Non-linear Iwasawa decomposition of stochastic flows: geometrical characterization and examples

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Abstract

Let φ_t be the stochastic flow of a stochastic differential equation on a Riemannian manifold M of constant curvature. For a given initial condition in the orthonormal frame bundle: $x_0 \in M$ and u an orthonormal frame in $T_{x_0}M$, there exists a unique decomposition $\varphi_t = \xi_t \circ \Psi_t$ where ξ_t is isometry, Ψ_t fixes x_0 and $D\Psi_t(u) = u \cdot s_t$ where s_t is an upper triangular matrix process. We present the results and the main ideas by working in detailed examples.

Key words: stochastic flows, group of affine transformations, isometries, non-linear Iwasawa decomposition.

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1 Introduction

Consider a vector field in a smooth complete d-dimensional Riemannian manifold M. If the vector field are sufficiently smooth (and bounded derivatives) then it determines a dynamical system which can be represented as a one-parameter family of (global) diffeomorphisms of M into itself. The description of stochastic flows is exactly the same, except that, in this case we have a family of *random* diffeomorphisms. Hence, dynamical systems in general are determined by this path (random or not) in the group of diffeomorphisms $\mathrm{Dif}(M)$, therefore, naturally, all informations regarding a given particular system, including the asymptotic behavior like stability (Lyapunov exponents), rotation numbers, stable/unstable submanifolds are contained in this trajectory. In general, the group $\mathrm{Dif}(M)$ is a fantastically

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rich infinite dimensional Lie group with a pretty much interesting topology which depends on the geometry of M.

The fact that $\mathrm{Dif}(M)$ is such a huge group motivates one to factorize a given flow into components which lie in smaller subgroups. Among finite dimensional subgroups of $\mathrm{Dif}(M)$ we are going to focus in the groups which provide information of the asymptotic properties of the dynamics: the group of affine transformations and the group of isometries. The purpose of this article is to present results which guarantee the existence of such factorizations for certain flows. For details of the applications of these decompositions on asymptotic parameters (Lyapunov exponents and rotation numbers) we refer to the articles M. Liao [6] and Ruffino [8]. For detailed description of Lyapunov exponents in stochastic dynamical systems we recommend L. Arnold [1] (and the references therein), and for a detailed description of matrix of rotation, we would suggest [7].

We address the article to readers who are not specialists in geometry, hence we are not going to present the details of the proofs, instead, we intend to emphasize the method and the results by showing detailed examples. The main theoretical results are exposed in M. Liao [6] with extensions in Ruffino [8].

We shall consider the following Stratonovich stochastic differential equation (sde) in the manifold M:

$$d\,x_t = \sum_{j=0}^{m} X^j(x_t) \circ dW_t^j \tag{1}$$

where $X^0, \ldots, X^m$ are (smooth) vector fields, $W_t^0 = dt$ and $(W^1, \ldots, W^m)$ is a Brownian motion on $\mathbb{R}^m$. We shall denote by $\varphi_t : \Omega \times M \to M$ the associated stochastic flow of C^∞–diffeomorphisms, which we shall assume complete (say, assuming that the derivatives of the vector fields X^j are bounded, see e.g. Kunita [4] or [5]).

What we call the "non-linear Iwasawa decomposition" is precisely the following. Under certain conditions on the vector fields [6], or if the manifold M has constant curvature [8] then, for an initial condition $x_0 \in M$ and an initial orthonormal frame u in the tangent space $T_{x_0}M$, there exists a unique factorization of the flow $\varphi_t = \xi_t \circ \Psi_t$, where ξ_t lies in the group of isometries, Ψ_t fixes the starting point x_0 for all $t \geq 0$, i.e., $\Psi_t(x_0) \equiv x_0$ and the derivative $D\Psi(u) = u\, s_t$ where s_t is an upper triangular matrix process.

Adding some other restrictions in the vector fields (or assuming that M is flat, [8]) we can go further in the decomposition and write

$$\varphi_t = \xi_t \circ \Psi_t \circ \Theta_t,$$

with ξ_t as above, Ψ_t satisfying the same derivative property as above, $(\xi_t \circ \Psi_t)$ is a process in the group of affine transformations of M, but now Θ_t is the component which fixes x_0 for all $t \geq 0$, i.e. $\Theta_t(x_0) \equiv x_0$ and the derivative $D\Theta_t \equiv Id_{(T_{x_0}M)}$.

Next section presents the theoretical results which we are going to illustrate in Section 3.

