)) \Big\}$$

for $k = 0, \ldots, N$,

$$J_\phi^* = \Big\{ j \in I_\phi^* : \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0) + \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1) = 0 \quad \forall (s,t) \in L(Q, (x^*, u^*)) \Big\}.$$

3.1 Construction of cones

To obtain first-order necessary conditions for (DOCP) using the Dubovitskii–Milyutin formalism we need to determine, at the optimal process (x^*, u^*), the cones of directions of decrease $D(\Psi, (x^*, u^*))$, feasible directions $V(Q_G, (x^*, u^*))$, and tangent directions $T(Q_F, (x^*, u^*))$ together with their respective dual cones $D(\Psi, (x^*, u^*))^*$, $V(Q_G, (x^*, u^*))^*$, and $T(Q_F, (x^*, u^*))^*$. Such cones are determined as follows.

3.1.1 The cone of directions of decrease

As we have already seen, the set $D(\Psi, (x^*, u^*))$ is an open cone with vertex at the origin, generated by the directions of decrease of Ψ at (x^*, u^*). As Ψ is differentiable at (x^*, u^*), if $\nabla \Psi(x^*, u^*) \neq 0$, by Proposition 2.5, the cone of directions of decrease is given by

$$\begin{aligned} D(\Psi, (x^*, u^*)) &= \Big\{ (s,t) \in X \times U : \nabla \Psi(x^*, u^*) \cdot (s,t) < 0 \Big\} \\ &= \Big\{ (s,t) \in X \times U : \sum_{k=0}^{N} \nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \\ &\quad + \sum_{k=0}^{N} \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k) \\ &\quad + \nabla \zeta(x^*(N+1))^\top s(N+1) < 0 \Big\}, \end{aligned} \tag{35}$$

where we used (3) in the second equality.

By Proposition 2.3, its dual cone is given by

$$D(\Psi, (x^*, u^*))^* = \{ -\xi \nabla \Psi(x^*, u^*), \ \xi \geq 0 \}.$$

Thence, given $\chi_\Psi \in D(\Psi, (x^*, u^*))^*$ and $(s, t) \in X \times U$, we have

$$\begin{aligned}\chi_\Psi \cdot (s,t) &= -\xi \nabla \Psi(x^*, u^*) \cdot (s,t) \\ &= -\xi \sum_{k=0}^{N} \nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \\ &\quad -\xi \sum_{k=0}^{N} \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k) \\ &\quad -\xi \nabla \zeta(x^*(N+1))^\top s(N+1) \end{aligned} \tag{36}$$

for some $\xi \geq 0$.

3.1.2 The cone of feasible directions

Now, we will determine the feasible cone $V(Q_G, (x^*, u^*))$. Let us denote, for each $k = 0, \ldots, N$ and $j \in I_g^*(k)$,

$$Q_g^j(k) = \Big\{(x, u) \in X \times U : g_j(x(k), u(k), k) \leq 0\Big\},$$

and for $j \in I_\phi^*$,

$$Q_\phi^j = \Big\{(x, u) \in X \times U : \phi_j(x(0), x(N+1)) \leq 0\Big\}.$$

By item iv. of Proposition 2.6 we obtain

$$V\Big(Q_G, (x^*, u^*)\Big) = \bigcap_{k=0}^{N} \bigcap_{j \in I_g^*(k)} V\Big(Q_g^j(k), (x^*, u^*)\Big) \bigcap_{j \in I_\phi^*} V\Big(Q_\phi^j, (x^*, u^*)\Big). \tag{37}$$

Note that when an inequality constraint is inactive at (x^*, u^*), by item i. of Proposition 2.6, we have that the cone of feasible directions is the entire space. So, such constraints may be disregarded.

Suppose that $\nabla g_j(x^*(k), u^*(k), k) \neq 0$ for all $j \in I_g^*(k)$, $k = 0, \ldots, N$, and $\nabla \phi_j(x^*(0), x^*(N+1)) \neq 0$ for all $j \in I_\phi^*$. Under these assumptions, we apply item iii. of Proposition 2.6 to get, for $k = 0, \ldots, N$ and $j \in I_g^*(k)$,

$$\begin{aligned} V\Big(Q_g^j(k), (x^*, u^*)\Big) = \Big\{(s,t) \in X \times U : \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s(k) \\ + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^\top t(k) < 0\Big\}, \end{aligned} \tag{38}$$

and for $j \in I_\phi^*$,

$$V\Big(Q_\phi^j, (x^*, u^*)\Big) = \Big\{(s,t) \in X \times U : \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0)$$

$$+\nabla_{x(N+1)}\phi_j(x^*(0),x^*(N+1))^\top s(N+1)<0\Big\}. \tag{39}$$

Now, applying Proposition 2.1, from (37) we see that

$$\begin{aligned} V\Big(Q,(x^*,u^*)\Big)^* &= \left(\bigcap_{k=0}^{N}\bigcap_{j\in I_g^*(k)} V\Big(Q_g^j(k),(x^*,u^*)\Big)\bigcap_{j\in I_\phi^*} V\Big(Q_\phi^j,(x^*,u^*)\Big)\right)^* \\ &= \sum_{k=0}^{N}\sum_{j\in I_g^*(k)} V\Big(Q_g^j(k),(x^*,u^*)\Big)^* + \sum_{j\in I_\phi^*} V\Big(Q_\phi^j,(x^*,u^*)\Big)^*. \end{aligned}$$

Then, applying Proposition 2.3, it follows, for each $k=0,\ldots,N$ and $j\in I_g^*(k)$, that

$$V\Big(Q_g^j(k),(x^*,u^*)\Big)^* = \Big\{-\mu_j(k)\nabla g_j(x^*(k),u^*(k),k),\ \mu_j(k)\geq 0\Big\},$$

and, for $j\in I_\phi^j$, that

$$V\Big(Q_\phi^j,(x^*,u^*)\Big)^* = \Big\{-\eta_j\nabla\phi_j(x^*(0),x^*(N+1)),\ \eta_j\geq 0\Big\}.$$

Thus, if $\chi_G\in V(Q_G,(x^*,u^*))^*$, given $(s,t)\in X\times U$, we have

$$\begin{aligned} \chi_G\cdot(s,t) = &-\sum_{k=0}^{N}\sum_{j\in I_g^*(k)}\mu_j(k)\nabla_{x(k)}g_j(x^*(k),u^*(k),k)^\top s(k) \\ &-\sum_{k=0}^{N}\sum_{j\in I_g^*(k)}\mu_j(k)\nabla_{u(k)}g_j(x^*(k),u^*(k),k)^\top t(k) \\ &-\sum_{j\in I_\phi^*}\eta_j\nabla_{x(0)}\phi_j(x^*(0),x^*(N+1))^\top s(0) \\ &-\sum_{j\in I_\phi^*}\eta_j\nabla_{x(N+1)}\phi_j(x^*(0),x^*(N+1))^\top s(N+1), \end{aligned}$$

for some $\mu_j(k)\geq 0$, $j\in I_g^*(k)$, $k=0,\ldots,N$, and $\eta_j\geq 0$, $j\in I_\phi^*$. If we define $\mu_j(k)=0$ for $j\notin I_g^*(k)$, $k=0,\ldots,N$, and $\eta_j=0$ for $j\notin I_\phi^*$, we can write

$$\begin{aligned} \chi_G\cdot(s,t) = &-\sum_{k=0}^{N}\sum_{j=1}^{q_k}\mu_j(k)\nabla_{x(k)}g_j(x^*(k),u^*(k),k)^\top s(k) \\ &-\sum_{k=0}^{N}\sum_{j=1}^{q_k}\mu_j(k)\nabla_{u(k)}g_j(x^*(k),u^*(k),k)^\top t(k) \end{aligned}$$

$$- \sum_{j=1}^{r_\phi} \eta_j \nabla_{x(0)} \phi_j (x^*(0), x^*(N+1))^\top s(0)$$
$$- \sum_{j=1}^{r_\phi} \eta_j \nabla_{x(N+1)} \phi_j (x^*(0), x^*(N+1))^\top s(N+1).$$