2 Decomposition of flows

Before we present the main results concerning the non-linear Iwasawa decomposition of stochastic flows, we introduce some geometric preliminaries to establish the framework and the objects which we need to carry on the examples.

We denote by $GL(M)$ the fibre bundle of linear frames on M. It is a principal bundle over M with projection $\pi : GL(M) \to M$ and structural group given by $Gl(d, \mathbb{R})$. A Riemannian metric in M determines a subbundle of orthonormal frames denoted by OM. The canonical Iwasawa decomposition (Gram-Schmidt orthonormalization) defines another projection $\perp: GL(M) \to OM$ with structural group $S \subset Gl(d, \mathbb{R})$, the subgroup of upper triangular matrices. Given a vector field X on M, its covariant derivative at $x \in M$ (with respect to the Levi-Civita connection) is a linear map $\nabla X : T_xM \to T_xM$, denoted by $\nabla X(Y)$ or $\nabla_Y X$ for a vector $Y \in T_xM$.

We can associate to ∇X, via adjoint, an element in the structural group $Gl(d, \mathbb{R})$ of the principal bundle $GL(M)$ given by a matrix $\tilde{X}(u) = \mathrm{ad}(u^{-1})\nabla X$, which acts on the right such that $\nabla X(u) = u\tilde{X}(u)$. Note that, different from ∇X, the right action of the matrix $\tilde{X}(u)$ does depend on u.

The natural lift of X to $GL(M)$ is the vector field δX in $GL(M)$ defined by the following: Let η_t be the local flow of diffeomorphisms associated to X, then

$$\delta X(u) = \frac{d}{dt}[D\,\eta_t(u)]_{t=0}.$$

Note that the vector field δX is equivariant by the right action of $Gl(d, \mathbb{R})$ in the fibres. The connection defines the vertical and horizontal components $\delta X(u) = V\delta X(u) + H\delta X(u)$, with $V\delta X(u) = \nabla X(u)$, see e.g. Elworthy [2] or Kobayashi and Nomizu [3].

Consider the canonical Cartan decomposition of matrices $\mathcal{G} = \mathcal{K} \oplus \mathcal{S}$ into a skew-symmetric and upper triangular component respectively. By projecting in each of these two components, we write $\tilde{X}(u) = [\tilde{X}(u)]_{\mathcal{K}} +$

$[\tilde{X}(u)]_{\mathcal{S}}$. With this notation, the vertical component splits into:

$$V\delta X(u) = u[\tilde{X}(u)]_{\mathcal{K}} + u[\tilde{X}(u)]_{\mathcal{S}}. \tag{2}$$

Analogous to the natural lift of the vector field X to $GL(M)$, there is a natural lift of X to OM, denoted by $(\delta X)^{\perp}$, it corresponds to the projection of δX onto OM, i.e. for $k \in OM$,

$$(\delta X)^{\perp}(k) := \frac{d}{dt}\, [D\,\eta_t(k)]^{\perp}_{t=0}.$$

Again, in terms of the right action of $\tilde{X}(k)$, the vertical component is simply $V(\delta X)^{\perp}(k) = k[\tilde{X}(k)]_{\mathcal{K}}$. In terms of the left action of (∇X) we shall denote $V(\delta X)^{\perp}(k) = (\nabla X(k))^{\perp}k$, where $(\nabla X(k))^{\perp}$ is a skew-symmetric matrix. For a detailed characterization of $(\nabla X(k))^{\perp}$ see [8, Lemma 2.2].

We shall denote by $A(M)$ and $a(M)$ respectively the Lie group of affine transformations of M and its corresponding Lie algebra of infinitesimal affine transformations (vector fields). This is the subgroup of $\mathrm{Dif}(M)$ whose derivatives preserve horizontal curves in the tangent bundle TM. It has dimension at most $d(d+1)$ (maximal dimension if and only if M is flat, see, e.g. Kobayashi and Nomizu [3]). For a fixed $u \in GL(M)$, the linear map

$$\begin{array}{rcl} i_1 : a(M) & \rightarrow & T_u GL(M) \\ X & \mapsto & \delta X(u) \end{array} \tag{3}$$

is injective, see e.g. [3, Thm VI.2.3]. We denote by $\delta a(u)$ its image in $T_u GL(M)$.

We shall denote by $I(M)$ and $\mathcal{I}(M)$ respectively the Lie group of isometries of M and its corresponding Lie algebra of infinitesimal isometries (Killing vector fields). It has dimension at most $d(d+1)/2$ (maximal dimension if and only if M constant curvature, see, e.g. [3]). Note that, if X is a Killing vector field then for any orthonormal frame u we have that $(\nabla X(u))^{\perp} = \nabla X$ and $(\delta X)^{\perp}(u) = \delta X(u)$. For a fixed $u \in OM$, the linear map

$$\begin{array}{rcl} i_2 : \mathcal{I}(M) & \rightarrow & T_u OM \\ X & \mapsto & \delta X(u) \end{array} \tag{4}$$

is just a restriction of the map i_1 defined above, hence it is also injective. We denote by $\delta\mathcal{I}(u)$ its image in $T_u OM$.

The main interesting results which we are going to illustrate in the next section are Theorem 2.2 and Corollary 2.3. To put these results in a context and to introduce the necessary geometrical objects, firstly we present

Theorem 2.1. Fix an element $u \in GL(M)$ with $\pi(u) = x_0$, we shall assume the following hypothesis on the vector fields X^i, $i = 0, 1, \ldots, m$ involved in the sde (1):

H1) $\delta D\Delta(X^i)(u) \in \delta a(u)$, for all affine transformations $\Delta \in A(M)$.

Theorem 2.1 *Assume that for a certain $u \in GL(M)$ hypothesis (H1) holds. Then, the associated stochastic flow φ_t has a unique decomposition:*

$$\varphi_t = \Delta_t \circ \Theta_t,$$

where Δ_t is a diffusion in the group of affine transformations $A(M)$, $\Theta_t(x_0) = x_0$ and $D\Theta_t = Id_{(T_{x_0}M)}$ for all $t \geq 0$.

We recall that since the linear map i_1 of equation (3) is injective, the map

$$X \longmapsto X^a \tag{5}$$

given by the unique infinitesimal affine transformation which satisfies $\delta X^a(u) = \delta X(u)$ is well defined. Note that, in this case:

$$X^a(x_0) = X(x_0) \quad \text{and} \quad \nabla X^a(x_0) = \nabla X(x_0). \tag{6}$$

We recall from [8] that the process Δ_t is the solution of the following equation in the group $A(M)$:

$$d\Delta_t = \sum_{j=0}^{n} \Delta_t [D\Delta_t^{-1}(X^j)]^a \circ dW_t^j, \tag{7}$$

with $\Delta_0 = Id_M$, where the elements in the Lie algebra $a(M)$ act on the right in $A(M)$. By Itô formula in the identity $\Delta_t \Delta_t^{-1} = Id_M$ one finds the Stratonovich equation with the elements of the Lie algebra acting on the left:

$$d\,\Delta_t^{-1} = -\sum_{j=0}^{m} [D\Delta_t^{-1}(X^j)]^a \Delta_t^{-1} \circ dW_t^j.$$

Next theorem guarantees the existence of the so-called non-linear Iwasawa decomposition.

Theorem 2.2 *Let M be a simply connected Riemannian manifold with constant curvature (or its quotient by discrete groups). Then, for every stochastic differential equation and every orthonormal frame $u_0 \in OM$, the associated stochastic flow φ_t has a unique non-linear Iwasawa decomposition*

$$\varphi_t = \xi_t \circ \Psi_t.$$

Conversely, if every flow φ_t on M has this decomposition then the space M has constant curvature.

Again, we recall that since the linear map i_1 of equation (4) is injective, the map

$$X \longmapsto X^i \tag{8}$$

given by the unique infinitesimal isometry which satisfies $\delta X^i(u) = \delta X(u)$ is well defined. Note that, analogous to equations (6), we have that

$$X^i(x_0) = X(x_0) \quad \text{and} \quad \nabla X^i(u) = (\nabla X(u))^{\perp} u. \tag{9}$$

The process ξ_t is the solution of the following equation in the group of isometries $I(M)$:

$$d\xi_t = \sum_{j=0}^{n} \xi_t [D\xi_t^{-1}(X^j)]^i \circ dW_t^j, \tag{10}$$

with $\xi_0 = Id_M$, where the elements in the Lie algebra $\mathcal{I}(M)$ act on the right in $I(M)$.

All systems in flat spaces satisfies simultaneously the hypotheses of the two theorems above, hence, juxtaposing these results we have the following corollary:

Corollary 2.3 *If M is flat, simply connected (or its quotient by discrete groups) then for every sde (1) and every orthonormal frame $u \in OM$, the associated stochastic flow φ_t has a unique decomposition*

$$\varphi = \xi_t \circ \Psi_t \circ \Theta_t$$

where each of the components ξ_t, Ψ_t, Θ_t have the properties stated in Theorems 2.1 and 2.2. Moreover, $\Delta_t = (\xi_t \circ \Psi_t)$ is a diffusion in the group of affine transformations. Conversely, if every flow φ_t has this decomposition then M is flat.