Hence, setting $\mu = (\mu(0), \mu(1), \ldots, \mu(N))$, we obtain, in the compact notation,

$$\begin{aligned} \chi_G \cdot (s,t) &= \Big(- \nabla_x \hat{G}(x^*, u^*)^\top \mu - \nabla_x \tilde{G}(x^*, u^*)^\top \eta \Big) \cdot s \\ &\quad + \Big(- \nabla_u \hat{G}(x^*, u^*)^\top \mu - \nabla_u \tilde{G}(x^*, u^*)^\top \eta \Big) \cdot t \\ &= \Big(- \nabla \hat{G}(x^*, u^*)^\top \mu - \nabla \tilde{G}(x^*, u^*)^\top \eta \Big) \cdot (s,t) \\ &= \Big(\nabla G(x^*, u^*)^\top \cdot (-\mu, -\eta) \Big) \cdot (s,t). \end{aligned} \tag{40}$$

3.1.3 The cone of tangent directions

The tangent cone to Q_F at a feasible process (x^*, u^*) will be obtained under a constant rank condition. We assume that there exists a neighborhood V of (x^*, u^*) such that $\nabla F(x, u)$ has the same rank for all $(x, u) \in V$. By Proposition 2.7, we have

$$\begin{aligned} T\Big(Q_F, (x^*, u^*)\Big) &= \ker \Big(\nabla F(x^*, u^*) \Big) \\ &= \{(s,t) \in X \times U : \nabla F(x^*, u^*) \cdot (s,t) = 0\} \\ &= \{(s,t) \in X \times U : \nabla_x \bar{F}(x^*, u^*) \cdot s + \nabla_u \bar{F}(x^*, u^*) \cdot t = 0, \\ &\qquad \nabla_x \hat{F}(x^*, u^*) \cdot s + \nabla_u \hat{F}(x^*, u^*) \cdot t = 0, \\ &\qquad \nabla_x \tilde{F}(x^*, u^*) \cdot s + \nabla_u \tilde{F}(x^*, u^*) \cdot t = 0\}. \end{aligned} \tag{41}$$

So, if $(s,t) \in T(Q_F, (x^*, u^*))$, from (18)–(20), we get

$$s(k+1) - \nabla_{x(k)} f(x^*(k), u^*(k), k) s(k) - \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k) = 0, \tag{42}$$
$$\nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k) = 0, \tag{43}$$

for $k = 0, \ldots, N$, and

$$\nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1) = 0. \tag{44}$$

By Proposition 2.2,

$$\Big(T(Q_F, (x^*, u^*)) \Big)^* = \Big(\ker(\nabla F(x^*, u^*)) \Big)^* = \Big(\ker(\nabla F(x^*, u^*)) \Big)^\perp.$$

From linear algebra we know that $(\ker(\nabla F(x^*, u^*)))^\perp = \text{range}(\nabla F(x^*, u^*)^\top)$. Therefore,

$$\Big(T(Q_F, (x^*, u^*))\Big)^* = \text{range}\Big(\nabla F(x^*, u^*)^\top\Big).$$

Thus, for every $\chi_F \in (T(Q_F, (x^*, u^*)))^*$, there exits $(p, \lambda, \gamma) \in \mathbb{R}^{n(N+1)} \times \mathbb{R}^{r_0+\ldots+r_N} \times \mathbb{R}^{r_\varphi}$ such that $\chi_F = \nabla F(x^*, u^*)^\top \cdot (-p, -\lambda, -\gamma)$. Hence, given $(s, t) \in X \times U$, we have

$$\begin{aligned}\chi_F \cdot (s, t) &= \Big(\nabla F(x^*, u^*)^\top \cdot (-p, -\lambda, -\gamma)\Big) \cdot (s, t) \\ &= (-p, -\lambda, -\gamma) \cdot \Big(\nabla F(x^*, u^*) \cdot (s, t)\Big).\end{aligned} \tag{45}$$

From

$$\nabla F(x^*, u^*) \cdot (s, t) = \Big(\nabla \bar{F}(x^*, u^*) \cdot (s, t), \nabla \hat{F}(x^*, u^*) \cdot (s, t), \nabla \tilde{F}(x^*, u^*) \cdot (s, t)\Big)$$

we obtain, using (18)–(20),

$$\begin{aligned}\chi_F \cdot (s, t) = -p \cdot \Big(\nabla \bar{F}(x^*, u^*) \cdot (s, t)\Big) - \lambda \cdot \Big(\nabla \hat{F}(x^*, u^*) \cdot (s, t)\Big) \\ -\gamma \cdot \Big(\nabla \tilde{F}(x^*, u^*) \cdot (s, t)\Big),\end{aligned}$$

so that

$$\begin{aligned}\chi_F \cdot (s, t) = &-\sum_{k=0}^{N} p(k+1)^\top s(k+1) \\ &-\sum_{k=0}^{N} p(k+1)^\top \nabla_{x(k)} f(x^*(k), u^*(k), k) s(k) \\ &-\sum_{k=0}^{N} p(k+1)^\top \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k) \\ &-\sum_{k=0}^{N} \lambda(k)^\top \nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) \\ &-\sum_{k=0}^{N} \lambda(k)^\top \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k) \\ &-\gamma^\top \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) \\ &-\gamma^\top \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1).\end{aligned}$$

4 The discrete maximum principle

In this section, we will provide first-order necessary optimality conditions for (DOCP), which consist of a discrete version of the maximum principle. These conditions will be obtained via the Dubovitskii–Milyutin formalism.

The Hamiltonian function associated with (DOCP)

$$H(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot,k):\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{r_k}\times\mathbb{R}^{q_k}\to\mathbb{R}$$

is defined, for each $k=0,\ldots,N$, as

$$H(x,u,p,\xi,\lambda,\mu,k)=p\cdot f(x,u,k)-\xi\psi(x,u,k)-\lambda\cdot b(x,u,k)-\mu\cdot g(x,u,k).$$

Theorem 4.1 (Discrete Weak Maximum Principle) *Let (x^*,u^*) be a local optimal process of (DOCP). Then, there exist $\xi\geq 0$, $\gamma\in\mathbb{R}^{r_\varphi}$, $\eta\in\mathbb{R}^{r_\phi}$, and for each $k=0,\ldots,N$, $(p(k+1),\lambda(k),\mu(k))\in\mathbb{R}^n\times\mathbb{R}^{r_k}\times\mathbb{R}^{q_k}$, not all null, with $\mu(k)\geq 0$, $k=0,\ldots,N$, and $\eta\geq 0$, such that the following conditions are satisfied:*

(i) *Adjoint equation:*

$$p(k)=\nabla_{x(k)}H(x^*(k),u^*(k),p(k+1),\xi,\lambda(k),\mu(k),k),\ \ k=1,\ldots,N;$$

(ii) *Transversality conditions:*

$$\begin{aligned}&\nabla_{x(0)}H(x^*(0),u^*(0),p(1),\xi,\lambda(0),\mu(0),0)\\&\quad=\gamma\cdot\nabla_{x(0)}\varphi(x^*(0),x^*(N+1))+\eta\cdot\nabla_{x(0)}\phi(x^*(0),x^*(N+1)),\\&p(N+1)=-\xi\nabla_{x(N+1)}\zeta(x^*(N+1))-\gamma\cdot\nabla_{x(N+1)}\varphi(x^*(0),x^*(N+1))\\&\qquad-\eta\cdot\nabla_{x(N+1)}\phi(x^*(0),x^*(N+1));\end{aligned}$$

(iii) *Stationarity condition:*

$$\nabla_{u(k)}H(x^*(k),u^*(k),p(k+1),\xi,\lambda(k),\mu(k),k)=0,\ \ k=0,\ldots,N;$$

(iv) *Complementarity conditions:*

$$\begin{aligned}&\mu_j(k)g_j(x^*(k),u^*(k),k)=0,\ \ j=1,\ldots,q_k,\ \ k=0,\ldots,N,\\&\eta_j\phi_j(x^*(0),x^*(N+1))=0,\ \ j=1,\ldots,r_\phi.\end{aligned}$$

Proof The proof will be divided into several cases.

Case 1. If $\nabla\Psi(x^*,u^*)=0$, by considering $\xi=1$, $p(k+1)=0$, $\lambda(k)=0$, $\mu(k)=0$, $k=0,\ldots,N$, $\gamma=0$, and $\eta=0$, conditions (i)–(iv) of the theorem are satisfied.

Case 2. If $\nabla\Psi(x^*,u^*)\neq 0$ and (x^*,u^*) is not a regular process, we have two subcases:

Case 2.1. Suppose that item 1 of Definition 3.1 is not satisfied, that is, the rank of $\nabla F(x^*,u^*)$ is not constant. Hence, $\nabla F(x^*,u^*)$ does not have full rank. Then,

$$\begin{aligned}\operatorname{rank}\big(\nabla F(x^*, u^*)\big) &< \min\big\{n(N+1) + (r_0 + \cdots + r_N) + r_\varphi, n(N+2) + m(N+1)\big\} \\ &\leq n(N+1) + (r_0 + \cdots + r_N) + r_\varphi \\ &= \dim\Big(\operatorname{range}\big(\nabla F(x^*, u^*)\big)\Big) + \dim\Big(\operatorname{range}\big(\nabla F(x^*, u^*)\big)\Big)^{\perp}.\end{aligned}$$

From the fact that $\operatorname{rank}(\nabla F(x^*, u^*)) = \dim(\operatorname{range}(\nabla F(x^*, u^*)))$, we see that

$$\dim\Big(\operatorname{range}\big(\nabla F(x^*, u^*)\big)\Big)^{\perp} > 0.$$

Then, $(\operatorname{range}(\nabla F(x^*, u^*)))^{\perp} \neq \{0\}$. Provided

$$\ker\big(\nabla F(x^*, u^*)^{\top}\big) = \Big(\operatorname{range}\big(\nabla F(x^*, u^*)\big)\Big)^{\perp},$$

it follows that there exists $l_F = (\bar{l}_F, \hat{l}_F, \tilde{l}_F) \in \mathbb{R}^{n(N+1)} \times \mathbb{R}^{(r_0+\ldots+r_N)} \times \mathbb{R}^{r_\varphi}$, $l_F \neq 0$, such that $\nabla F(x^*, u^*)^{\top} \cdot l_F = 0$, that is

$$\begin{bmatrix} \nabla_x \bar{F}(x^*, u^*)^{\top} & \nabla_x \hat{F}(x^*, u^*)^{\top} & \nabla_x \tilde{F}(x^*, u^*)^{\top} \\ \nabla_u \bar{F}(x^*, u^*)^{\top} & \nabla_u \hat{F}(x^*, u^*)^{\top} & \nabla_u \tilde{F}(x^*, u^*)^{\top} \end{bmatrix} \begin{bmatrix} \bar{l}_F \\ \hat{l}_F \\ \tilde{l}_F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let us denote $\bar{l}_F = (\bar{l}_F(0), \ldots, \bar{l}_F(N)) \in \mathbb{R}^{n(N+1)}$, $\hat{l}_F = (\hat{l}_F(0), \ldots, \hat{l}_F(N)) \in \mathbb{R}^{(r_0+\ldots+r_N)}$, and $\tilde{l}_F \in \mathbb{R}^{r_\varphi}$. Defining $\xi = 0$, $p(k+1) = \bar{l}_F(k)$, $\lambda(k) = \hat{l}_F(k)$, $\mu(k) = 0, k = 0, \ldots, N$, $\gamma = \tilde{l}_F$, $\eta = 0$, it is easy to see from (4)–(10) and (11)–(17) that conditions (i)–(iv) are satisfied.

Case 2.2. Suppose that in Definition 3.1, item 1 is fulfilled, but item 2 is not. Then, we have that the following system

$$\begin{aligned}
0 &= s(k+1) - \nabla_{x(k)} f(x^*(k), u^*(k), k) s_k \\
&\quad - \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k),\ k = 0, \ldots, N, \\
0 &= \nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k),\ k = 0, \ldots, N, \\
0 &= \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1), \\
0 &> \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^{\top} s(k) \\
&\quad + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^{\top} t(k),\ k = 0, \ldots, N,\ j \in I_g^*(k), \\
0 &> \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^{\top} s(0) \\
&\quad + \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^{\top} s(N+1),\ j \in I_\phi^*,
\end{aligned}$$

has no solution $(s, t) \in X \times U$. In other words, system

$$\nabla F(x^*, u^*) \cdot (s, t) = 0, \quad -\nabla G^{I^*}(x^*, u^*) \cdot (s, t) > 0$$

does not have any solution $(s, t) \in X \times U$, where

$$I^* = \bigcup_{k=0}^{N} I_g^*(k) \bigcup I_\phi^*,$$

and $\nabla G^{I^*}(x^*, u^*)$ denotes the matrix obtained from $\nabla G(x^*, u^*)$ after removing the rows whose indices do not belong to I^*. It follows from Proposition 2.4 that there exist $l_F = (\bar{l}_F, \hat{l}_F, \tilde{l}_F) \in \mathbb{R}^{n(N+1)} \times \mathbb{R}^{(r_0+\ldots+r_N)} \times \mathbb{R}^{r_\varphi}$, and $l_G = (\hat{l}_G, \tilde{l}_G) \in \mathbb{R}_+^{(|I_g^*(0)|+\ldots+|I_g^*(N)|)} \times \mathbb{R}_+^{|I_\phi^*|}$, $l_F \neq 0$, such that

$$\nabla F(x^*, u^*)^\top \cdot l_F - \nabla G^{I^*}(x^*, u^*)^\top \cdot l_G = 0,$$

that is

$$\begin{bmatrix} \nabla_x \bar{F}(x^*, u^*)^\top & \nabla_x \hat{F}(x^*, u^*)^\top & \nabla_x \tilde{F}(x^*, u^*)^\top \\ \nabla_u \bar{F}(x^*, u^*)^\top & \nabla_u \hat{F}(x^*, u^*)^\top & \nabla_u \tilde{F}(x^*, u^*)^\top \end{bmatrix} \begin{bmatrix} \bar{l}_F \\ \hat{l}_F \\ \tilde{l}_F \end{bmatrix} + \begin{bmatrix} \nabla_x \hat{G}(x^*, u^*)^\top & \nabla_x \tilde{G}(x^*, u^*)^\top \\ \nabla_u \hat{G}(x^*, u^*)^\top & \nabla_u \tilde{G}(x^*, u^*)^\top \end{bmatrix} \begin{bmatrix} \hat{l}_G \\ \tilde{l}_G \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let us define $\gamma = -\tilde{l}_F$ and, for each $k = 0, \ldots, N$, $p(k+1) = -\bar{l}_F(k)$, $\lambda(k) = -\hat{l}_F(k)$,

$$\mu_j(k) = \begin{cases} (\hat{l}_G)_j(k), & j \in I_g^*(k), \\ 0, & j \in \{q_1, \ldots, q_k\} \setminus I_g^*(k), \end{cases}$$

and

$$\eta_j = \begin{cases} (\tilde{l}_G)_j, & j \in I_\phi^*, \\ 0, & j \in \{1, \ldots, r_\phi\} \setminus I_\phi^*. \end{cases}$$

Therefore, by using (4)–(17) and (21)–(29), conditions (i)–(iv) of the theorem follow with $\xi = 0$.

Case 3. Now consider the case where $\nabla \Psi(x^*, u^*) \neq 0$ and (x^*, u^*) is a regular process of the extended Mangasarian–Fromovitz type, that is (x^*, u^*) satisfies Definition 3.1. Note that the cone of directions of decrease, the cone of feasible directions, and the cone of tangent directions are given by Eqs. (35), (37)–(39) and (41)–(44), respectively. By the convexity of the cones, Theorem 2.1 can be applied. Thus, we have that there exist $\chi_\Psi \in D(\Psi, (x^*, u^*))^*$, $\chi_G \in V(Q_G, (x^*, u^*))^*$ and $\chi_F \in T(Q_F, (x^*, u^*))^*$, not all identically null, such that

$$\left(\chi_\Psi + \chi_G + \chi_F\right) \cdot (s, t) = 0$$

for all $(s,t) \in X \times U$. From (36), (40) and (45), we get the existence of $\xi \geq 0$, $\mu \in \mathbb{R}^{q_0+\cdots+q_N}$, $\eta \in \mathbb{R}^{r_\phi}$, $p \in \mathbb{R}^{n(N+1)}$, $\lambda \in \mathbb{R}^{r_0+\ldots+r_N}$, and $\gamma \in \mathbb{R}^{r_\varphi}$, with

$$\mu_j(k) \geq 0,\ j = 1,\ldots,q_k, \quad \mu_j(k) = 0,\ j \notin I_g^*(k),\ k = 0,\ldots,N, \tag{46}$$

$$\eta_j \geq 0,\ j = 1,\ldots,r_\phi, \quad \eta_j = 0,\ j \notin I_\phi^*, \tag{47}$$

such that

$$\begin{aligned}\Big(-\xi\nabla\Psi(x^*,u^*) + \nabla G(x^*,u^*)^\top\cdot(-\mu,-\eta)\Big)\cdot(s,t)\\ + \Big(\nabla F(x^*,u^*)^\top\cdot(-p,-\lambda,-\gamma)\Big)\cdot(s,t) = 0\end{aligned} \tag{48}$$

for all $(s,t) \in X \times U$. Particularly, for $(s,t) = (s,0)$, we obtain

$$\big(-\xi\nabla_x\Psi(x^*,u^*) + \nabla_x G(x^*,u^*)^\top\cdot(-\mu,-\eta) + \nabla_x F(x^*,u^*)^\top\cdot(-p,-\lambda,-\gamma)\big)\cdot s{=}0$$

for all $s \in X$, which implies that

$$-\xi\nabla_x\Psi(x^*,u^*) + \nabla_x G(x^*,u^*)^\top\cdot(-\mu,-\eta) + \nabla_x F(x^*,u^*)^\top\cdot(-p,-\lambda,-\gamma) = 0.$$