3 Examples:

The diffusion ξ_t in the group of isometries presented in Theorem 2.2 become well defined by equation (10). In this section we shall give a description of the calculation of the vector fields X^i involved in this equation in each one of the three possibilities of simply connected manifolds with constant curvature. In the case of flat spaces, the coefficients X^a of equation (7) for the diffusion $\Delta_t = \xi_t \circ \Psi_t$ (Corollary 2.3) will also be described.

3.1 Flat spaces:

We recall that the group $A(\mathbb{R}^d)$ of affine transformations in $\mathbb{R}^d$ (or any of its quotient space by discrete subgroup) can be represented as a subgroup of $Gl(d+1,\mathbb{R})$:

$$A(\mathbb{R}^d) = \left\{ \begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix} \text{ with } g \in Gl(d,\mathbb{R}) \text{ and } v \text{ is a column vector} \right\}.$$

It acts on the left in $\mathbb{R}^d$ through its natural embedding on $\mathbb{R}^{d+1}$ given by $x \mapsto (1,x)$. The group of isometries is the subgroup of $A(M)$ where $g \in O(n,\mathbb{R})$. Given a vector field X, assume that the initial condition x_0 is the origin and that u is an orthonormal frame in the tangent space at x_0. One can easily compute the vector fields $X^a \in a(\mathbb{R}^d)$ and $X^i \in \mathcal{I}(\mathbb{R}^d)$ using the properties established in equations (6) and (9):

$$X^a(x) = X(0) + (D_0X)x$$

and

$$X^i(x) = X(0) + (D_0X(u))^{\perp}x$$

We shall fix u to be the canonical basis $\{e_1,\ldots,e_d\}$ of $\mathbb{R}^d$ then the matrix $(D_0X(u))^{\perp}$ is simply the skew-symmetric component $(D_0X)_{\mathcal{K}}$.

In terms of the Lie algebra action of $a(\mathbb{R}^d)$, the vector fields X^a and X^i are given by the action of the elements:

$$X^a = \begin{pmatrix} 1 & 0 \\ X & D_0X \end{pmatrix}$$

and

$$X^i = \begin{pmatrix} 1 & 0 \\ X & (D_0X)_{\mathcal{K}} \end{pmatrix}$$

Let φ_t be the flow associated with the vector field X. One checks by inspection and by uniqueness that the component Δ_t in the group of affine transformations (Theorem 2.1) and the component ξ_t (Theorem 2.2) which solve equations (7) and (10), respectively, are given by:

$$\Delta_t = \begin{pmatrix} 1 & 0 \\ \varphi_t & (D_0\varphi_t) \end{pmatrix}, \tag{11}$$

$$\xi_t = \begin{pmatrix} 1 & 0 \\ \varphi_t & (D_0\varphi_t)^{\perp} \end{pmatrix} \tag{12}$$

and

$$\Psi_t = \begin{pmatrix} 1 & 0 \\ 0 & (D_0\varphi_t)^u \end{pmatrix}, \tag{13}$$

where $D_0\varphi_t = (D_0\varphi_t)^{\perp} \cdot (D_0\varphi_t)^u$ is the canonical Iwasawa decomposition of the derivative $D_0\varphi_t$.

For linear systems the decomposition is straightforward. Consider for example the Ornstein-Uhlenbeck system given by

$$dx_t = -\frac{1}{2}x_t \ dt + dW_t$$

where $x_t \in \mathbb{R}^d$ and W_t is a Brownian motion in $\mathbb{R}^d$. The components given by equations (11), (12) and (13) are obtained substituting the derivative $(D_0\varphi_t)$ by $e^{-t/2}\ Id$. In particular the free component Θ_t degenerates to the trivial element in $\mathrm{Dif}(M)$, i.e. $\Theta_t \equiv Id$ for all $t \geq 0$.

Note that, in general, though the X^a corresponds to the first two elements of the Taylor series of a vector field X, the factor Δ_t presents a strong non-linear behavior (in time) due to the fact that the coefficients of equation (7) are non-autonomous.