From (1), (5), (7), (9), (11), (13), (15), (16), (17), (22), (24), (26), (28), and (29), it follows that

$$\begin{aligned}&\nabla_{x(0)}f(x^*(0),u^*(0),0)^\top p(1) - \xi\nabla_{x(0)}\psi(x^*(0),u^*(0),0)\\ &\quad - \nabla_{x(0)}b(x^*(0),u^*(0),0)^\top\lambda(0) - \nabla_{x(0)}g(x^*(0),u^*(0),0)^\top\mu(0)\\ &= \nabla_{x(0)}\varphi(x^*(0),x^*(N+1))^\top\gamma + \nabla_{x(0)}\phi(x^*(0),x^*(N+1))^\top\eta,\end{aligned}$$

for $k = 0$,

$$\begin{aligned}p(k) = {}&\nabla_{x(k)}f(x^*(k),u^*(k),k)^\top p(k+1) - \xi\nabla_{x(k)}\psi(x^*(k),u^*(k),k)\\ &-\nabla_{x(k)}b(x^*(k),u^*(k),k)^\top\lambda(k) - \nabla_{x(k)}g(x^*(k),u^*(k),k)^\top\mu(k),\end{aligned}$$

for $k = 1,\ldots,N$, and

$$\begin{aligned}p(N+1) = {}&-\xi\nabla_{x(N+1)}\zeta(x^*(N+1)) - \nabla_{x(N+1)}\varphi(x^*(0),x^*(N+1))^\top\gamma\\ &-\nabla_{x(N+1)}\phi(x^*(0),x^*(N+1))^\top\eta,\end{aligned}$$

for $k = N+1$. Therefore, the adjoint equation and the transversality conditions are satisfied. The complementary conditions follow from (46)–(47).

Taking $(s,t) = (0,t)$ in (48), we have

$$\begin{aligned}\Big(-\xi\nabla_u\Psi(x^*,u^*) + \nabla_u G(x^*,u^*)^\top\cdot(-\mu,-\eta)\\ +\nabla_u F(x^*,u^*)^\top\cdot(-p,-\lambda,-\gamma)\Big)\cdot t = 0\end{aligned}$$

for all $t \in U$, so that

$$-\xi \nabla_u \Psi(x^*, u^*) + \nabla_u G(x^*, u^*)^\top \cdot (-\mu, -\eta) + \nabla_u F(x^*, u^*)^\top \cdot (-p, -\lambda, -\gamma) = 0.$$

From (2), (6), (8), (10), (12), (14), (23), (25), and (27), we see that

$$\begin{aligned} 0 = \nabla_{u(k)} f(x^*(k), u^*(k), k)^\top p(k+1) - \xi \nabla_{u(k)} \psi(x^*(k), u^*(k), k) \\ -\nabla_{u(k)} b(x^*(k), u^*(k), k)^\top \lambda(k) - \nabla_{u(k)} g(x^*(k), u^*(k), k)^\top \mu(k), \end{aligned}$$

for $k = 0, \ldots, N$. Thus, the stationarity condition is satisfied. □

4.1 Nondegenerate necessary optimality conditions

It can be seen from the proof of Theorem 4.1 that if (x^*, u^*) is a regular process of the extended Mangasarian–Fromovitz type (Definition 3.1), then the multiplier associated with the objective function is nonzero. Thus, we have the following corollary.

Corollary 4.1 *Let (x^*, u^*) be a local optimal process of (DOCP). Assume that the extended Mangasarian–Fromovitz regularity condition is satisfied at (x^*, u^*). Then, the conditions (i)–(iv) of Theorem 4.1 are satisfied with $\xi > 0$.*

Proof If $\nabla \Psi(x^*, u^*) = 0$, it follows from Case 1 of the proof of Theorem 4.1 that $\xi = 1 > 0$. If $\nabla \Psi(x^*, u^*) \neq 0$, from Definition 3.1, Remark 3.4, (37), (38), (39), and (41), we have that there exists $(s, t) \in X \times U$ such that $(s, t) \in T(Q_F, (x^*, u^*)) \cap V(Q_G, (x^*, u^*))$, that is,

$$T\big(Q_F, (x^*, u^*)\big) \cap V\big(Q_G, (x^*, u^*)\big) \neq \emptyset.$$

Then, from Remark 2.1, we have that $\chi_\Psi \neq 0$, that is, $\xi \nabla \Psi(x^*, u^*) \neq 0$. As $\xi \geq 0$ and $\nabla \Psi(x^*, u^*) \neq 0$, it follows that $\xi > 0$. □

Next, we will establish the nondegenerate discrete maximum principle under the constant rank of the subspace component regularity condition, which is weaker than the extended Mangasarian–Fromovitz one. The following auxiliary discrete optimal control problem will be used:

$$\begin{aligned} \min\ & \sum_{k=0}^{N} \psi(x(k), u(k), k) + \zeta(x(N+1)) \\ \text{s.t.}\ & x(k+1) = f(x(k), u(k), k),\ k = 0, \ldots, N, \\ & b(x(k), u(k), k) = 0,\ k = 0, \ldots, N, \\ & g_j(x(k), u(k), k) = 0,\ j \in J_g^*(k),\ k = 0, \ldots, N, \\ & \varphi(x(0), x(N+1)) = 0, \\ & \phi_j(x(0), x(N+1)) = 0,\ j \in J_\phi^*, \\ & g_j(x(k), u(k), k) \leq 0,\ j \in \{1, \ldots, q_k\} \setminus J_g^*(k),\ k = 0, \ldots, N, \\ & \phi_j(x(0), x(N+1)) \leq 0,\ j \in \{1, \ldots, r_\phi\} \setminus J_\phi^*. \end{aligned} \tag{AOCP}$$

We will denote the set of all feasible processes of (AOCP) as $\mathcal{Q} = \mathcal{Q}_{\mathcal{F}} \cap \mathcal{Q}_{\mathcal{G}}$, where $\mathcal{Q}_{\mathcal{F}}$ is related to the equality constraints and $\mathcal{Q}_{\mathcal{G}}$ to the inequality ones.

Lemma 4.1 *Let $(x^*, u^*) \in Q$. If the constraints of (DOCP) satisfy the constant rank of the component subspace regularity condition at (x^*, u^*), then (x^*, u^*) is a regular process of extended Mangasarian–Fromovitz type for (AOCP).*

Proof We know from Remark 3.5 that

$$J_g^*(k) = \mathcal{J}_g^*(k),\ k = 0, \ldots, N, \quad \text{and} \quad J_\phi^* = \mathcal{J}_\phi^*.$$

So, it is enough to show that item 2 of Definition 3.1 is satisfied. Given $k \in \{0, \ldots, N\}$, for each $j \in I_g^*(k) \backslash J_g^*(k)$, by the definition of $L(Q, (x^*, u^*))$ and $J_g^*(k)$, we know that there exists $(s^{kj}, t^{kj}) \in L(Q, (x^*, u^*))$ such that

$$0 > \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s^{kj}(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^\top t^{kj}(k). \quad (49)$$

Similarly, given $j \in I_\phi^* \setminus J_\phi^*$, there exists $(s^j, t^j) \in L(Q, (x^*, u^*))$ such that

$$\begin{aligned} 0 > &\nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s^j(0) \\ &+ \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s^j(N+1). \end{aligned} \quad (50)$$

Let $(s, t) \in X \times U$ be given as

$$(s, t) = \sum_{k=0}^{N} \sum_{I_g^*(k) \backslash J_g^*(k)} (s^{kj}, t^{kj}) + \sum_{I_\phi^* \backslash J_\phi^*} (s^j, t^j).$$

It is clear that $(s, t) \in L(Q, (x^*, u^*))$. Therefore, for $k = 0, \ldots, N$,

$$s(k+1) = \nabla_{x(k)} f(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k), \quad (51)$$
$$0 = \nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k), \quad (52)$$

and

$$0 = \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1). (53)$$

For each $k \in \{0, \ldots, N\}$, from the very definition of $J_g^*(k)$, we see that

$$0 = \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^\top t(k) \quad (54)$$

for $j \in J_g^*(k)$. Analogously, by the definition of J_ϕ^*,

$$\begin{aligned} 0 = &\nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0) \\ &+ \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1) \end{aligned} \quad (55)$$

for $j \in J_\phi^*$. From the Eqs. (51)–(55) above we get $(s, t) \in \ker\big(\nabla \mathcal{F}(x^*, u^*)\big)$. From (49), (50) and the fact that $(s, t) \in L(Q, (x^*, u^*))$, we obtain

$$0 > \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k)^\top t(k)$$

for $j \in I_\phi^* \backslash J_g^*(k)$, $k = 0, \ldots, N$, and

$$0 > \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0) + \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1)$$

for $j \in I_\phi^* \setminus J_\phi^*$.

This shows that item 2 of Definition 3.1 is fulfilled and the proof is done. □

We are now in position to state and prove a nondegenerate form of the discrete maximum principle under the constant rank of the subspace component constraint qualification (Definition 3.2).

Theorem 4.2 (Discrete Weak Maximum Principle) *Let (x^*, u^*) be a local optimal process of (DOCP). Assume that the constant rank of the subspace component constraint qualification is satisfied at (x^*, u^*). Then, there exist $\xi > 0$, $\gamma \in \mathbb{R}^{r_\varphi}$, $\eta \in \mathbb{R}^{r_\phi}$, and for each $k = 0, \ldots, N$, $(p(k+1), \lambda(k), \mu(k)) \in \mathbb{R}^n \times \mathbb{R}^{r_k} \times \mathbb{R}^{q_k}$, not all null, with $\mu(k) \geq 0$, $k = 0, \ldots, N$, and $\eta \geq 0$, such that the following conditions are satisfied:*

(i) *Adjoint equation:*

$$p(k) = \nabla_{x(k)} H(x^*(k), u^*(k), p(k+1), \xi, \lambda(k), \mu(k), k), \;\; k = 1, \ldots, N;$$

(ii) *Transversality conditions:*

$$\begin{aligned} &\nabla_{x(0)} H(x^*(0), u^*(0), p(1), \xi, \lambda(0), \mu(0), 0) \\ &\quad = \gamma \cdot \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) + \eta \cdot \nabla_{x(0)} \phi(x^*(0), x^*(N+1)), \\ &p(N+1) = -\xi \nabla_{x(N+1)} \zeta(x^*(N+1)) - \gamma \cdot \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) \\ &\quad - \eta \cdot \nabla_{x(N+1)} \phi(x^*(0), x^*(N+1)); \end{aligned}$$

(iii) *Stationarity condition:*

$$\nabla_{u(k)} H(x^*(k), u^*(k), p(k+1), \xi, \lambda(k), \mu(k), k) = 0, \;\; k = 0, \ldots, N;$$

(iv) *Complementarity conditions:*

$$\begin{aligned} &\mu_j(k) g_j(x^*(k), u^*(k), k) = 0, \;\; j = 1, \ldots, q_k, \;\; k = 0, \ldots, N, \\ &\eta_j \phi_j(x^*(0), x^*(N+1)) = 0, \;\; j = 1, \ldots, r_\phi. \end{aligned}$$

Proof It is clear that (x^*, u^*) is a local optimal process of (AOCP). By Lemma 4.1 and Corollary 4.1 we know that there exist $\xi > 0$, $\gamma \in \mathbb{R}^{r_\varphi}$, $\eta \in \mathbb{R}^{r_\phi}$, and for each $k = 0, \ldots, N$, $(p(k+1), \lambda(k), \mu(k)) \in \mathbb{R}^{n(N+1)} \times \mathbb{R}^{r_k} \times \mathbb{R}^{q_k}$, not all null, with

$\mu_j(k) \geq 0$, $j \in \{1, \dots, q_k\} \backslash J_g^*(k)$, $k = 0, \dots, N$, and $\eta_j \geq 0$, $j \in \{1, \dots, r_\phi\} \backslash J_\phi^*$, such that

$$p(k) = \nabla_{x(k)} H(x^*(k), u^*(k), p(k+1), \xi, \lambda(k), \mu(k), k), \; k = 1, \dots, N; \tag{56}$$

$$\begin{aligned} &\nabla_{x(0)} H(x^*(0), u^*(0), p(1), \xi, \lambda(0), \mu(0), 0) \\ &\quad = \gamma \cdot \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) + \eta \cdot \nabla_{x(0)} \phi(x^*(0), x^*(N+1)), \end{aligned} \tag{57}$$

$$\begin{aligned} p(N+1) &= -\xi \nabla_{x(N+1)} \zeta(x^*(N+1)) - \gamma \cdot \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) \\ &\quad - \eta \cdot \nabla_{x(N+1)} \phi(x^*(0), x^*(N+1)); \end{aligned} \tag{58}$$

$$\nabla_{u(k)} H(x^*(k), u^*(k), p(k+1), \xi, \lambda(k), \mu(k), k) = 0, \; k = 0, \dots, N; \tag{59}$$

$$\mu_j(k) g_j(x^*(k), u^*(k), k) = 0, \; j \in \{1, \dots, q_k\} \setminus J_g^*(k), \; k = 0, \dots, N, \tag{60}$$

$$\eta_j \phi_j(x^*(0), x^*(N+1)) = 0, \; j \in \{1, \dots, r_\phi\}. \tag{61}$$

Let $(s, t) \in L(Q, (x^*, u^*))$. From (56)–(59), we get

$$\begin{aligned} p(k)^\top s(k) = \; & p(k+1)^\top \nabla_{x(k)} f(x^*(k), u^*(k), k) s(k) \\ & - \xi \nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \\ & - \lambda(k)^\top \nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) \\ & - \sum_{j \in I_g^*(k)} \mu_j(k) \nabla_{x(k)} g_j(x^*(k), u^*(k), k)^\top s(k), \end{aligned} \tag{62}$$

for $k = 1, \dots, N$,

$$\begin{aligned} & p(1)^\top \nabla_{x(0)} f(x^*(0), u^*(0), 0) s(0) - \xi \nabla_{x(0)} \psi(x^*(0), u^*(0), 0)^\top s(0) \\ & \quad - \lambda(0)^\top \nabla_{x(0)} b(x^*(0), u^*(0), 0) s(0) \\ & \quad - \sum_{j \in I_g^*(0)} \mu_j(0) \nabla_{x(0)} g_j(x^*(0), u^*(0), 0)^\top s(0) \\ & = -\gamma^\top \nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) \\ & \quad - \sum_{I_\phi^*} \eta_j \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0), \end{aligned} \tag{63}$$

$$\begin{aligned} & p(N+1)^\top s(N+1) \\ & = -\xi \nabla_{x(N+1)} \zeta(x^*(0), x^*(N+1)) s(N+1) \\ & \quad - \gamma^\top \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1) \\ & \quad - \sum_{j \in I_\phi^*} \eta_j \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1), \end{aligned} \tag{64}$$

$$\begin{aligned} 0 = \; & p(k+1)^\top \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k) - \xi \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k) \\ & - \lambda(k)^\top \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k) \\ & + \sum_{j \in I_g^*(k)} \mu_j(k) \nabla_{u(k)} g_j(x^*(k), u^*(k))^\top t(k), \; k = 0, \dots, N. \end{aligned} \tag{65}$$

Summing (62)–(65) and rearranging terms, we obtain

$$
\begin{aligned}
&\xi \sum_{k=0}^{N} \left[\nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k)\right] \\
&\quad + \xi \nabla_{x(N+1)} \zeta(x^*(0), x^*(N+1)) s(N+1) \\
&\quad = - \sum_{k=1}^{N+1} p(k)^\top s(k) \\
&\quad + \sum_{k=0}^{N} p(k+1)^\top \left[\nabla_{x(k)} f(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} f(x^*(k), u^*(k), k) t(k)\right] \\
&\quad - \sum_{k=0}^{N} \lambda(k)^\top \left[\nabla_{x(k)} b(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} b(x^*(k), u^*(k), k) t(k)\right] \\
&\quad - \sum_{k=0}^{N} \sum_{j \in I_g^*(k)} \mu_j(k) \left[\nabla_{x(k)} g_j(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k) t(k)\right] \\
&\quad - \gamma^\top \left[\nabla_{x(0)} \varphi(x^*(0), x^*(N+1)) s(0) + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1)) s(N+1)\right] \\
&\quad - \sum_{I_\phi^*} \eta_j \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0) \\
&\quad - \sum_{I_\phi^*} \eta_j \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1).
\end{aligned}
$$