3.2 Spheres S^d

Let X be a vector field in the sphere S^d. Assume that the starting point is the north pole $N = (0, 0, \ldots, 1) \in S^d$ and that the orthonormal frame is the canonical basis $u = (e_1, \ldots, e_d)$. One way to calculate X^i is finding the element A in the Lie algebra of skew-symmetric matrices $so(d+1)$ whose vector field $\widetilde{A}$ induced in S^d satisfies equations (9), i.e.:

$$\widetilde{A}(e_{d+1}) = X(N),$$

and

$$\frac{d}{dt}[\ e^{At}u]_{t=0} = (\nabla X(u))^{\perp}u.$$

Hence,

$$A = \begin{pmatrix} (\nabla X(N))_{\mathcal{K}} & X(N) \\ X(N)^t & 0 \end{pmatrix},$$

where $X(N)^t$ is the transpose of the column vector $X(N)$.

To complement this description of the vector X^i, we would suggest the reader to see the calculations in Liao [6] in terms of the partial derivatives of the components of X. In that (rather analytical) description, however, one misses the geometrical insight which our description in terms of the action of the skew-symmetry matrix A has.

North-south flow: Let $S^2 - \{N\}$ be parametrized by the stereographic projection π from $\mathbb{R}^2$ which intersects S^2 in the equator. The north-south flow is given by the projection on S^2 of the linear exponential contraction on $\mathbb{R}^2$, precisely: $\varphi_t(p) = \pi \circ e^{-t}\pi^{-1}(p)$. It is associated to the vector field $X(x) = \pi_x(-e_3)$, where π_x is the orthogonal projection into the tangent space $T_x S^d$. For a point $(x, y, z) \in S^2$, one checks that the flow is given by

$$\varphi_t(x, y, z) = \frac{1}{\cosh(t) - z\sinh(t)} \left(x, y, z\cosh(t) - \sinh(t)\right).$$

Let $x_0 = e_1$ and $u = (e_2, e_3)$. For these initial conditions we have the decomposition: $\varphi_t = \xi_t \circ \Psi_t$ where

$$\xi_t = \begin{pmatrix} \operatorname{sech}(t) & 0 & \tanh(t) \\ 0 & 1 & 0 \\ -\tanh(t) & 0 & \operatorname{sech}(t) \end{pmatrix}$$

and, using the double-angle formulas $\sinh(2t) = 2\sinh(t)\cosh(t)$ and $\cosh(2t) = 2\cosh^2(t) - 1$, we find

$$\Psi_t = \left(\frac{2x - 2}{\cosh(2t) - z\sinh(2t) + 1} + 1, \frac{y}{\cosh(t) - z\sinh(t)}, \frac{2(z\cosh(t) + (x-1)\sinh(t))}{\cosh(2t) - z\sinh(2t) + 1} \right).$$

Hence, the derivative of Ψ_t at $(1, 0, 0)$ is

$$D_{(1,0,0)}\Psi_t = \begin{pmatrix} \operatorname{sech}^2(t) & 0 & 0 \\ 0 & \operatorname{sech}(t) & 0 \\ \tanh(t) & 0 & \operatorname{sech}(t) \end{pmatrix}.$$

One sees that

$$D_{(1,0,0)}\Psi_t\ (u) = u\ s_t,$$

where s_t is the upper triangular matrix

$$s_t = \begin{pmatrix} \operatorname{sech}(t) & 0 \\ 0 & \operatorname{sech}(t) \end{pmatrix}.$$

3.3 Hyperbolic spaces

This example has already been worked out in [8], where we deal with the hyperboloid H^n in $\mathbb{R}^{n+1}$ with the metric invariant by the Lorentz group

$O(1,n)$. In this case, a global parametrization centered at $N=(1,0,\ldots,0)\in H^n$ is given by the graph of the map $x^1=\sqrt{1+\sum_{j=2}^{n+1}(x^j)^2}$. We just recall the formula which states that given a vector field $X(x)=a_1(x)\,\partial_1+\ldots+a_{n+1}(x)\,\partial_{n+1}$ with respect to the coordinates above, then, at the point $N=(1,0,\ldots,0)\in H^n$ and an orthonormal frame u in T_NM we have:

$$X^i(u)=\left(\begin{array}{c|ccc} 0 & a_2(N) & \ldots & a_{n+1}(N) \\ \hline a_2(N) & & & \\ \vdots & & ([\partial_j\, a_i](u))^{\perp} & \\ a_{n+1}(N) & & & \end{array}\right).$$

Note that, if u is the canonical basis in T_NM then $([\partial_j\, a_i](u))^{\perp}$ is simply $[(\partial_j a_i)]_{\mathcal{K}}$.

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