By using the fact that $(s, t) \in L(Q, (x^*, u^*))$, from (30)–(32) and the definition of the sets $J_g^*(k)$ and J_ϕ^* (see Remark 3.5), we get

$$
\begin{aligned}
&\xi \sum_{k=0}^{N} \left[\nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k)\right] \\
&\quad + \xi \nabla_{x(N+1)} \zeta(x^*(0), x^*(N+1)) s(N+1) \\
&\quad = - \sum_{k=1}^{N+1} p(k)^\top s(k) + \sum_{k=0}^{N} p(k+1)^\top s(k+1) \\
&\quad - \sum_{k=0}^{N} \sum_{j \in I_g^*(k) \setminus J_g^*(k)} \mu_j(k) \left[\nabla_{x(k)} g_j(x^*(k), u^*(k), k) s(k) + \nabla_{u(k)} g_j(x^*(k), u^*(k), k) t(k)\right] \\
&\quad - \sum_{j \in I_\phi^* \setminus J_\phi^*} \eta_j \nabla_{x(0)} \phi_j(x^*(0), x^*(N+1))^\top s(0) \\
&\quad - \sum_{j \in I_\phi^* \setminus J_\phi^*} \eta_j \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1))^\top s(N+1).
\end{aligned}
$$

The two first sums after the equality sign above cancel each other out. By using the fact that $(s, t) \in L(Q, (x^*, u^*))$ one more time, from (33)–(34) and observing that

$\xi > 0$, $\mu_j(k) \geq 0$, $j \in I_g^*(k) \backslash J_g^*(k)$, $\eta_j \geq 0$, $j \in I_\phi^* \backslash J_\phi^*$, we conclude that

$$\begin{aligned}\nabla \Psi(x^*, u^*) = &\sum_{k=0}^{N} \left[\nabla_{x(k)} \psi(x^*(k), u^*(k), k)^\top s(k) \nabla_{u(k)} \psi(x^*(k), u^*(k), k)^\top t(k) \right] \\ &+ \nabla_{x(N+1)} \zeta(x^*(0), x^*(N+1)) s(N+1) \geq 0\end{aligned}$$

for all $(s, t) \in L(Q, (x^*, u^*))$. This means that $\nabla \Psi(x^*, u^*) \in L(Q, (x^*, u^*))^\circ$.

From Remark 3.3 and (1)–(2), there exist $\gamma \in \mathbb{R}^{r_\varphi}$, $\eta_j \geq 0$, $j \in I_\phi^*$, and for each $k = 0, \ldots, N$, $p(k+1) \in \mathbb{R}^n$, $\lambda(k) \in \mathbb{R}^{r_k}$ and $\mu_j(k) \geq 0$, $j \in I_g^*(k)$, such that

$$\begin{aligned}\nabla_{x(0)} \psi(x^*(0), u^*(0), 0) = &-\nabla_{x(0)} f(x^*(0), u^*(0), 0)^\top p(1) \\ &+\nabla_{x(0)} b(x^*(0), u^*(0), 0)^\top \lambda(0) \\ &+\nabla_{x(0)} \varphi(x^*(0), x^*(N+1))^\top \gamma \\ &+ \sum_{j \in I_g^*(0)} \mu_j(0) \nabla_{x(0)} g_j(x^*(0), u^*(0), 0) \\ &+ \sum_{I_\phi^*} \eta_j \nabla_{x(0)} \phi(x^*(0), x^*(N+1)), \\ &\nabla_{x(k)} \psi(x^*(k), u^*(k), k) \\ &= p(k) - \nabla_{x(k)} f(x^*(k), u^*(k), k)^\top p(k+1) \\ &+\nabla_{x(k)} b(x^*(k), u^*(k), k)^\top \lambda(k) \\ &+ \sum_{j \in I_g^*(k)} \mu_j(k) \nabla_{x(k)} g_j(x^*(k), u^*(k), k), \;\; k = 1, \ldots, N, \\ &\nabla \zeta(x^*(N+1)) = p(N+1) + \nabla_{x(N+1)} \varphi(x^*(0), x^*(N+1))^\top \gamma \\ &+ \sum_{j \in I_\phi^*} \eta_j \nabla_{x(N+1)} \phi_j(x^*(0), x^*(N+1)), \\ \nabla_{u(k)} \psi(x^*(k), u^*(k), k) = &-\nabla_{u(k)} f(x^*(k), u^*(k), k)^\top p(k+1) \\ &+\nabla_{u(k)} b(x^*(k), u^*(k), k)^\top \lambda(k) \\ &+ \sum_{j \in I_g^*(k)} \mu_j(k) \nabla_{u(k)} g_j(x^*(k), u^*(k)), \;\; k = 0, \ldots, N.\end{aligned}$$

By defining $\mu_j(k) = 0$ for $j \in \{q_1, \ldots, q_k\} \backslash I_g^*(k)$, $k = 0, \ldots, N$, and $\eta_j = 0$, for $j \in \{1, \ldots, r_\phi\} \backslash I_\phi^*$, we see that conditions (i)–(iv) hold with $\xi = 1$. □

4.2 Example

In the following example, we present a simple problem where CRSC is satisfied, but neither the extended MFCQ nor any other full rank-based CQs are verified.

Consider the following discrete optimal control problem with $N = 1$, $n = m = 2$, $r_0 = 1$, $r_1 = 2$, $q_0 = 2$, $q_1 = 4$, $r_\varphi = 1$, and $r_\phi = 2$. We will denote $x(k) = (x_1(k), x_2(k)) \in \mathbb{R}^2$, $k = 0, 1, 2$, and $u(k) = (u_1(k), u_2(k)) \in \mathbb{R}^2$, $k = 0, 1$.

$$\begin{aligned}
&\text{Minimize } \psi(x(0), u(0), 0) + \psi(x(1), u(1), 1) + \zeta(x(2)) \\
&\text{subject to } x(k+1) = f(x(k), u(k), k),\ k = 0, 1, \\
&\qquad b(x(k), u(k), k) = 0,\ k = 0, 1, \\
&\qquad g(x(k), u(k), k) \le 0,\ k = 0, 1, \\
&\qquad \varphi(x(0), x(2)) = 0, \\
&\qquad \phi(x(0), x(2)) \le 0,
\end{aligned}$$

in which

$$\begin{aligned}
\psi(x(0), u(0), 0) &= (x_2(0))^2 + (u_2(0))^2, \\
\psi(x(1), u(1), 1) &= x_1(1) + u_1(1), \\
\zeta(x(2)) &= x_1(2) + x_2(2), \\
f(x(0), u(0), 0) &= \begin{bmatrix} x_1(0) + u_1(0) \\ x_2(0) + u_2(0) \end{bmatrix}, \\
f(x(1), u(1), 1) &= \begin{bmatrix} x_1(1) + u_1(1) \\ x_2(1) + u_2(1) \end{bmatrix}, \\
b(x(0), u(0), 0) &= (x_1(0) + u_1(0))^2, \\
b(x(1), u(1), 1) &= \begin{bmatrix} x_2(1) + u_1(1) + u_2(1) \\ u_1(1) + u_2(1) \end{bmatrix}, \\
g(x(0), u(0), 0) &= \begin{bmatrix} -u_1(0) \\ x_1(0) + u_1(0) \end{bmatrix}, \\
g(x(1), u(1), 1) &= \begin{bmatrix} x_2(1) - u_1(1) \\ -x_2(1) + u_1(1) \\ (x_2(1) - u_1(1))^2 \\ x_1(1) + u_2(1) - 1 \end{bmatrix}, \\
\varphi(x(0), x(2)) &= x_1(0) + x_1(2) + 2x_2(2), \\
\phi(x(0), x(2)) &= \begin{bmatrix} x_1(2) + 2x_2(2) \\ -x_1(2) - 2x_2(2) \end{bmatrix}.
\end{aligned}$$

It can be seen that $(x^*, u^*) = (0, 0)$ is a global optimal process. Indeed, let (x, u) be a feasible process. From the dynamic equation

$$x_1(1) = f_1(x(0), u(0), 0) = x_1(0) + u_1(0)$$

and the equality mixed constraint

$$b(x(0), u(0), 0) = (x_1(0) + u_1(0))^2 = 0$$

we see that

$$x_1(1) = 0. \tag{66}$$

From the equality mixed constraint

$$b(x(1), u(1), 1) = 0 \Leftrightarrow \begin{cases} x_2(1) + u_1(1) + u_2(1) = 0 \\ u_1(1) + u_2(1) = 0 \end{cases}$$

we see that

$$x_2(1) = 0. \tag{67}$$

From the dynamic equation $x(2) = f(x(1), u(1), 1)$, (66) and (67), we get

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} x_1(1) + u_1(1) \\ x_2(1) + u_2(1) \end{bmatrix} = \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix}, \tag{68}$$

which, together with $b_2(x(1), u(1), 1) = 0$, implies that

$$x_1(2) + x_2(2) = u_1(1) + u_2(1) = 0. \tag{69}$$

From the boundary condition $\phi_1(x(0), x(2)) = x_1(2)+2x_2(2) \leq 0$, $\phi_2(x(0), x(2)) = -x_1(2) - 2x_2(2) \leq 0$ and (68), we have $u_1(1)+2u_2(1) = 0$. By using (69), we obtain

$$u_2(1) = 0. \tag{70}$$

Finally, from (69) and (70), we get

$$u_1(1) = 0. \tag{71}$$

It follows from the definition of the objective function $\Psi(x, u)$, (66), (69), and (71), that $\Psi(x, u) \geq 0$ for all feasible processes (x, u). It is clear that (x^*, u^*) is a feasible process with $\Psi(x^*, u^*) = 0$. Thus, (x^*, u^*) is a global optimal process.

We will show that (CRSC) is satisfied at (x^*, u^*). We have that

$$\begin{aligned}
&\bar{F}_{01}(x, u) = x_1(1) - x_1(0) - u_1(0), && \bar{F}_{02}(x, u) = x_2(1) - x_2(0) - u_2(0), \\
&\bar{F}_{11}(x, u) = x_1(2) - x_1(1) - u_1(1), && \bar{F}_{12}(x, u) = x_2(2) - x_2(1) - u_2(1), \\
&\hat{F}_{01}(x, u) = (x_1(0) + u_1(0))^2, && \hat{F}_{11}(x, u) = x_2(1) + u_1(1) + u_2(1), \\
&\hat{F}_{12}(x, u) = u_1(1) + u_2(1), && \tilde{F}_1(x, u) = x_1(0) + x_1(2) + 2x_2(2), \\
&\hat{G}_{01}(x, u) = -u_1(0), && \hat{G}_{02}(x, u) = x_1(0) + u_1(0), \\
&\hat{G}_{11}(x, u) = x_2(1) - u_1(1), && \hat{G}_{12}(x, u) = -x_2(1) + u_1(1), \\
&\hat{G}_{13}(x, u) = (x_2(1) - u_1(1))^2, && \hat{G}_{14}(x, u) = x_1(1) + u_2(1) - 1, \\
&\tilde{G}_1(x, u) = x_1(2) + 2x_2(2), && \tilde{G}_2(x, u) = -x_1(2) - 2x_2(2).
\end{aligned}$$

Then,

$$
\begin{aligned}
\nabla \bar{F}(x,u) &= \begin{bmatrix} \nabla \bar{F}_{01}(x,u) \\ \nabla \bar{F}_{02}(x,u) \\ \nabla \bar{F}_{11}(x,u) \\ \nabla \bar{F}_{12}(x,u) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \\
\nabla \hat{F}(x,u) &= \begin{bmatrix} \nabla \hat{F}_{01}(x,u) \\ \nabla \hat{F}_{11}(x,u) \\ \nabla \hat{F}_{12}(x,u) \end{bmatrix} \\
&= \begin{bmatrix} 2(x_1(0)+u_1(0)) & 0\,0\,0\,0\,0 & 2(x_1(0)+u_1(0)) & 0\,0\,0 \\ 0 & 0\,0\,1\,0\,0 & 0 & 0\,1\,1 \\ 0 & 0\,0\,0\,0\,0 & 0 & 0\,1\,1 \end{bmatrix}, \\
\nabla \tilde{F}(x,u) &= \begin{bmatrix} \nabla \tilde{F}_1(x,u) \end{bmatrix} \\
&= \begin{bmatrix} 1\;0\;0\;0\;1\;2\;0\;0\;0\;0 \end{bmatrix}, \\
\nabla \hat{G}(x,u) &= \begin{bmatrix} \nabla \hat{G}_{01}(x,u) \\ \nabla \hat{G}_{02}(x,u) \\ \nabla \hat{G}_{11}(x,u) \\ \nabla \hat{G}_{12}(x,u) \\ \nabla \hat{G}_{13}(x,u) \\ \nabla \hat{G}_{14}(x,u) \end{bmatrix} \\
&= \begin{bmatrix} 0\,0\,0 & 0 & 0\,0 & -1 & 0 & 0 & 0 \\ 1\,0\,0 & 0 & 0\,0 & 1 & 0 & 0 & 0 \\ 0\,0\,0 & 1 & 0\,0 & 0 & 0 & -1 & 0 \\ 0\,0\,0 & -1 & 0\,0 & 0 & 0 & 1 & 0 \\ 0\,0\,0 & 2(x_2(1)-u_1(1)) & 0\,0 & 0 & 0 & -2(x_1(1)-u_1(1)) & 0 \\ 0\,0\,1 & 0 & 0\,0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
\nabla \tilde{G}(x,u) &= \begin{bmatrix} \nabla \tilde{G}_1(x,u) \\ \nabla \tilde{G}_2(x,u) \end{bmatrix} = \begin{bmatrix} 0\,0\,0\,0 & 1 & 2 & 0\,0\,0\,0 \\ 0\,0\,0\,0 & -1 & -2 & 0\,0\,0\,0 \end{bmatrix}.
\end{aligned}
$$

It is clear that $I_g^*(0) = \{1, 2\}$, $I_g^*(1) = \{1, 2, 3\}$ and $I_\phi^* = \{1, 2\}$. Since

$$
\begin{aligned}
&-\nabla \hat{G}_{01}(x^*,u^*) = \nabla \hat{G}_{02}(x^*,u^*) - \nabla \tilde{F}_1(x^*,u^*) + \nabla \tilde{G}_1(x^*,u^*), \\
&-\nabla \hat{G}_{02}(x^*,u^*) = \nabla \hat{G}_{01}(x^*,u^*) - \nabla \tilde{F}_1(x^*,u^*) + \nabla \tilde{G}_1(x^*,u^*), \\
&-\nabla \hat{G}_{11}(x^*,u^*) = \nabla \hat{G}_{12}(x^*,u^*), \\
&-\nabla \hat{G}_{12}(x^*,u^*) = \nabla \hat{G}_{11}(x^*,u^*), \\
&-\nabla \hat{G}_{13}(x^*,u^*) = 0, \\
&-\nabla \tilde{G}_1(x^*,u^*) = \nabla \tilde{G}_2(x^*,u^*), \\
&-\nabla \tilde{G}_2(x^*,u^*) = \nabla \tilde{G}_1(x^*,u^*),
\end{aligned}
$$

we see that

$$
\begin{aligned}
&-\nabla \hat{G}_{01}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,\\
&-\nabla \hat{G}_{02}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,\\
&-\nabla \hat{G}_{11}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,\\
&-\nabla \hat{G}_{12}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,\\
&-\nabla \hat{G}_{13}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,\\
&-\nabla \tilde{G}_{1}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,\\
&-\nabla \tilde{G}_{2}(x^*, u^*) \in L(Q, (x^*, u^*))^\circ,
\end{aligned}
$$

so that $\mathcal{J}_g^*(0) = \{1, 2\}$, $\mathcal{J}_g^*(1) = \{1, 2, 3\}$ and $\mathcal{J}_\phi^* = \{1, 2\}$. The matrix $\begin{bmatrix} \nabla F(x,u) \\ \nabla G^{\mathcal{J}_-^*}(x,u) \end{bmatrix}$ is given by

$$
\begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1\\
\frac{\partial \hat{F}_{01}(x,u)}{\partial x_1(0)} & 0 & 0 & 0 & 0 & 0 & \frac{\partial \hat{F}_{01}(x,u)}{\partial u_1(0)} & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & \frac{\partial \hat{G}_{13}(x,u)}{\partial x_2(1)} & 0 & 0 & 0 & 0 & \frac{\partial \hat{G}_{13}(x,u)}{\partial u_1(1)} & 0\\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

where $\frac{\partial \hat{F}_{01}(x,u)}{\partial x_1(0)} = \frac{\partial \hat{F}_{01}(x,u)}{\partial u_1(0)} = 2(x_1(0) + u_1(0))$, $\frac{\partial \hat{G}_{13}(x,u)}{\partial x_2(1)} = 2(x_2(1) - u_1(1))$ and $\frac{\partial \hat{G}_{13}(x,u)}{\partial u_1(1)} = -2(x_2(1) - u_1(1))$. It is easy to see that this matrix has a constant rank equal to 9 in any neighborhood of the feasible process (x^*, u^*). Thus, CRSC (Definition 3.2) is satisfied at (x^*, u^*).

By Theorem 4.2, there exist $\xi > 0$, $p(1) \in \mathbb{R}^2$, $p(2) \in \mathbb{R}^2$, $\lambda(0) \in \mathbb{R}$, $\lambda(1) \in \mathbb{R}^2$, $\mu(0) \in \mathbb{R}^2$, $\mu(1) \in \mathbb{R}^4$, $\gamma \in \mathbb{R}$ and $\eta \in \mathbb{R}^2$ such that the conditions (i)–(iv) of the discrete maximum principle are satisfied. For example, it is easy to see that conditions (i)–(iv) are valid with $\xi = 1$, $p(1) = (-1, 0)$, $p(2) = (0, -3)$, $\lambda(0) = 0$, $\lambda(1) = (-1, -2)$, $\mu(0) = (1, 0)$, $\mu(1) = (0, 2, 1, 0)$, $\gamma = -1$ and $\eta = (1, 1)$.

We note that RCPLD is not satisfied at (x^*, u^*). Indeed, it is clear that the rows of

$$\begin{bmatrix} \nabla F(x,u) \\ \nabla \hat{G}_{13}(x,u) \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 & 1 & 0 & 0\,0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0\,0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1\,0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0\,1 & 0 & 0 & 0 & -1 \\ \frac{\partial \hat{F}_{01}(x,u)}{\partial x_1(0)} & 0 & 0 & 0 & 0\,0 & \frac{\partial \hat{F}_{01}(x,u)}{\partial u_1(0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0\,0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0\,0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{\partial \hat{G}_{13}(x,u)}{\partial x_2(1)} & 0\,0 & 0 & 0 & \frac{\partial \hat{G}_{13}(x,u)}{\partial u_1(1)} & 0 \end{bmatrix}$$

are positively linearly dependent at (x^*, u^*), but for $(x, u) \neq (0, 0)$ near (x^*, u^*) they are linearly independent. Thence, other stronger constraint qualifications, such as LICQ, MFCQ, extended MFCQ, CPLD, CRCQ and RCRCQ, are not satisfied either.

5 Conclusions

The contributions of this study are from obtaining the non-degenerated discrete maximum principle for a class of discrete optimal control problems with mixed equality and inequality constraints under the constant rank of the component subspace (CRCS) constraint qualification by using the Dubovitskii–Milyutin formalism. CRSC is more appropriate than other constraint qualifications found in the literature in the following sense: (1) while it is weaker than both MFCQ and CRCQ, it is simpler to verify than those based in abstract assumptions, like Abadie and Guignard, for example; (2) provided it is a constant-rank like CQ, it allows the presence of redundancy on modeling the optimization problem, which is highly valued for nonlinear problems (removing redundancy in nonlinear systems is, in general, a hard task); (3) among the constant-rank based CQs, CRSC is the one that precisely describes the set of gradients that need to be taken into account, namely, those associated to the equality constraints plus the gradients of the active inequality constraints that behave locally as an equality one (in CRCQ and RCRCQ it is necessary to consider all subsets of the index set of active constraints); (4) CRSC has a great practical importance due to its relevance in designing optimization computational methods, it implies the global convergence of augmented Lagrangian methods, for example. The Dubovitskii–Milyutin formalism offers a significant advantage in that it presents a unified approach to examining a broad range of optimization problems with constraints. Its use enables our results to be easily generalized for DOC problems posed in Banach spaces, problems with pure (abstract or functional) state constraints, problems with unilateral control constraints (both in abstract or functional form), among other types of constraints.

With respect to future work, a natural continuation of this study is the development of methods for solving numerically DOC problems. We mean a dedicated method for optimal control, not the mere application of general nonlinear programming methods.

Once the nondegenerate necessary optimality conditions were established here under CRSC, we believe that a method of augmented Lagrangian type should be one of the first attempts. We may also investigate establishing second-order necessary optimality conditions for DOC problems. CRCQ is known to be a second-order constraint qualification for nonlinear programming problems. It would be interesting to investigate if the same passes for DOC problems.

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Declarations

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References

1. Andreani, R., Haeser, G., Schuverdt, M.L., Silva, P.J.S.: Two new weak constraint qualifications and applications. SIAM J. Optim. **22**(3), 1109–1135 (2012)
2. Boltyanskii, V.G.: Optimal Control of Discrete Systems. Wiley, New York (1978)
3. Butkovski, A.G.: Necessary and sufficient conditions of optimality of discrete control systems. Autom. Remote. Control. **24**(8), 963–970 (1963)
4. Canon, M.D., Cullum, C.D., Polak, E.: Theory of Optimal Control and Mathematical Programming. McGraw-Hill, New York (1970)
5. Cullum, J.: An explicit procedure for discretizing continuous, optimal control problems. J. Optim. Theory Appl. **8**(1), 15–34 (1971)
6. Girsanov, I.V.: Lecture Notes in Economics and Mathematical Systems. Springer, Berlin, Heidelberg, New York (1972)
7. Jordan, B., Polak, E.: Theory of a class of discrete optimal control systems. J. Electr. Control **17**, 697–711 (1964)
8. Kleindorfer, P.R., Kriebel, C., Thompson, G., Kleindorfer, G.: Discrete optimal control of production plans. Manag. Sci. **22**, 261–273 (1975)
9. Kruger, A.Y., Minchenko, L., Outrata, J.V.: On relaxing the Mangasarian–Fromovitz constraint qualification. Positivity **18**(1), 171–189 (2014)
10. Magnanti, T.L.: Non-linear programming and the maximum principle for discrete time optimal control problems. Revue française d'automatique, informatique, recherche opérationnelle. Recherche opérationnelle **9**(V3), 75–91 (1975)
11. Mangasarian, O.L.: Nonlinear Programming, Classics in Applied Mathematics, vol. 10. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1994)
12. Marinković, B.: Optimality conditions in discrete optimal control problems. Optim. Methods Softw. **22**, 959–969 (2007)
13. Marinković, B.: Optimality conditions in discrete optimal control problems with state constraints. Numer. Funct. Anal. Optim. **28**(7–8), 945–955 (2007)
14. Marinković, B.: Optimality conditions for discrete optimal control problems with equality and inequality type constraints. Positivity **12**, 535–545 (2008)
15. Minchenko, L., Stakhovski, S.: About generalizing the Mangasarian–Fromovitz regularity condition. Dokl. BGUIR **8**, 104–109 (2010)
16. Minchenko, L., Stakhovski, S.: On relaxed constant rank regularity condition in mathematical programming. Optimization **60**, 429–440 (2011)

17. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F.: The Mathematical Theory of Optimal Processes. Interscience, New York (1962)
18. Rojas-Medar, M.A., Isoton, C., dos Santos, L.B., Vivanco-Orellana, V.: Optimality conditions for discrete-time control problems. J. Optim. Theory Appl. **185**, 113–115 (2020)
19. Rozonoer, L.I.: The maximum principle of L.S. Pontryagin in the theory of optimal systems. Autom. Remote Control **20**, 1517–1532 (1959)
20. Thuy, L.Q., Thanh, B.T., Toan, N.T.: On the no-gap second-order optimality conditions for a discrete optimal control problem with mixed constraints. J. Optim. Theory Appl. **173**(2), 421–442 (2017)
21. Toan, N.T., Ansari, Q.H., Yao, J.C.: Second-order necessary optimality conditions for a discrete optimal control problem. J. Optim. Theory Appl. **165**(3), 812–836 (2015)
22. Toan, N.T., Thuy, L.Q.: Second-order necessary optimality conditions for a discrete optimal control problem with mixed constraints. J. Glob. Optim. **64**, 533–562 (2016)
23. Toan, N.T., Thuy, L.Q., Ansari, Q.H., Yao, J.: Second-order necessary optimality conditions for a discrete optimal control problem with nonlinear state equations. Optim. Control Appl. Meth. **41**, 2250–2281 (2020)